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# L(d,j,s) Minimal and Surjective Graph Labeling

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# $L(d, j, s)$ MINIMAL AND SURJECTIVE GRAPH LABELING

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Abstract. Interference between radio signals can be modeled using distance labeling where the vertices on the graph represent the radio towers and the edges represent the interference between the towers. The distance between vertices affects the labeling of the vertices to account for the strength of interference. In this paper we consider three levels of interference between signals on a given graph, $G$. Define $D(x, y)$ to represent the distance between vertex $x$ and vertex $y$. An $L(d, j, s)$ labeling of graph $G$ is a function $f$ from the vertex set of a graph to the set of positive integers, where $|f(x) - f(y)| \geq d$ if $D(x, y) = 1$, $|f(x) - f(y)| \geq j$ if $D(x, y) = 2$, and $|f(x) - f(y)| \geq s$ if $D(x, y) = 3$ for positive integers $m$ and $d$ where $d > j > s$. In this paper we will examine surjective and minimal labeling of different families of graphs including paths, cycles, caterpillars, complete graphs, and complete bipartite graphs.

## 1. Introduction and Definitions

An $L(d, j, s)$ labeling is a simplified model for the channel assignment problem. A summary of the history of the channel assignment problem can be found in $L(3, 2, 1)$-Labeling of Simple Graphs [3].

Define $D(x, y)$ to represent the distance between vertex $x$ and vertex $y$. Let $d$, $j$, and $s$ be positive integers where $d > j > s$. An $L(d, j, s)$ labeling of graph G is a function $f$ from the vertex set of a graph to the set of positive integers such that for any two vertices $x, y$, if $D(x, y) = 1$, then $| f(x)-f(y) | \geq d$; if $D(x, y) = 2$, then $| f(x)-f(y) | \geq j$; and if $D(x, y) = 3$, then $| f(x)-f(y) | \geq s$. For example, consider Figure 1. If vertex $w$ is labeled 1, then because the distance between vertex $w$ and vertex $x$ is 1, the labels of vertex $w$ and vertex $x$ must differ by at least $d$. In other words, $| f(w) - f(x) | \geq d$. The label of vertex $y$ must satisfy $| f(w) - f(y) | \geq j$ and $| f(x) - f(y) | \geq s$ since $D(w, x) = 2$ and $D(x, y) = 1$. The remaining labels must be labeled considering all vertices of distance 3 or less. One possible labeling of the graph in Figure 1 is depicted in Figure 2.

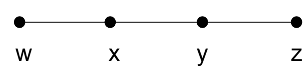


Figure 1. Path with four vertices.

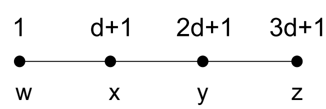


Figure 2. An $L(d, j, s)$ labeling of a path with four vertices.

This paper will examine two types of $L(d, j, s)$ labeling; minimal labeling and surjective labeling. Minimal labeling finds the smallest largest number, $k(G)$, required to label a given graph. For instance, consider Figure 3 which is an $L(3, 2, 1)$ labeling of a path with four vertices, a special case of an $L(d, j, s)$ labeling of Figure 1. However, Figure 3 is a minimal $L(3, 2, 1)$ labeling since there is no labeling that has a smaller largest label. In this case, the smallest largest number is 6. This paper will introduce special cases of $L(d, j, s)$ minimal labeling for uniform caterpillars, paths, cycles, complete graphs, and complete bipartite graphs in sections, 4, 6, 8, 9, and 10.

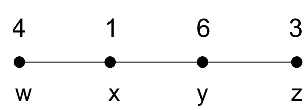


Figure 3. a minimal $L(3, 2, 1)$ labeling of a path with four vertices.

A surjective labeling of a graph requires that every label, $\{1, 2, 3, \ldots, m\}$, is used exactly once, where the graph has $m$ vertices. For example, consider Figure 4 which is an $L(3, 2, 1)$ surjective labeling, a special case of $L(d, j, s)$ labeling, of a path of length 7. Note that the labels 1 through 7 have all been used exactly once. We will discuss special cases of $L(d, j, s)$ surjective labeling of paths, cycles, uniform caterpillars, complete graphs, and complete bipartite graphs in sections 2, 3, 5, 7, and 11.

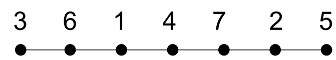


FIGURE 4. A surjective $L(3, 2, 1)$ labeling of a path with seven vertices.

We will use the following families of graphs throughout the paper: complete graphs, complete bipartite graphs, paths, cycles, and uniform caterpillars.

**Definition.** A *complete graph* is a graph in which every vertex is adjacent to every other vertex and is denoted by $K_n$ where $n$ is the number of vertices in the graph.

**Definition.** A *complete bipartite graph* is a graph in which the set of vertices can be decomposed into two disjoint sets $A$ and $B$ such that no two vertices within the same set are adjacent and every vertex in set $A$ is adjacent to every vertex in set $B$.

**Definition.** A graph G, where $G = (v, E)$, is called a path, denoted by $P_n$, if $v = \{v_1, v_2, ..., v_n\}$ such that only $(v_i, v_{i+1}) \in E$ where $1 \leq i \leq n - 1$.

**Definition.** A graph $G$, where $G = (V, E)$, is called a cycle, denoted by $C_n$, if $V = \{v_1, v_2, ..., v_n\}$ such that only $(v_i, v_{i+1}) \in E$ where $1 \leq i \leq n - 1$ and $(v_1, v_n) \in E$.

**Definition.** A caterpillar is a tree in which every vertex is on a central path, called the spine, or adjacent to a vertex on the spine. A graph is a caterpillar if the removal of the degree one vertices produces a path.

**Definition.** A uniform caterpillar is a caterpillar in which every vertex is either of degree one or degree $\Delta$. We denote a uniform caterpillar with $n$ vertices on the spine by $Cat_n$.

## 2. $L(3, 2, 1)$ SURJECTIVE LABELING OF PATHS, CYCLES, AND UNIFORM CATERPILLARS

$L(3, 2, 1)$ minimal labeling of paths, cycles and uniform caterpillars can be found in $L(3, 2, 1)$-*Labeling and Surjective Labeling of Simple Graphs.* [2]. In this section we will only consider $L(3, 2, 1)$ surjective labeling.

**Paths.** In this subsection we will discuss the shortest non-trivial path that can be labeled using surjective labeling. We will also show that all paths of length greater then or equal to 7 can be surjectively labeled. A proof for Theorem 2.1 are from the paper $L(3, 2, 1)$-*Labeling on Simple Graphs* [1] and Theorem 2.2, and its proof, can be found in $L(3, 2, 1)$-*Labeling and Surjective Labeling of Simple Graphs.* [2]

**Theorem 2.1.** *For any path,* $P_n$*,*

$$k(P_n) = \begin{cases} 1, & \textit{if } n = 1; \\ 4, & \textit{if } n = 2; \\ 6, & \textit{if } n = 3, 4; \\ 7, & \textit{if } n = 5, 6, 7; \\ 8, & \textit{if } n \geq 8; \end{cases}$$

*using $L(3, 2, 1)$ labeling.* [1]

**Theorem 2.2.** *The shortest non-trivial path that can be surjectively labeled using $L(3, 2, 1)$ labeling is $P_7$.*

*Proof.* The labeling {3,6,1,4,7,2,5} shows that $P_7$ can be surjectively labeled. We know by Theorem 2.1 that $k(P_6) = 7$, $k(P_5) = 7$, $k(P_4) = 6$, $k(P_3) = 6$, and $k(P_2) = 4$. So for $n < 7$, a path cannot be labeled with a surjective label. Therefore, the shortest non-trivial path that can be surjectively labeled is $P_7$. [2] □

**Theorem 2.3.** *Paths of length greater than or equal to 7 can be surjectively labeled using $L(3, 2, 1)$ labeling.*

*Proof.* By Theorem 2.2 we know that $P_7$ can be surjectively labeled. Assume the path $P_{n-1}$ can be surjectively labeled where $n \geq 8$ . Call the vertex labeled $n-3$ in $P_{n-1}$, $v_i$. Then if vertices $v_{i-1}$ and $v_{i+1}$ exist, they must be labeled less than $n-6$. Also, if vertices $v_{i-2}$ and $v_{i+2}$ exist, they must be labeled less than $n-9$ or labeled $n-1$. If vertex $v_i$ is of degree 1 in $P_{n-1}$, then append an additional vertex to $v_i$ and label this new vertex $n$. This creates a surjective labeling of $P_n$. If vertex $v_i$ is of degree 2 in $P_{n-1}$ and $v_{i+2}$ is not labeled $n-1$, then add an additional vertex between $v_i$ and $v_{i+1}$. Label this new vertex $n$. This creates a surjective labeling of $P_n$. If vertex $v_i$ is of degree 2 in $P_{n-1}$ and $v_{i+2}$ is labeled $n-1$, then add an additional vertex between $v_i$ and $v_{i-1}$. Label this new vertex $n$. This creates a surjective labeling of $P_n$. Thus, paths of length greater than or equal to 7 can be surjectively labeled. □

**Cycles.** In this subsection we will discuss the shortest non-trivial cycle that can be labeled using surjective labeling. We will also show that all cycles of length greater then or equal to 8 can be surjectively labeled. A proof of Theorem 2.4 can be found in paper $L(3, 2, 1)$-*Labeling on Simple Graphs* [1] and Theorem 2.5, and it's proof, are from the paper $L(3, 2, 1)$-*Labeling and Surjective Labeling of Simple Graphs.* [2]

**Theorem 2.4.** *For any cycle, $C_n$, with $n \geq 3$,*

$$k(C_n) = \begin{cases} 7, & \textit{if } n = 3; \\ 8, & \textit{if } n \textit{ is even}; \\ 9, & \textit{if } n \textit{ is odd and } n \neq 3, 7; \\ 10, & \textit{if } n = 7; \end{cases}$$

*using $L(3, 2, 1)$ labeling.*[1]

**Theorem 2.5.** *The shortest cycle that can be surjectively labeled using $L(3, 2, 1)$ labeling is $C_8$.*

*Proof.* The labeling $\{3, 6, 1, 4, 7, 2, 5, 8\}$ shows that $C_8$ can be labeled with a surjective label. We know by Theorem 2.4 that $k(C_n) > n$ for $3 \leq n \leq 7$. This shows that the shortest cycle that can be labeled using surjective labeling is $C_8$. [2] □

**Theorem 2.6.** *Cycles of length greater than or equal to 8 can be surjectively labeled using $L(3, 2, 1)$ labeling.*

*Proof.* By Theorem 2.5 we know that $C_8$ can be surjectively labeled. Assume the cycle $C_{n-1}$ can be surjectively labeled, where $n \geq 9$. Call the vertex labeled $n-3$ in $C_{n-1}$, $v_i$. Then vertices $v_{i+1}$ and $v_{i-1}$ must be labeled less than or equal to $n-3-3 = n-6$. Also, vertices $v_{i+2}$ and $v_{i-2}$ must be labeled less than $n-6-3 = n-9$ or $n-1$. If vertex $v_{i+2}$ is not labeled $n-1$, then add an additional vertex between $v_i$ and $v_{i+1}$. Label this vertex $n$. This creates a surjective labeling of $C_n$. If vertex $v_{i+2}$ is labeled $n-1$, then add an additional vertex between $v_i$ and $v_{i-1}$. Label this vertex $n$. This creates a surjective labeling of $C_n$. Thus, cycles of length greater than or equal to 8 can be surjectively labeled. [2] □

**Uniform Caterpillars.** In this subsection we will explore how to surjectively label uniform caterpillars. We will consider the special case when $\Delta = 2$ in Theorem 2.7 and $\Delta > 2$ in Theorem 2.8 and Theorem 2.9.

**Theorem 2.7.** *A uniform caterpillar with $\Delta = 2$ can be surjectively labeled using $L(3, 2, 1)$ labeling if and only if $n \geq 5$.*

*Proof.* A caterpillar with $\Delta = 2$ is a path. A path can be surjectively labeled using $L(3, 2, 1)$ labeling if and only if it has a length of greater than or equal to 7 vertices by Theorem 2.3. Therefore, a uniform caterpillar with $\Delta = 2$ can be surjectively labeled using $L(3, 2, 1)$ labeling if and only if $n \geq 5$. □

**Theorem 2.8.** *A uniform caterpillar with $n \leq 3$ cannot be surjectively labeled using $L(3, 2, 1)$ labeling.*

*Proof.* Case I: $n = 3$.

Let $i$ be the label of the middle vertex on the spine. Every other vertex is at most two vertices away from the vertex labeled $i$. Therefore, this caterpillar cannot be surjectively labeled because $(i + 1)$ and $(i - 1)$ cannot be placed anywhere on the graph.

Case II: $n \leq 2$.

Let $i$ be the label of any vertex on the spine. Every other vertex is at most two vertices away from the vertex labeled $i$. Therefore, this caterpillar cannot be surjectively labeled because $(i + 1)$ and $(i - 1)$ cannot be placed anywhere on the graph. □

**Theorem 2.9.** *Any uniform caterpillar of $Cat_n$, with $n \geq 4$ and $\Delta \geq 3$ can be surjectively labeled using $L(3, 2, 1)$ labeling.*

*Proof.* Case 1: $n = 4$ and $\Delta \geq 3$.

The spine can be labeled $\{4, 7, 10, 3\}$. The unlabeled vertices adjacent to the vertex labeled 4 can be labeled 9 and 1, the unlabeled vertex adjacent to the vertex labeled 7 can be labeled 2, the unlabeled vertex adjacent to the vertex labeled 10 can be labeled 5, and the unlabeled vertices adjacent to the vertex labeled 3 can be labeled 6 and 8. If $\Delta = 3$, then this is a surjective labeling of the caterpillar. If

$\Delta > 3$ label the $\Delta - 3$ unlabeled vertices adjacent to vertex $v_n$ using the expression $(k+1)n + 2 + i$ where $0 \leq k \leq \Delta - 3$ and $1 \leq i \leq n$.

Case II: $n \geq 5$ and $\Delta \geq 3$.

The spine of a caterpillar is a path. Let $V = \{v_1, v_2, v_3 ... v_n\}$ be the set of vertices on the spine of the caterpillar with $v_i$ adjacent to $v_{i+1}$ for $1 \leq i \leq n-1$. Let $v_0$ be a vertex not on the spine that is adjacent to $v_1$ and $v_{n+1}$ a vertex not on the spine that is adjacent to $v_n$. The path $\{v_0, v_1, \dots, v_n, v_{n+1}\}$ can be surjectively labeled with the labels 1 through $n+2$ by Theorem 2.3. Surjectively label this path in such a way that the vertex with label $n+2$, call it $v_m$, has the largest possible index $m$. Label the $\Delta - 2$ unlabeled vertices adjacent to vertex $v_n$ using the formula $(k+1)n + 2 + i$ where $0 \leq k \leq \Delta - 3$ and $1 \leq i \leq n$. If vertex $v_1$ is not labeled $n+1$, then this gives a surjective labeling of the caterpillar. If vertex $v_1$ is labeled $n+1$, then switch the labels of the vertices labeled $n+3$ and $n+4$. This gives the surjective labeling of the uniform caterpillar. □

## 3. $L(d, 2, 1)$ Surjective Labeling of Paths

In this section we will find the smallest path, $P_n$, that can be surjectively labeled using $L(d, 2, 1)$ labeling for certain values of $d$. We will also show that if a path $P_k$ can be surjectively labeled for a particular value $d$, then $P_n$ where $n > k$ can also be surjectively labeled. We developed a computer program to quickly and exhaustively check permutations of varying lengths to determine which permutations represented $L(d, 2, 1)$, $L(md, d, 1)$, $L(d + m, d, 1)$, or $L(d, j, s)$ surjective labels of paths.

This program begins by creating an array of $n$ positions and places a 1 in the first position. Once a number, in this case 1, has been used, it is then marked unavailable and is no longer able to be used in the remaining positions of the array. Next the smallest number, which both remains to be used and is valid with a given labeling type, is placed in the second position. This algorithm is repeated until all the positions are filled. Once this process is completed through the $n$th position, the array is printed, but if no valid labeling exists for the $i$th spot, the previous spot is adjusted to the next smallest available number. The computer repeats this process until either all the possible valid labels are printed or if none exist it will tell us so.

This data was gathered using the computer program described above. Conjecture 3.1 is a summary of the pattern discovered from Table 1.

**Conjecture 3.1.** *For $L(d, 2, 1)$ labeling where $d \geq 3$, the shortest path, $P_n$, that can be surjectively labeled is $P_{2d+2}$ if $n$ is even and $P_{2d+1}$ if $n$ is odd.*

**Theorem 3.2.** *If there exists a surjective $L(d, 2, 1)$ labeling of path $P_k$ for some positive integer $k$, then path $P_n$, with $n > k$ can also be surjectively labeled.*

*Proof.* Assume the path $P_{n-1}$ can be surjectively labeled. Call the vertex labeled $n - d$ in $P_{n-1}$, $v_i$. Then if vertices $v_{i-1}$ and $v_{i+1}$ exist, they must be labeled less than $n - 2d$. Also, if vertices $v_{i-2}$ and $v_{i+2}$ exist, they must be labeled less than $n - 3d$ or greater than $n - d + 2$. If vertex $v_i$ is of degree 1 in $P_{n-1}$, then append an additional vertex to $v_i$ and label this new vertex $n$. If vertex $v_i$ is of degree 2 in $P_{n-1}$ and $v_{i+2}$ is not labeled $n-1$, then add an additional vertex between $v_i$ and $v_{i+1}$. Label this new vertex $n$. This creates a surjective labeling of $P_n$. If vertex $v_i$ is of degree 2 in $P_{n-1}$ and $v_{i+2}$ is labeled $n-1$, then add an additional

| $d$ | $n$ |
|---|---|
| 3 | 7 |
| 4 | 10 |
| 5 | 11 |
| 6 | 14 |
| 7 | 15 |
| 8 | 18 |
| 9 | 19 |
| 10 | 22 |
| 11 | 23 |

TABLE 1. This table shows the length of the shortest path, $P_n$, that can be surjectively labeled using $L(d,2,1)$ labeling.

vertex between $v_i$ and $v_{i-1}$. Label this new vertex $n$. This also creates a surjective labeling of $P_n$. □

## 4. $L(md,d,1)$ Minimal Labeling of Paths and Cycles

In this section we will find $k(G_n)$ for paths and cycles of length $n$ using $L(md,d,1)$ labeling where $m$ and $d$ are positive integers and $md > d > 1$. A summary of the results for paths can be found in Theorem 4.2. When considering cycles, we must consider 2 cases. We will begin by considering $L(md,d,1)$ labeling where $m \geq 3$ and $d \geq 2$. A summary of those results can be found in Theorem 4.10. We will conclude the section by examining the special case of $L(2d,d,1)$ labeling where $d \geq 2$. The results of $k(C_n)$ using $L(2d,d,1)$ labeling can be found in Theorem 4.14.

**Lemma 4.1.** *For a path on $n$ vertices, $P_n$, with $n \geq 5$, $d \geq 2$, and $m \geq 2$, $k(P_n) \geq md + 2d + 1$ using $L(md,d,1)$ labeling.*

*Proof.* Let $f$ be a minimal $L(md,d,1)$ labeling for a path on $n$ vertices, $P_n$. Consider vertex $v_i$ with label 1. There is an induced subpath of at least 3 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}\}$ be this subpath. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $md + 1 \leq f(v_{i+1}) \leq md + d$.
Then $f(v_{i+2}) \geq 2md + 1$, which is greater than or equal to $md + 2d + 1$ when $m \geq 2$.

Case II: $md + d + 1 \leq f(v_{i+1}) \leq md + 2d$.
Then $d+1 \leq f(v_{i+2}) \leq 2d$. Now there are at least two vertices not yet labeled so we know either $v_{i-1}$ or $v_{i+3}$ exists. If $v_{i+3}$ exists then $f(v_{i+3})$ must be greater than or equal to $md + 2d + 1$. If $v_{i-1}$ exists and $v_{i+3}$ does not, then consequently $v_{i-2}$ exists. Then $md + 1 \leq f(v_{i-1}) \leq md + d$. This result forces $f(v_{i-2}) \geq 2md + 1$ which is greater than or equal to $md + 2d + 1$ for $m \geq 2$.

Therefore, we can conclude that $k(P_n) \geq md + 2d + 1$ when $n \geq 5$, $d \geq 2$, and $m \geq 2$ using $L(md,d,1)$ labeling. □

**Theorem 4.2.** *For any path, $P_n$, when $d \geq 2$ and $m \geq 2$*

$$k(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ md+1, & \text{if } n = 2; \\ md+d+1, & \text{if } n = 3,4; \\ md+2d+1, & \text{if } n \geq 5; \end{cases}$$

*using $L(md, d, 1)$ labeling.*

*Proof.* Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices on $P_n$, with $v_i$ adjacent to $v_{i+1}$ for $1 \leq i \leq n-1$. For each $P_n$ we proceed with the following cases.

Case I: $n = 1$.
This is evidently true.

Case II: $n = 2$.
The labeling pattern $\{md+1, 1\}$ shows that $k(P_n) = md+1$ for $n = 2$.

Case III: $n = 3, 4$.
Consider vertex $v_i$ such that $f(v_i) = 1$. If $v_i$ is of degree 2, then we know that vertices $v_{i+1}$ and $v_{i-1}$ exist such that $f(v_{i+1}) \geq md+1$ and $f(v_{i-1}) \geq md+d+1$. If $v_i$ is of degree 1, then we know that either vertices $v_{i+1}$ and $v_{i+2}$ or $v_{i-1}$ and $v_{i-2}$ exist. Assume without the loss of generality, that $v_{i+1}$ and $v_{i+2}$ exist. Then $md+1 \leq f(v_{i+1}) \leq md+d$. This forces $f(v_{i+2}) \geq 2md+1$, which is greater than or equal to $md+2d+1$ when $m \geq 2$. Thus, the labeling pattern $\{md+1, 1, md+d+1, d+1\}$ shows that $k(P_n) = md+d+1$ for $n = 3, 4$. Observe that this pattern is not repeatable.

Case IV: $n \geq 5$.
Let $f$ be defined as $f(\{v_1, v_2, v_3, v_4\}) = \{1, md+d+1, d+1, md+2d+1\}$ and $f(v_i) = f(v_j)$ if $i \equiv j \pmod 4$. Therefore we can conclude by the definition of $f$ that $k(P_n) \leq md+2d+1$ for $n \geq 5$. By combining this result with the results of Lemma 4.1, we obtain $k(P_n) = md+2d+1$ for $n \geq 5$. □

**Lemma 4.3.** *For a cycle on 4 vertices, $C_4$, with $d \geq 2$ and $m \geq 2$, $k(C_4) \geq md+2d+1$ using $L(md, d, 1)$ labeling.*

*Proof.* Let $f$ be a minimal $L(md, d, 1)$ labeling for a cycle with 4 vertices, $C_4$. Consider vertex $v_i$ with label 1. There is an induced subpath of 4 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ be this subpath. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $md+1 \leq f(v_{i+1}) \leq md+d$.
Then $f(v_{i+2}) \geq 2md+1$, which is greater than or equal to $md+2d+1$ when $m \geq 2$.

Case II: $md+d+1 \leq f(v_{i+1}) \leq md+2d$.
Then $d+1 \leq f(v_{i+2}) \leq 2d$. This forces $f(v_{i+3})$ to be greater than or equal to $md+2d+1$.

Therefore, we can conclude that $k(C_4) \geq md+2d+1$ when $d \geq 2$ and $m \geq 2$ using $L(md, d, 1)$ labeling. □

**Lemma 4.4.** *For a cycle with $n$ vertices where $n$ is an odd integer greater than or equal to 3, $k(C_n) \geq 2md + 1$ when $d \geq 2$ and $m \geq 2$ using $L(md, d, 1)$ labeling.*

*Proof.* Let $f$ be a minimal $L(md, d, 1)$ labeling for a cycle with $n$ vertices where $n$ is an odd number. Given the nature of odd cycles, two vertices with labels greater than or equal to $md$ must be adjacent to one another in the graph. Therefore, $k(C_n) \geq 2md$. Assume $k(C_n) = 2md$. Then a vertex labeled $md$ must be adjacent to a vertex labeled $2md$. However this would force another vertex to be labeled $3md$. Thus, $k(C_n) \geq 2md + 1$ for $2 \nmid n$. □

**Lemma 4.5.** *For a cycle on 6 vertices, $C_6$, with $d \geq 2$ and $m \geq 3$, $k(C_6) \geq md + 3d + 1$ using $L(md, d, 1)$ labeling.*

*Proof.* Let $f$ be a minimal $L(md, d, 1)$ labeling for a cycle with 6 vertices, $C_6$. Consider vertex $v_i$ with label 1. There is an induced subpath of 6 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}$ be this subpath. Note that no value can be repeated in this subpath given that in the cycle each vertex is at most a distance of 3 from every other vertex. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $md + 1 \leq f(v_{i+1}) \leq md + d$.
Then $f(v_{i+2}) \geq 2md + 1$ which is greater than or equal to $md + 3d + 1$ when $m \geq 3$.

Case II: $md + d + 1 \leq f(v_{i+1}) \leq md + 2d$.
Then $d + 1 \leq f(v_{i+2}) \leq 2d$, $md + 2d + 1 \leq f(v_{i+3}) \leq md + 3d$, and $2d + 1 \leq f(v_{i+4}) \leq 3d$. This result forces $f(v_{i+5}) \geq md + 3d + 1$.

Case III: $md + 2d + 1 \leq f(v_{i+1}) \leq md + 3d$.
Then $d + 1 \leq f(v_{i+2}) \leq 3d$. If $d + 1 \leq f(v_{i+2}) \leq 2d$, then $md + d + 1 \leq f(v_{i+3}) \leq md + 2d$. This result forces $f(v_{i+4}) \geq 2md + d + 1$ which is greater than $md + 3d + 1$ when $m \geq 3$. If $2d + 1 \leq f(v_{i+2}) \leq 3d$, then $f(v_{i+3}) \geq md + 3d + 1$.

Therefore, we can conclude that $k(C_6) \geq md + 3d + 1$ when $d \geq 2$ and $m \geq 3$ using $L(md, d, 1)$ labeling. □

**Lemma 4.6.** *For a cycle with $2 \mid n$, $4 \nmid n$ and $n \geq 10$, $k(C_n) \geq md + 3d + 1$ using $L(md, d, 1)$ labeling when $d \geq 2$ and $m \geq 3$.*

*Proof.* Let $f$ be a minimal $L(md, d, 1)$ labeling for a cycle $2 \mid n$, $4 \nmid n$, and $n \geq 10$. From Theorem 4.2 we know that for a path with $n \geq 5$, $k(P_n) = md + 2d + 1$ using $L(md, d, 1)$ labeling. Therefore, for any cycle with $n \geq 5$, $k(C_n) \geq md + 2d + 1$. Assume that $md + 2d + 1 \leq k(C_n) \leq md + 3d$. Assume $md + 2d + 1 \leq f(v_i) \leq md + 3d$. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $1 \leq f(v_{i+1}) \leq d$.
Then $md + 1 \leq f(v_{i+2}) \leq md + 2d$. If $md + 1 \leq f(v_{i+2}) \leq md + d$, then $f(v_{i+3}) \geq 2md + 1$ which is greater than or equal to $md + 3d + 1$ when $m \geq 3$. If $md + d + 1 \leq f(v_{i+2}) \leq md + 2d$, then $d + 1 \leq f(v_{i+3}) \leq 2d$, and $md + 2d + 1 \leq f(v_{i+4}) \leq md + 3d$. This forces $1 \leq f(v_{i+5}) \leq d$ or $2d + 1 \leq f(v_{i+5}) \leq 3d$. If $2d + 1 \leq f(v_{i+5}) \leq 3d$, then $f(v_{i+6}) \geq md + 3d + 1$. If $1 \leq f(v_{i+5}) \leq d$, then $md + 1 \leq f(v_{i+6}) \leq md + 2d$. If $md + 1 \leq f(v_{i+6}) \leq md + d$, then $f(v_{i+7}) \geq 2md + 1$ which is greater than or equal to $md + 3d + 1$ when $m \geq 3$. If $md + d + 1 \leq$

$f(v_{i+6}) \leq md + 2d$, then $d + 1 \leq f(v_{i+7}) \leq 2d$. Notice that this labeling pattern is a repeated pattern of four labels. Since $n$ is not divisible by 4, we must have two vertices in our cycle that cannot be labeled using this pattern. It follows that one of these vertices must be labeled with a label greater than or equal to $md + 3d + 1$.

Case II: $d + 1 \leq f(v_{i+1}) \leq 2d$.

Then $md+d+1 \leq f(v_{i+2}) \leq md + 2d$, $1 \leq f(v_{i+3}) \leq d$, and $md+1 \leq f(v_{i+4}) \leq md + d$ or $md + 2d + 1 \leq f(v_{i+4}) \leq md + 3d$. If $md + 1 \leq f(v_{i+4}) \leq md + d$, then $f(v_{i+5}) \geq 2md + 1$ which is greater than or equal to $md + 3d + 1$ when $m \geq 3$. If $md + 2d + 1 \leq f(v_{i+4}) \leq md + 3d$, then $d + 1 \leq f(v_{i+5}) \leq 3d$. If $2d + 1 \leq f(v_{i+5}) \leq 3d$, then $f(v_{i+6}) \geq md + 3d + 1$. If $d + 1 \leq f(v_{i+5}) \leq 2d$, then $md + d + 1 \leq f(v_{i+6}) \leq md + 2d$ and $1 \leq f(v_{i+7}) \leq d$. Notice that this labeling pattern is a repeated pattern of four labels. Since $n$ is not divisible by 4, we must have two vertices in our cycle that cannot be labeled using this pattern. It follows that one of these vertices must be labeled greater than $md + 3d + 1$.

Case III: $2d + 1 \leq f(v_{i+1}) \leq 3d$.

Then $f(v_{i+2}) \geq md + 3d + 1$.

Since assuming $md+2d+1 \leq k(C_n) \leq md + 3d$ leads to a contradiction, we can conclude that $k(C_n) \geq md + 3d + 1$ for $2 \mid n$, $4 \nmid n$ and $n \geq 10$. □

**Fact 4.7.** *Let $n$ be an even integer. If $n \geq 4$, then $n = 4a+6b$ for some non-negative integers a,b.*

**Fact 4.8.** *Let $n$ be an even integer. If $n \geq 8$, then $n = 4a+5b$ for some non-negative integers a,b.*

**Fact 4.9.** *Let $n$ be an odd integer. If $n \geq 9$ and $n \neq 11$, then $n = 4a + 5b$ for some non-negative integers a,b.*

**Theorem 4.10.** *For any cycle, $C_n$, where $n$ is a positive integer greater than or equal to 3, $d \geq 2$, and $m \geq 3$*

$$k(C_n) = \begin{cases} md + 2d + 1, & \text{if } 4 \mid n; \\ md + 3d + 1, & \text{if } 2 \mid n \text{ and } 4 \nmid n; \\ 2md + 1, & \text{if } 2 \nmid n; \end{cases}$$

*using $L(md, d, 1)$ labeling.*

*Proof.* Let $n \geq 3$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices on $C_n$, with $v_i$ adjacent to $v_{i+1}$ for $1 \leq i \leq n - 1$ and vertex $v_1$ adjacent to $v_n$. For $C_n$ we proceed with the following cases.

Case I: $4 \mid n$.

By Lemma 4.3 we know that $k(C_4) \geq md + 2d + 1$. The labeling pattern $\{1, md + d + 1, d + 1, md + 2d + 1\}$ shows that $k(C_4) = md + 2d + 1$. By Theorem 4.2 we know that for a path with $n \geq 5$, $k(P_n) = md+2d+1$ using $L(md, d, 1)$ labeling. Therefore for any cycle $n \geq 5$, $k(C_n) \geq md + 2d + 1$. Notice that we can repeat the labeling pattern of $C_4$ infinitely for all $C_n$ where $4 \mid n$. Therefore $k(C_n) = md + 2d + 1$ when $4 \mid n$.

Case II: $2 \mid n$ and $4 \nmid n$.

By Lemma 4.5 we know that $k(C_6) \geq md + 3d + 1$. The labeling pattern $\{1, md+d+1, d+1, md+2d+1, 2d+1, md+3d+1\}$ shows that $k(C_6) = md+3d+1$. From Lemma 4.6 we know that $k(C_n) \geq md + 3d + 1$ for $2 \mid n$, $4 \nmid n$, and $n \geq 10$. By Fact 4.7 we know that if $n$ is a positive integer and $n \geq 4$, then $n = 4a + 6b$ for some non-negative integers $a$,$b$. The labeling pattern

$$\{\underbrace{1, md+d+1, d+1, md+2d+1,}_{a \text{ times}}\underbrace{1, md+d+1, d+1, md+2d+1, 2d+1, md+3d+1}_{b \text{ times}}\}$$

can be used to label any cycle with $n \mid 2$ and $n \geq 4$. Therefore, $k(C_n) = md+3d+1$ when $2 \mid n$ and $4 \nmid n$.

Case III: $2 \nmid n$.

By Lemma 4.4 we know that $k(C_n) \geq 2md + 1$ when $2 \nmid n$, $d \geq 2$, and $m \geq 2$. The labeling pattern $\{1, md+1, 2md+1\}$ shows that $k(C_3) = 2md+1$. The labeling pattern $\{1, md + 1, 2md + 1, d + 1, md + d + 1\}$ shows that $k(C_5) = 2md + 1$. The labeling pattern $\{1, md+1, 2md+1, 2d+1, md+2d+1, d+1, md+d+1\}$ shows that $k(C_7) = 2md + 1$. The labeling pattern $\{1, md + d + 1, d + 1, md + 2d + 1, 1, md + d + 1, d + 1, md + 2d + 1, 2d + 1, 2md + 1, md + 1\}$ shows that $k(C_{11}) = 2md + 1$. We also know from Fact 4.9 that if $n$ is an odd integer and $n \geq 9$ and $n \neq 11$, then $n = 4a + 5b$ for some non-negative integers $a$,$b$. The labeling pattern

$$\{\underbrace{1, md+d+1, d+1, md+2d+1,}_{a \text{ times}}\underbrace{1, md+d+1, d+1, 2md+1, md+1}_{b \text{ times}}\}$$

can be used to label any cycle with $2 \nmid n$, $n \geq 9$, and $n \neq 11$. Therefore, $k(C_n) = 2md + 1$. □

**Lemma 4.11.** *For a cycle on 6 vertices, $C_6$, with $d \geq 2$, $k(C_6) \geq 4d + 2$ using $L(2d, d, 1)$ labeling.*

*Proof.* Let $f$ be a minimal $L(2d, d, 1)$ labeling for a cycle with 6 vertices, $C_6$. Consider vertex $v_i$ with label 1. There is an induced subpath of 6 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}$ be this subpath. Note that no value can be repeated in this subpath given that in the cycle each vertex is at most a distance of 3 from every other vertex. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $f(v_{i+1}) = 2d + 1$.

Then $f(v_{i+2}) = 4d + 1$, $2 \leq f(v_{i+3}) \leq d + 1$, and $2d + 2 \leq f(v_{i+4}) \leq 3d + 1$. This forces $f(v_{i+5})$ to be greater than or equal to $4d + 2$.

Case II: $2d + 2 \leq f(v_{i+1}) \leq 3d$.

Then $f(v_{i+2}) \geq 4d + 2$.

Case III: $f(v_{i+1}) = 3d + 1$.

Then $f(v_{i+2}) = d + 1$, $f(v_{i+3}) = 4d + 1$, and $f(v_{i+4}) = 2d + 1$. This forces $f(v_{i+5})$ to be greater than or equal to $5d + 1$ which is greater than $4d + 2$.

Case IV: $3d + 2 \leq f(v_{i+1}) \leq 4d$.

Then $d + 1 \leq f(v_{i+2}) \leq 2d$. This result forces $f(v_{i+3}) \geq 4d + 2$.

Case V: $f(v_{i+1}) = 4d + 1$.

Then $d+1 \leq f(v_{i+2}) \leq 2d+1$. If $f(v_{i+2}) = d+1$, then $f(v_{i+3}) = 3d+1$. This result forces $f(v_{i+4}) \geq 5d+1$. If $d+2 \leq f(v_{i+2}) \leq 2d+1$, then $f(v_{i+3}) \geq 5d+1$.

Therefore, we can conclude that $k(C_6) \geq 4d+2$ when $d \geq 2$ using $L(2d, d, 1)$ labeling. □

**Lemma 4.12.** *For a cycle on 7 vertices, $C_7$, when $d \geq 2$, $k(C_7) \geq 4d+3$ using $L(2d, d, 1)$ labeling.*

*Proof.* Let $f$ be a minimal $L(2d, d, 1)$ labeling for a cycle with 7 vertices, $C_7$. Consider vertex $v_i$ with label 1. There is an induced subpath of 7 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}\}$ be this subpath. Note that no value can be repeated in this subpath given that in the cycle each vertex is at most a distance of 3 away from every other vertex. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $f(v_{i+1}) = 2d+1$
Then $4d+1 \leq f(v_{i+2}) \leq 4d+2$, $2 \leq f(v_{i+3}) \leq d+1$, $2d+2 \leq f(v_{i+4}) \leq 3d+2$. If $f(v_{i+4}) = 2d+2$, then $f(v_{i+5}) = 4d+2$. This result forces $f(v_{i+6}) \geq 6d+2$ since $v_i$ and $v_{i+6}$ are adjacent in the cycle. If $2d+3 \leq f(v_{i+4}) \leq 3d+2$, then $f(v_{i+5}) \geq 4d+3$.

Case II: $f(v_{i+1}) = 2d+2$.
Then $f(v_{i+2}) = 4d+2$, $2 \leq f(v_{i+3}) \leq d+2$, and $2d+3 \leq f(v_{i+4}) \leq 3d+2$. If $2d+3 \leq f(v_{i+4}) \leq 3d+1$, then $f(v_{i+5}) = 4d+3$. If $f(v_{i+4}) = 3d+2$, then $f(v_{i+5}) = d+2$. This result forces $f(v_{i+6}) \geq 4d+3$.

Case III: $2d+3 \leq f(v_{i+1}) \leq 3d$ where $d > 2$.
If $d > 2$, then $f(v_{i+2}) \geq 4d+3$.

Case IV: $3d+1 \leq f(v_{i+1}) \leq 4d$.
Then $d+1 \leq f(v_{i+2}) \leq 2d$, $4d+1 \leq f(v_{i+3}) \leq 4d+2$, and $2d+1 \leq f(v_{i+4}) \leq 2d+2$. This forces $f(v_{i+5}) \geq 5d+1$ since $v_i$ and $v_{i+5}$ are of distance 2 in the cycle.

Case V: $f(v_{i+1}) = 4d+1$.
Then $d+1 \leq f(v_{i+2}) \leq 2d+1$. If $f(v_{i+2}) = d+1$, then $f(v_{i+3}) = 3d+1$. This forces $f(v_{i+4}) \geq 5d+1$ which is greater than or equal to $4d+3$ when $d \geq 2$. If $d+2 \leq f(v_{i+2}) \leq 2d+1$, then $f(v_{i+3})$ is greater than or equal to $5d+1$.

Case VI: $f(v_{i+1}) = 4d+2$.
Then $d+1 \leq f(v_{i+2}) \leq 2d+2$. If $f(v_{i+2}) = d+1$, then $3d+1 \leq f(v_{i+3}) \leq 3d+2$. This forces $f(v_{i+4}) \geq 5d+1$ which is greater than or equal to $4d+3$ when $d \geq 2$. If $f(v_{i+2}) = d+2$, then $f(v_{i+3}) = 3d+2$, $f(v_{i+4}) = 2$, and $f(v_{i+5}) = 2d+2$. This forces $f(v_{i+6}) \geq 5d+2$ which is greater than $4d+3$ when $d > 1$. If $d+3 \leq f(v_{i+2}) \leq 2d+1$, then $f(v_{i+3}) \geq 5d+2$. If $f(v_{i+2}) = 2d+2$, then $f(v_{i+3}) = 2$, $3d+2 \leq f(v_{i+4}) \leq 4d+1$, and $d+2 \leq f(v_{i+5}) \leq 2d+1$. This forces $f(v_{i+6}) \geq 5d+2$.

Therefore, we can conclude that $k(C_7) \geq 4d+3$ when $d \geq 2$ using $L(2d, d, 1)$ labeling. □

**Lemma 4.13.** *For a cycle with 11 vertices, $C_{11}$, when $d \geq 2$, $k(C_{11}) \geq 4d+2$ using $L(2d, d, 1)$ labeling.*

*Proof.* Let $f$ be a minimal $L(2d, d, 1)$ labeling for a cycle with 11 vertices, $C_{11}$. Consider vertex $v_i$ with label 1. There is an induced subpath of 11 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_{i+7}, v_{i+8}, v_{i+9}, v_{i+10}\}$ be this subpath. Now we can consider the possibilities for $f(v_{i+1})$. Note that in a cycle with 11 vertices a label can appear at most twice on the cycle.

Case I: $f(v_{i+1}) = 2d + 1$.

Then $f(v_{i+2}) = 4d + 1$ and $2 \leq f(v_{i+3}) \leq d + 1$. If $2 \leq f(v_{i+3}) \leq d$, then $2d + 2 \leq f(v_{i+4}) \leq 3d + 1$. This forces $f(v_{i+5})$ to be greater than or equal to $4d + 2$. If $f(v_{i+3}) = d + 1$, then $f(v_{i+4}) = 3d + 1$ and $f(v_{i+5}) = 1$. It follows that $f(v_{i+6}) = 2d + 1$ or $f(v_{i+6}) = 4d + 1$. If $f(v_{i+6}) = 4d + 1$, then $d + 1 \leq f(v_{i+7}) \leq 2d + 1$. If $d + 2 \leq f(v_{i+7}) \leq 2d + 1$, then $f(v_{i+8}) \geq 5d + 1$ which is greater than $4d + 2$ when $d > 1$. If $f(v_{i+7}) = d + 1$, then $f(v_{i+8}) = 3d + 1$. This forces $f(v_{i+9}) \geq 5d + 1$. If $f(v_{i+6}) = 2d + 1$, then $f(v_{i+7}) = 4d + 1$ and $2 \leq f(v_{i+8}) \leq d + 1$. Then $2d + 2 \leq f(v_{i+9}) \leq 3d + 1$ which forces $f(v_{i+10})$ to be greater than or equal to $4d + 2$.

Case II: $2d + 2 \leq f(v_{i+1}) \leq 3d$.

Then $f(v_{i+2})$ is greater than or equal to $4d + 2$.

Case III: $3d + 1 \leq f(v_{i+1}) \leq 4d$.

Then $d + 1 \leq f(v_{i+2}) \leq 2d$. If $d + 2 \leq f(v_{i+2}) \leq 2d$, then $f(v_{i+3}) \geq 4d + 2$. If $f(v_{i+2}) = d + 1$, then $f(v_{i+3}) = 4d + 1$. Then $f(v_{i+4}) = 1$ or $f(v_{i+4}) = 2d + 1$. If $f(v_{i+4}) = 1$, then $2d + 1 \leq f(v_{i+5}) \leq 3d + 1$. If $2d + 1 \leq f(v_{i+5}) \leq 3d$, then $f(v_{i+6}) \geq 4d + 2$. If $f(v_{i+5}) = 3d + 1$, then $f(v_{i+6}) = d + 1$, $f(v_{i+7}) = 4d + 1$, and $f(v_{i+8}) = 2d + 1$. This forces $f(v_{i+9}) \geq 5d + 1$ which is greater than $4d + 2$ when $d > 1$. If $f(v_{i+4}) = 2d + 1$, then $f(v_{i+5}) = 1$, $3d + 1 \leq f(v_{i+6}) \leq 4d$, and $d + 1 \leq f(v_{i+7}) \leq 2d$. If $d + 2 \leq f(v_{i+7}) \leq 2d$, then $f(v_{i+8}) \geq 4d + 2$. If $f(v_{i+7}) = d + 1$, then $f(v_{i+8}) = 4d + 1$ and $f(v_{i+9}) = 2d + 1$. This forces $f(v_{i+10})$ to be greater than or equal to $5d + 1$.

Case IV: $f(v_{i+1}) = 4d + 1$.

Then $d + 1 \leq f(v_{i+2}) \leq 2d + 1$. If $d + 2 \leq f(v_{i+2}) \leq 2d + 1$, then $f(v_{i+3}) \geq 5d + 1$. If $f(v_{i+2}) = d + 1$, then $f(v_{i+3}) = 3d + 1$, $f(v_{i+4}) = 1$, and $f(v_{i+5}) = 2d + 1$ or $f(v_{i+5}) = 4d + 1$. If $f(v_{i+5}) = 2d + 1$, then $f(v_{i+6}) = 4d + 1$, $2 \leq f(v_{i+7}) \leq d + 1$, and $2d + 2 \leq f(v_{i+8}) \leq 3d + 1$. This forces $f(v_{i+9}) \geq 4d + 2$. If $f(v_{i+5}) = 4d + 1$, then $d + 1 \leq f(v_{i+6}) \leq 2d + 1$. If $d + 2 \leq f(v_{i+6}) \leq 2d + 1$, then $f(v_{i+7}) \geq 5d + 1$. If $f(v_{i+6}) = d + 1$, then $f(v_{i+7}) = 3d + 1$ which forces $f(v_{i+8}) \geq 5d + 1$.

Therefore, we can conclude that $k(C_{11}) \geq 4d + 2$ when $d \geq 2$ using $L(2d, d, 1)$ labeling. □

**Theorem 4.14.** *For any cycle, $C_n$, where $n$ is a positive integer greater than or equal to 3 and $d \geq 2$*

$$k(C_n) = \begin{cases} 4d + 1, & \text{if } n \neq 6, 7, 11; \\ 4d + 2, & \text{if } n = 6;\ n = 11; \\ 4d + 3, & \text{if } n = 7; \end{cases}$$

*using $L(2d, d, 1)$ labeling.*

*Proof.* Let $n \geq 3$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices on $C_n$, with $v_i$ adjacent to $v_{i+1}$ for $1 \leq i \leq n-1$ and vertex $v_1$ adjacent to $v_n$. For $C_n$ we proceed with the following cases.

Case I: $2 \mid n$ and $n \neq 6$.

By Lemma 4.3 we know that $k(C_4) \geq 4d+1$ when $m = 2$. The labeling pattern $\{1, 3d+1, d+1, 4d+1\}$ shows that $k(C_4) = 4d+1$. From Theorem 4.2 we know that for a path with $n \geq 5$, $k(P_n) = 4d+1$ using $L(2d, d, 1)$ labeling. Therefore, for any cycle with $n \geq 5$, $k(C_n) \geq 4d+1$. By Fact 4.8 we know that if $n$ is an even integer and $n \geq 8$, then $n = 4a + 5b$ for some non-negative integers $a$,$b$. The labeling pattern

$$\{\underbrace{1, 3d+1, d+1, 4d+1,}_{a \text{ times}}\underbrace{1, 3d+1, d+1, 4d+1, 2d+1}_{b \text{ times}}\}$$

shows that for any cycle with $2 \mid n$ and $n \geq 8$, $k(C_n) = 4d+1$.

Case II: $2 \nmid n$ and $n \neq 7, 11$.

The labeling pattern $\{1, 2d+1, 4d+1\}$ shows that $k(C_3) = 4d+1$. By Lemma 4.4 we know that $k(C_n) \geq 4d+1$ when $2 \nmid n$, $m = 2$, and $d \geq 2$. The labeling pattern $\{1, 2d+1, 4d+1, d+1, 3d+1\}$ shows that $k(C_5) = 4d+1$. We know from Fact 4.9 that if $n$ is an odd integer and $n \geq 9$ and $n \neq 11$, then $n = 4a + 5b$ for some non-negative integers $a$,$b$. The labeling pattern

$$\{\underbrace{1, 3d+1, d+1, 4d+1,}_{a \text{ times}}\underbrace{1, 3d+1, d+1, 4d+1, 2d+1}_{b \text{ times}}\}$$

shows that for any cycle with $2 \nmid n$, $n \geq 9$, and $n \neq 11$ $k(C_n) = 4d+1$.

Case III: $n = 6$.

By Lemma 4.11 we know that $k(C_6) \geq 4d+2$. The labeling pattern $\{1, 2d+1, 4d+1, 2, 2d+2, 4d+2\}$ shows that $k(C_6) = 4d+2$.

Case IV: $n = 11$.

By Lemma 4.13 we know that $k(C_{11}) \geq 4d+2$. The labeling pattern $\{1, 2d+1, 4d+1, d+1, 3d+1, 1, 2d+1, 4d+1, 2, 2d+2, 4d+2\}$ shows that $k(C_{11}) = 4d+2$.

Case V: $n = 7$.

By Lemma 4.12 we know that $k(C_7) \geq 4d+3$. The labeling pattern $\{1, 2d+2, 4d+2, 2, 3d+2, d+2, 4d+3\}$ shows that $k(C_7) = 4d+3$. □

## 5. $L(md, d, 1)$ Surjective Labeling of Paths

Conjecture 5.1 was made by observing the data produced by the computer program described in Section 3. We can see the length of the shortest path, $n$, that can be surjectively labeled for the various parameters of $L(md, d, 1)$ labeling in Table 2. The bold numbers are the cases when $m = d$, while the strikethrough represents lengths of paths which cannot be surjectively labeled using $L(md, d, 1)$ labeling.

**Conjecture 5.1.** *For $L(md, d, 1)$ labeling, where positive integer $m$ and $d \geq 2$, the shortest path, $P_n$, that can be surjectively labeled is $P_{2md+d}$.*

| $d$ \ $m$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 2 | **10** | 14 | 18 | 22 | 26 |
| 3 | 15 | **21** | 27 | | |
| 4 | 20 | 28 | | | |
| 5 | 25 | ~~33 34~~ | | | |
| 6 | 30 | | | | |

TABLE 2. This table shows lengths of the shortest path, $n$, that can be surjectively labeled for the various parameters of $L(md, d, 1)$ labeling.

## 6. $L(d + m, d, 1)$ Labeling of Paths

In this section we will find $k(P_n)$ for all paths of length $n$ using $L(d + m, d, 1)$ labeling where $m$ and $d$ are positive integers and $d + m > d > 1$. Two cases need to be considered: $d > m > 0$ and $m \geq d \geq 2$. A summary of the results in this section can be found in Theorem 6.2 and Theorem 6.4.

**Lemma 6.1.** *For a path on $n$ vertices, $P_n$, with $n \geq 8$, $d \geq 2$, and $d > m > 0$, $k(P_n) \geq 2d + 2m + 2$ using $L(d + m, d, 1)$ labeling.*

*Proof.* Let $f$ be a minimal $L(d + m, d, 1)$ labeling for a path on $n$ vertices, $P_n$. Consider vertex $v_i$ with label 1. There is an induced subpath of at least 5 vertices with $v_i$ as an end. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$ be this subpath. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $f(v_{i+1}) = d + m + 1$.
Then $f(v_{i+2}) = 2d + 2m + 1$, $2 \leq f(v_{i+3}) \leq m + 1$, and $d + m + 2 \leq f(v_{i+4}) \leq d + 2m + 1$. Now there are 3 vertices not yet labeled so we know either $v_{i+5}$ or the subpath $\{v_{i-3}, v_{i-2}, v_{i-1}\}$ exists. If $v_{i+5}$ exists, then $f(v_{i+5})$ must be greater than or equal to $2d + 2m + 2$. If the subpath $\{v_{i-3}, v_{i-2}, v_{i-1}\}$ exists, then $2d + m + 1 \leq f(v_{i-1}) \leq 2d + 2m$ and $d + 1 \leq f(v_{i-2}) \leq d + m$. This result forces $f(v_{i-3}) \geq 3d + m + 1$, which is greater than or equal to $2d + 2m + 2$ when $d \geq m + 1$.

Case II: $d + m + 2 \leq f(v_{i+1}) \leq 2d + m$.
Then $f(v_{i+2}) \geq 2d + 2m + 2$.

Case III: $2d + m + 1 \leq f(v_{i+1}) \leq 2d + 2m$.
Then $d + 1 \leq f(v_{i+2}) \leq d + m$. This result forces $f(v_{i+3})$ to be greater than or equal to $3d + m + 1$, which is greater than or equal to $2d + 2m + 2$ when $d \geq m + 1$.

Case IV: $f(v_{i+1}) = 2d + 2m + 1$.
Then $d + 1 \leq f(v_{i+2}) \leq d + m + 1$. This leads to $f(v_{i+3}) \geq 3d + 2m + 1$.

Therefore, we can conclude that $k(P_n) \geq 2d + 2m + 2$ when $n \geq 8$ and $d > m > 0$ using $L(d + m, d, 1)$ labeling. □

**Theorem 6.2.** *For any path, $P_n$, when $d > m > 0$*

$$k(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ d+m+1, & \text{if } n = 2; \\ 2d+m+1, & \text{if } n = 3,4; \\ 2d+2m+1, & \text{if } n = 5,6,7; \\ 2d+2m+2, & \text{if } n \geq 8; \end{cases}$$

*using $L(d+m,d,1)$ labeling.*

*Proof.* Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices on $P_n$, with $v_i$ adjacent to $v_{i+1}$ for $1 \leq i \leq n-1$. For each $P_n$ we proceed with the following cases.

Case I: $n = 1$.
This is evidently true.

Case II: $n = 2$
The labeling pattern $\{1, d+m+1\}$ shows that $k(P_n) = d+m+1$ for $n = 2$.

Case III: $n = 3, 4$
Consider vertex $v_i$ such that $f(v_i) = 1$. If $v_i$ is of degree 2 then we know that vertices $v_{i+1}$ and $v_{i-1}$ exist such that $f(v_{i+1}) \geq d+m+1$ and $f(v_{i-1}) \geq 2d+m+1$. If $v_i$ is of degree 1, then we know that either vertices $v_{i+1}$ and $v_{i+2}$ or $v_{i-1}$ and $v_{i-2}$ exist. Assume without loss of generality, that $v_{i+1}$ and $v_{i+2}$ exist. Then $d+m+1 \leq f(v_{i+1}) \leq 2d+m$, which forces $f(v_{i+2})$ to be greater than $2d+m+1$. Thus, the labeling pattern $\{d+1, 2d+m+1, 1, d+m+1\}$ shows that $k(P_n) = 2d+m+1$ for $n = 3, 4$.

Case IV: $n = 5, 6, 7$
Consider vertex $v_i$ where $f(v_i) = 1$. Then $d+m+1 \leq f(v_{i+1}) \leq 2d+2m$. If $d+m+1 \leq f(v_{i+2}) \leq 2d+m$, then $f(v_{i+3}) \geq 2d+2m+1$. If $2d+m+1 \leq f(v_{i+1}) \leq 2d+2m$, then $d+1 \leq f(v_{i+2}) \leq d+m$. Now there are at least two vertices not yet labeled. If $v_{i+3}$ exists, then $f(v_{i+3}) \geq 3d+m+1$ which is greater than $2d+2m+1$ when $d > m$. If $v_{i-1}$ exists and $v_{i+3}$ does not, then consequently $v_{i-2}$ exists. Then $d+m+1 \leq f(v_{i-1}) \leq d+2m$. This forces $f(v_{i-2}) \geq 2d+2m+1$. Thus, $k(P_n) \geq 2d+2m+1$. The labeling pattern $\{d+1, 2d+m+1, 1, d+m+1, 2d+2m+1, 2, d+m+2\}$ shows that $k(P_n) = 2d+2m+1$ for $n = 5, 6, 7$. Observe that this pattern is not repeatable.

Case V: $n \geq 8$
Let $f$ be defined as $f(\{v_1, v_2, v_3, v_4, v_5, v_6\}) = \{1, d+m+1, 2d+2m+1, 2, d+m+2, 2d+2m+2\}$ and $f(v_i) = f(v_j)$ if $i \equiv j \pmod 6$. Therefore we can conclude by the definition of $f$ that $k(P_n) \leq 2d+2m+2$ for $n \geq 8$. By combining this result with the results of Lemma 6.1, we obtain $k(P_n) = 2d+2m+2$ for $n \geq 8$. □

**Lemma 6.3.** *For a path on $n$ vertices, $P_n$, with $n \geq 5$ and $m \geq d \geq 2$, $k(P_n) \geq 3d+m+1$ using $L(d+m,d,1)$ labeling.*

*Proof.* Let $f$ be a minimal $L(d+m,d,1)$ labeling for a path on $n$ vertices, $P_n$. Consider vertex $v_i$ with label 1. There is an induced subpath of at least 3 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}\}$ be this subpath. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $d + m + 1 \le f(v_{i+1}) \le 2d + m$.

Then $f(v_{i+2}) \ge 2d + 2m + 1$ which is greater than or equal to $3d + m + 1$ when $m \ge d$.

Case II: $2d + m + 1 \le f(v_{i+1}) \le 3d + m$.

Then $d+1 \le f(v_{i+2}) \le 2d$. Now there are at least two vertices not yet labeled so we know either $v_{i-1}$ or $v_{i+3}$ exists. If $v_{i+3}$ exists then $f(v_{i+3})$ must be greater than or equal to $3d+m+1$. If $v_{i-1}$ exists and $v_{i+3}$ does not, then consequently $v_{i-2}$ exists. Then $d + m + 1 \le f(v_{i-1}) \le 2d + m$. This result forces $f(v_{i-2}) \ge 2d + 2m + 1$ which is greater than or equal to $3d + m + 1$ when $m \ge d$.

Therefore we can conclude that $k(P_n) \ge 3d + m + 1$ when $n \ge 5$ and $m \ge d \ge 2$ using $L(d + m, d, 1)$ labeling. □

**Theorem 6.4.** *For any path, $P_n$, when $m \ge d \ge 2$*

$$k(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ d + m + 1, & \text{if } n = 2; \\ 2d + m + 1, & \text{if } n = 3, 4; \\ 3d + m + 1, & \text{if } n \ge 5; \end{cases}$$

*using $L(d + m, d, 1)$ labeling.*

*Proof.* Let $V = \{v_1, v_2, \dots, v_n\}$ be the set of vertices on $P_n$, with $v_i$ adjacent to $v_{i+1}$ for $1 \le i \le n - 1$. For each $P_n$ we proceed with the following cases.

Case I: $n = 1$.

This is evidently true.

Case II: $n = 2$.

The labeling pattern $\{1, d + m + 1\}$ shows that $k(P_n) = d + m + 1$ when $n = 2$.

Case III: $n = 3, 4$.

Consider vertex $v_i$ such that $f(v_i) = 1$. If $v_i$ is of degree 2 then we know that vertices $v_{i+1}$ and $v_{i-1}$ exist such that $f(v_{i+1}) \ge d + m + 1$ and $f(v_{i-1}) \ge 2d + m + 1$. If $v_i$ is of degree 1, then we know that either vertices $v_{i+1}$ and $v_{i+2}$ or $v_{i-1}$ and $v_{i-2}$ exist. Assume without the loss of generality, that $v_{i+1}$ and $v_{i+2}$ exist. Then $d + m + 1 \le f(v_{i+1}) \le 2d + m$, which forces $f(v_{i+2})$ to be greater than $2d + m + 1$. Thus, the labeling pattern $\{d + m + 1, 1, 2d + m + 1, d + 1\}$ shows that $k(P_n) = 2d + m + 1$ for $n = 3, 4$.

Case IV: $n \ge 5$.

Let $f$ be defined as $f(\{v_1, v_2, v_3, v_4\}) = \{1, 2d + m + 1, d + 1, 3d + m + 1\}$ and $f(v_i) = f(v_j)$ if $i \equiv j \pmod 4$. Therefore we can conclude by the definition of $f$ that $k(P_n) \le 3d + m + 1$ for $n \ge 5$. By combining this result with the results of Lemma 6.3, we obtain $k(P_n) = 3d + m + 1$ for $n \ge 5$. □

## 7. $L(d + m, d, 1)$ Surjective Labeling of Paths

Table 3 contains a list of the length of the shortest path that can be surjectively labeled using $L(d+m, d, 1)$ labeling. The data presented in this section was gathered using the computer program described in Section 3. Conjecture 7.1 is a summary of the date from Table 3.

| | | $m$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 2 | 3 | 4 | 5 | 6 | 7 |
| | 2 | **10** | 11 | 14 | 15 | 18 | 19 |
| | 3 | 12 | **15** | 16 | 18 | 21 | 22 |
| | 4 | 16 | 17 | **20** | 21 | 24 | 25 |
| $d$ | 5 | 18 | 20 | 22 | **25** | 26 | 28 |
| | 6 | 22 | 24 | 26 | 27 | **30** | 31 |
| | 7 | 24 | 26 | 28 | 30 | 32 | |
| | 8 | 28 | 29 | 32 | 33 | | |

TABLE 3. This table shows the length of the shortest path, $n$, that can be surjectively labeled for the various parameters of $L(d+m,d,1)$ labeling.

**Conjecture 7.1.** *For $L(d+m,d,1)$ labeling, where integers $m \geq 2$, $d \geq 2$, and $m = d$ , the shortest path, $P_n$, that can be surjectively labeled is $P_{5m}$.*

## 8. $L((m+1)d, md, d)$ LABELING OF PATHS AND CYCLES

In this section we will find $k(G)$ for paths and cycles of length $n$ using $L((m+1)d, md, d)$ labeling. A summary of the results for paths can be found in Theorem 8.2 and in Theorem 8.9 for cycles.

**Lemma 8.1.** *For a path on $n$ vertices, $P_n$, with $n \geq 8$, $m \geq 2$, and $d \geq 1$, $k(P_n) \geq 2md + 3d + 1$ using $L((m+1)d, md, d)$ labeling.*

*Proof.* Let $f$ be the minimal $L((m+1)d, md, d)$ labeling for a path on $n$ vertices, $P_n$. Consider vertex $v_i$ with label 1. There is an induced subpath of at least 5 vertices with $v_i$ as an end. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$ be this subpath. Now we consider the possibilities for $f(v_{i+1})$.

Case I: $md + d + 1 \leq f(v_{i+1}) \leq md + 2d$.

Then $2md+2d+1 \leq f(v_{i+2}) \leq 2md + 3d$, $d+1 \leq f(v_{i+3}) \leq 2d$, and $md+2d+1 \leq f(v_{i+4}) \leq md + 3d$. Now there are 3 vertices not yet labeled so we know either $v_{i+5}$ exists or the subpath $\{v_{i-3}, v_{i-2}, v_{i-1}\}$ exists. If $v_{i+5}$ exists, then $f(v_{i+5})$ must be greater than or equal to $2md+3d+1$. If the subpath $\{v_{i-3}, v_{i-2}, v_{i-1}\}$ exists, then $2md + d + 1 \leq f(v_{i-1}) \leq 2md + 2d$ and $md + 1 \leq f(v_{i-2}) \leq md + d$. This result forces $f(v_{i-3})$ to be greater than or equal to $3md + d + 1$, which is greater than or equal to $2md + 3d + 1$ when $m \geq 2$.

Case II: $md + 2d + 1 \leq f(v_{i+1}) \leq md + 3d$.

Then $f(v_{i+2}) \geq 2md + 3d + 1$.

Case III: $2md + d + 1 \leq f(v_{i+1}) \leq 2md + 3d$.

Then $md + 1 \leq f(v_{i+2}) \leq md + 2d$. This forces $f(v_{i+3}) \geq 3md + d + 1$, which is greater than or equal to $2md + 3d + 1$ when $m \geq 2$.

Therefore, we can conclude that $k(P_n) \geq 2md + 3d + 1$ when $n \geq 8$, $m \geq 2$, and $d \geq 1$ using $L((m+1)d, md, d)$ labeling. □

**Theorem 8.2.** *For any path $P_n$, when $m \geq 2$ and $d \geq 1$*

$$k(P_n) = \begin{cases} 1, & \textit{if } n = 1; \\ md + d + 1, & \textit{if } n = 2; \\ 2md + d + 1, & \textit{if } n = 3, 4; \\ 2md + 2d + 1, & \textit{if } n = 5, 6, 7; \\ 2md + 3d + 1, & \textit{if } n \geq 8; \end{cases}$$

*using $L((m+1)d, md, d)$ labeling.*

*Proof.* Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices on $P_n$, with $v_i$ adjacent to $v_{i+1}$ for $1 \leq i \leq n-1$. For each $P_n$ we proceed with the following cases.

Case I: $n = 1$
This is evidently true.

Case II: $n = 2$
The labeling pattern $\{1, md + d + 1\}$ shows that $k(P_n) = md + d + 1$ for $n = 2$.

Case III: $n = 3, 4$
Consider vertex $v_i$ where $f(v_i) = 1$. If $v_i$ is of degree 2, then we know that vertices $v_{i+1}$ and $v_{i-1}$ exist such that $f(v_{i+1}) \geq md + d + 1$ and $f(v_{i-1}) \geq 2md + d + 1$. If $v_i$ is of degree 1, then we know that either vertices $v_{i+1}$ and $v_{i+2}$ or $v_{i-1}$ and $v_{i-2}$ exist. Assume without the loss of generality, that $v_{i+1}$ and $v_{i+2}$ exist. Then $md + d + 1 \leq f(v_{i+1}) \leq 2md + d$, which forces $f(v_{i+2})$ to be greater than $2md + d + 1$. Thus, the labeling pattern $\{md + 1, 2md + d + 1, 1, md + d + 1\}$ shows that $k(P_n) = 2md + d + 1$ for $n = 3, 4$.

Case IV: $n = 5, 6, 7$
Consider vertex $v_i$ where $f(v_i) = 1$. Then $md + d + 1 \leq f(v_{i+1}) \leq 2md + 2d$. If $md + d + 1 \leq f(v_{i+1}) \leq 2md + d$, then $f(v_{i+2}) \geq 2md + 2d + 1$. If $2md + d + 1 \leq f(v_{i+1}) \leq 2md + 2d$, then $md + 1 \leq f(v_{i+2}) \leq md + d$. Now there are at least two vertices not yet labeled. If $v_{i+3}$ exists, then $f(v_{i+3}) \geq 3md + d + 1$ which is greater than $2md + 2d + 1$ when $m > 1$. If $v_{i-1}$ exists and $v_{i+3}$ does not, then consequently $v_{i-2}$ exists. Then $md + d + 1 \leq f(v_{i-1}) \leq md + 2d$. This forces $f(v_{i-2}) \geq 2md + 2d + 1$. Thus, $k(P_n) \geq 2md + 2d + 1$. The labeling pattern $\{md + 1, 2md + d + 1, 1, md + d + 1, 2md + 2d + 1, d + 1, md + 2d + 1\}$ shows that $k(P_n) = 2md + 2d + 1$ for $n = 5, 6, 7$. Observe that this pattern is not repeatable.

Case V: $n \geq 8$
Let $f$ be defined as $f(\{v_1, v_2, v_3, v_4, v_5, v_6\}) = \{1, md+d+1, 2md+2d+1, d+1, md+2d+1, 2md+3d+1\}$ and $f(v_i) = f(v_j)$ if $i \equiv j \pmod 6$. We can conclude by the definition of $f$ that $k(P_n) \leq 2md + 3d + 1$ for $n \geq 8$. By combining this result with the results of Lemma 8.1, we obtain $k(P_n) = 2md + 3d + 1$ for $n \geq 8$. □

**Lemma 8.3.** *For a cycle on 4 vertices, $C_4$, with $d \geq 1$ and $m \geq 2$, $k(C_4) \geq 3md + d + 1$ using $L((m+1)d, md, d)$ labeling.*

*Proof.* Let $f$ be a minimal $L((m+1)d, md, d)$ labeling for a cycle on 4 vertices, $C_4$. Consider vertex $v_i$ with label 1. There is an induced subpath of 4 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ be this subpath. Note that since every vertex is at most a distance of 2 away, every pair of vertices must have labels that differ by at least $md$. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $md + d + 1 \leq f(v_{i+1}) \leq 2md$.
Then $2md + 2d + 1 \leq f(v_{i+2}) \leq 3md + d$. This forces $f(v_{i+3}) \geq 3md + 3d + 1$ since $v_i$ is adjacent to $v_{i+3}$ in $C_4$.

Case II: $2md + 1 \leq f(v_{i+1}) \leq 2md + d$.
Then $f(v_{i+2}) \geq 3md + d + 1$.

Case III: $2md + d + 1 \leq f(v_{i+1}) \leq 2md + d$.
Then $md + 1 \leq f(v_{i+2}) \leq 2md$. This forces $f(v_{i+3}) \geq 3md + d + 1$.

Therefore, we can conclude that $k(C_4) \geq 3md + d + 1$ when $d \geq 1$ and $m \geq 2$ using $L((m + 1)d, md, d)$ labeling. □

**Lemma 8.4.** *For a cycle on 5 vertices, $C_5$, with $d \geq 1$ and $m \geq 2$, $k(C_5) \geq 4md + 1$ using $L((m + 1)d, md, d)$ labeling.*

*Proof.* Since every vertex is at most a distance of two from every other vertex all labels must differ by at least $md$. So labeling $C_5$ requires at least $4md + 1$. Therefore, $k(C_5) \geq 4md + 1$. □

**Lemma 8.5.** *For a cycle on 6 vertices, $C_6$, with $d \geq 1$ and $m \geq 2$, $k(C_6) \geq 2md + 3d + 1$ using $L((m + 1)d, md, d)$ labeling.*

*Proof.* Let $f$ be a minimal $L((m + 1)d, md, d)$ labeling for a cycle with 6 vertices, $C_6$. Consider vertex $v_i$ with label 1. There is an induced subpath of 6 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}$ be this subpath. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $md + d + 1 \leq f(v_{i+1}) \leq md + 2d$.
Then $2md+2d+1 \leq f(v_{i+2}) \leq 2md + 3d$, $d+1 \leq f(v_{i+3}) \leq 2d$, and $md+2d+1 \leq f(v_{i+4}) \leq md + 3d$. This forces $f(v_{i+5}) \geq 2md + 3d + 1$.

Case II: $md + 2d + 1 \leq f(v_{i+1}) \leq 2md + d$.
This forces $f(v_{i+2}) \geq 2md + 3d + 1$.

Case III: $2md + d + 1 \leq f(v_{i+1}) \leq 2md + 3d$.
Then $md + 1 \leq f(v_{i+2}) \leq md + 2d$. This forces $f(v_{i+3}) \geq 3md + d + 1$ which is greater than or equal to $2md + 3d + 1$ when $m \geq 2$.

Therefore, we can conclude that $k(C_6) \geq 2md + 3d + 1$ when $d \geq 1$ and $m \geq 2$ using $L((m + 1)d, md, d)$ labeling. □

**Lemma 8.6.** *For a cycle on 7 vertices, $C_7$, with $d \geq 1$ and $m \geq 2$, $k(C_7) \geq 3md + 3d + 1$ using $L((m + 1)d, md, d)$ labeling.*

*Proof.* Let $f$ be a minimal $L((m + 1)d, md, d)$ labeling for a cycle with 7 vertices, $C_7$. Consider $v_i$ with label 1. There is an induced subpath of 7 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}\}$ be this subpath. Note that no value can be repeated in this subpath given that in the cycle each vertex is at most a distance of 3 away from every other vertex. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $md + d + 1 \leq f(v_{i+1}) \leq 2md + d$.
Then $2md + 2d + 1 \leq f(v_{i+2}) \leq 3md + 3d$, $d + 1 \leq f(v_{i+3}) \leq md + d$, and $md + 2d + 1 \leq f(v_{i+4}) \leq 2md + 3d$. If $md + 2d + 1 \leq f(v_{i+4}) \leq 2md + 2d$, then

$2md+3d+1 \le f(v_{i+5}) \le 3md+3d$. This forces $f(v_{i+6}) \ge 3md+4d+1$. If $2md+2d+1 \le f(v_{i+4}) \le 2md+3d$, then $f(v_{i+5}) \ge 3md+3d+1$.

Case II: $2md+d+1 \le f(v_{i+1}) \le 2md+2d$.

Then $md+1 \le f(v_{i+2}) \le md+d$ or $3md+2d+1 \le f(v_{i+2}) \le 3md+3d$. If $md+1 \le f(v_{i+2}) \le md+d$, then $3md+d+1 \le f(v_{i+3}) \le 3md+3d$, and $2md+1 \le f(v_{i+4}) \le 2md+d$. This forces $f(v_{i+5}) \ge 4md+d+1$ which is greater than or equal to $3md+3d+1$ when $m \ge 2$. Since $v_i$ is adjacent to $v_{i+6}$ in the cycle, if $3md+2d+1 \le f(v_{i+2}) \le 3md+3d$, then $md+d+1 \le f(v_{i+6}) \le md+2d$ or $3md+d+1 \le f(v_{i+6}) \le 3md+2d$. If $md+d+1 \le f(v_{i+6}) \le md+2d$, then $2md+2d+1 \le f(v_{i+5}) \le 3md+2d$ and $d+1 \le f(v_{i+4}) \le 2d$. This forces $f(v_{i+3}) \ge 4md+3d+1$. If $3md+d+1 \le f(v_{i+6}) \le 3md+2d$, then $md+1 \le f(v_{i+5}) \le 2md+d$. If $md+1 \le f(v_{i+5}) \le md+2d$, then $f(v_{i+4}) \ge 4md+2d+1$ which is greater than $3md+3d+1$ when $m>1$. If $md+2d+1 \le f(v_{i+5}) \le 2md+d$, then $d+1 \le f(v_{i+4}) \le md$. This forces $f(v_{i+3}) \ge 4md+3d+1$.

Case III: $2md+2d+1 \le f(v_{i+1}) \le 2md+3d$.

Then $md+1 \le f(v_{i+2}) \le md+2d$. If $md+1 \le f(v_{i+2}) \le md+d$, then $3md+2d+1 \le f(v_{i+3}) \le 3md+3d$ and $2md+1 \le f(v_{i+4}) \le 2md+2d$. This forces $f(v_{i+5}) \ge 4md+2d+1$ which is greater than $3md+3d+1$ when $m>1$. If $md+d+1 \le f(v_{i+2}) \le md+2d$, then $3md+2d+1 \le f(v_{i+3}) \le 3md+3d$ and $d+1 \le f(v_{i+4}) \le 2d$ or $2md+d+1 \le f(v_{i+4}) \le 2md+2d$. If $d+1 \le f(v_{i+4}) \le 2d$, then $md+2d+1 \le f(v_{i+5}) \le 2md+2d$. This forces $f(v_{i+6}) \ge 3md+3d+1$. If $2md+d+1 \le f(v_{i+4}) \le 2md+2d$, then $md+1 \le f(v_{i+5}) \le md+d$. This forces $f(v_{i+6}) \ge 3md+3d+1$.

Case IV: $2md+3d+1 \le f(v_{i+1}) \le 3md+3d$

Then $md+1 \le f(v_{i+2}) \le 2md+2d$. If $md+1 \le f(v_{i+2}) \le md+2d$, then $f(v_{i+3}) \ge 3md+3d+1$. If $md+2d+1 \le f(v_{i+2}) \le 2md+2d$, then $d+1 \le f(v_{i+3}) \le md+d$, $2md+2d+1 \le f(v_{i+4}) \le 3md+2d$, and $md+d+1 \le f(v_{i+5}) \le 2md+d$. This forces $f(v_{i+6}) \ge 3md+3d+1$.

Therefore, we can conclude that $k(C_7) \ge 3md+3d+1$ when $d \ge 1$ and $m \ge 2$ using $L((m+1)d, md, d)$ labeling. □

**Lemma 8.7.** *For a cycle on 9 vertices, $C_9$, with $d \ge 1$, $k(C_9) \ge 8d+1$ using $L(3d,2d,d)$ labeling.*

*Proof.* Let $f$ be a minimal $L(3d,2d,d)$ labeling for a cycle on 9 vertices, $C_9$. Consider vertex $v_i$ with label 1. There is an induced subpath of 9 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_{i+7}, v_{i+8}\}$ be this subpath. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $3d+1 \le f(v_{i+1}) \le 4d$.

Then $6d+1 \le f(v_{i+2}) \le 8d$. If $6d+1 \le f(v_{i+2}) \le 7d$, then $d+1 \le f(v_{i+3}) \le 2d$, $4d+1 \le f(v_{i+4}) \le 5d$, $7d+1 \le f(v_{i+5}) \le 8d$, $2d+1 \le f(v_{i+6}) \le 3d$, and $5d+1 \le f(v_{i+7}) \le 6d$. This forces $f(v_{i+8}) \ge 8d+1$. If $7d+1 \le f(v_{i+2}) \le 8d$, then $d+1 \le f(v_{i+3}) \le 2d$ and $4d+1 \le f(v_{i+4}) \le 6d$. This forces $f(v_{i+5}) \ge 8d+1$.

Case II: $4d+1 \le f(v_{i+1}) \le 5d$.

Then $7d+1 \le f(v_{i+2}) \le 8d$, $d+1 \le f(v_{i+3}) \le 3d$, and $5d+1 \le f(v_{i+4}) \le 6d$. This forces $f(v_{i+5}) \ge 8d+1$.

Case III: $5d+1 \le f(v_{i+1}) \le 6d$.

Then $2d+1 \le f(v_{i+2}) \le 3d$, $7d+1 \le f(v_{i+3}) \le 8d$, and $1 \le f(v_{i+4}) \le d$ or $4d+1 \le f(v_{i+4}) \le 5d$. If $1 \le f(v_{i+4}) \le d$, then $3d+1 \le f(v_{i+5}) \le 6d$. If $3d+1 \le f(v_{i+5}) \le 4d$, then $6d+1 \le f(v_{i+6}) \le 7d$. This forces $f(v_{i+7}) \ge 9d+1$. If $4d+1 \le f(v_{i+5}) \le 5d$, then $f(v_{i+6}) \ge 8d+1$. If $5d+1 \le f(v_{i+5}) \le 6d$, then $2d+1 \le f(v_{i+6}) \le 3d$ and $7d+1 \le f(v_{i+7}) \le 8d$. This forces $f(v_{i+8}) \ge 10d+1$ since $v_i$ is adjacent to $v_{i+8}$. If $4d+1 \le f(v_{i+4}) \le 5d$, then $1 \le f(v_{i+5}) \le 2d$, $6d+1 \le f(v_{i+6}) \le 7d$, and $2d+1 \le f(v_{i+7}) \le 4d$. Then $f(v_{i+8}) \ge 8d+1$.

Case IV: $6d+1 \le f(v_{i+1}) \le 7d$.

Then $2d+1 \le f(v_{i+2}) \le 4d$. This forces $f(v_{i+3}) \ge 8d+1$.

Case V: $7d+1 \le f(v_{i+1}) \le 8d$.

Then $2d+1 \le f(v_{i+2}) \le 5d$. If $2d+1 \le f(v_{i+2}) \le 3d$, then $5d+1 \le f(v_{i+3}) \le 6d$, $1 \le f(v_{i+4}) \le d$, and $3d+1 \le f(v_{i+5}) \le 4d$ or $7d+1 \le f(v_{i+5}) \le 8d$. If $3d+1 \le f(v_{i+5}) \le 4d$, then $6d+1 \le f(v_{i+6}) \le 8d$. This forces $f(v_{i+7}) \ge 9d+1$ since vertex $v_i$ is adjacent to vertex $v_{i+8}$. If $7d+1 \le f(v_{i+5}) \le 8d$, then $2d+1 \le f(v_{i+6}) \le 5d$. If $2d+1 \le f(v_{i+6}) \le 3d$, then $5d+1 \le f(v_{i+7}) \le 6d$. This forces $f(v_{i+8}) \le 8d+1$. If $3d+1 \le f(v_{i+6}) \le 5d$, then $f(v_{i+7}) \ge 9d+1$. If $3d+1 \le f(v_{i+2}) \le 4d$, then $f(v_{i+3}) \ge 9d+1$. If $4d+1 \le f(v_{i+2}) \le 5d$, then $d+1 \le f(v_{i+3}) \le 2d$, $6d+1 \le f(v_{i+4}) \le 7d$, and $3d+1 \le f(v_{i+5}) \le 4d$. This forces $f(v_{i+6}) \ge 8d+1$.

Therefore, we can conclude that $k(C_9) \ge 8d+1$ when $d \ge 1$ using $L(3d, 2d, d)$ labeling. □

**Lemma 8.8.** *For a cycle on 9 vertices, $C_9$, with $d \ge 1$ and $m \ge 3$, $k(C_9) \ge 2md+4d+1$ using $L((m+1)d, md, d)$ labeling.*

*Proof.* Let $f$ be a minimal $L((m+1)d, md, d)$ labeling for a cycle with 9 vertices, $C_9$. Consider vertex $v_i$ with label 1. There is an induced subpath of 9 with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_{i+7}, v_{i+8}\}$ be this subpath. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $md+d+1 \le f(v_{i+1}) \le md+2d$.

Then $2md+2d+1 \le f(v_{i+2}) \le 2md+4d$. If $2md+2d+1 \le f(v_{i+2}) \le 2md+3d$, then $d+1 \le f(v_{i+3}) \le 2d$, $md+2d+1 \le f(v_{i+4}) \le md+3d$, $2md+3d+1 \le f(v_{i+5}) \le 2md+4d$, $2d+1 \le f(v_{i+6}) \le 3d$, and $md+3d+1 \le f(v_{i+7}) \le md+4d$. This forces $f(v_{i+8}) \ge 2md+4d+1$. If $2md+3d+1 \le f(v_{i+2}) \le 2md+4d$, then $d+1 \le f(v_{i+3}) \le 2d$ and $md+2d+1 \le f(v_{i+4}) \le md+4d$. This forces $f(v_{i+5}) \ge 2md+4d+1$.

Case II: $md+2d+1 \le f(v_{i+1}) \le md+3d$.

Then $2md+3d+1 \le f(v_{i+2}) \le 2md+4d$, $d+1 \le f(v_{i+3}) \le 3d$, and $md+3d+1 \le f(v_{i+4}) \le md+4d$. This forces $f(v_{i+5}) \ge 2md+4d+1$.

Case III: $md+3d+1 \le f(v_{i+1}) \le 2md+d$.

Then $f(v_{i+2}) \ge 2md+4d+1$.

Case IV: $2md+d+1 \le f(v_{i+1}) \le 2md+3d$.

Then $md+1 \le f(v_{i+2}) \le md+2d$. This forces $f(v_{i+3}) \ge 3md+d+1$ which is greater than or equal to $2md+4d+1$ when $m \ge 3$.

Case V: $2md+3d+1 \le f(v_{i+1}) \le 2md+4d$.

Then $md + 1 \leq f(v_{i+2}) \leq md + 3d$. If $md + 1 \leq f(v_{i+2}) \leq md + 2d$, then $f(v_{i+3}) \geq 3md + 3d + 1$ which is greater than $2md + 4d + 1$ when $m > 1$. If $md + 2d + 1 \leq f(v_{i+2}) \leq md + 3d$, then $d + 1 \leq f(v_{i+3}) \leq 2d$, $2md + 2d + 1 \leq f(v_{i+4}) \leq 2md + 3d$, and $md + d + 1 \leq f(v_{i+5}) \leq md + 2d$. This forces $f(v_{i+6}) \geq 3md + 2d + 1$ which is greater than $2md + 4d + 1$ when $m > 2$.

Therefore, we can conclude that $k(C_9) \geq 2md + 4d + 1$ when $d \geq 1$ and $m \geq 3$ using $L((m + 1)d, md, d)$ labeling. □

**Theorem 8.9.** *For cycle, $C_n$*

$$k(C_n) = \begin{cases} 2md + 2d + 1, & \text{if } n = 3; \\ 3md + d + 1, & \text{if } n = 4; \\ 4md + 1, & \text{if } n = 5; \\ 2md + 3d + 1, & \text{if } n = 6; \\ 3md + 3d + 1, & \text{if } n = 7; \\ 2md + 4d + 1, & \text{if } n = 9; \end{cases}$$

*using $L((m + 1)d, md, d)$ labeling where $d \geq 1$ and $m \geq 2$.*

*Proof.* Let $n \geq 3$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices $C_n$, with $v_i$ adjacent to $v_{i+1}$ for $1 \leq i \leq n - 1$ and vertex $v_1$ adjacent to $v_n$. For $C_n$ we proceed with the following cases.

Case I: $n = 3$

The labeling pattern $\{1, md+d+1, 2md+2d+1\}$ shows that $k(C_3) = 2md+2d+1$.

Case II: $n = 4$

By Lemma 8.3 we know that $k(C_4) \geq 3md + d + 1$. The labeling pattern $\{1, 2md + d + 1, md + 1, 3md + d + 1\}$ shows that $k(C_4) = 3md + d + 1$.

Case III: $n = 5$

By Lemma 8.4 we know that $k(C_5) \geq 4md + 1$. The labeling pattern $\{1, 3md + 1, md + 1, 4md + 1, 2md + 1\}$ shows that $k(C_5) = 4md + 1$.

Case IV: $n = 6$

By Lemma 8.5 we know that $k(C_6) \geq 2md + 3d + 1$. The labeling pattern $\{1, md + d + 1, 2md + 2d + 1, d + 1, md + 2d + 1, 2md + 3d + 1\}$ shows that $k(C_6) = 2md + 3d + 1$.

Case V: $n = 7$

By Lemma 8.6 we know that $k(C_7) \geq 3md + 3d + 1$. The labeling pattern $\{1, 2md + 2d + 1, md + d + 1, 3md + 2d + 1, d + 1, md + 2d + 1, 3md + 3d + 1\}$ shows that $k(C_7) = 3md + 3d + 1$.

Case VI: $n = 9$

By Lemma 8.7 we know that $k(C_9) \geq 8d + 1$ using $L(3d, 2d, d)$ labeling, which is a special case of $L((m + 1)d, md, d)$ labeling when $m = 2$. By Lemma 8.8 we know that $k(C_9) \geq 2md + 4d + 1$ when $m \geq 3$. The labeling pattern $\{1, md + d + 1, 2md + 2d + 1, d + 1, md + 2d + 1, 2md + 3d + 1, 2d + 1, md + 3d + 1, 2md + 4d + 1\}$ shows that $k(C_9) = 2md + 4d + 1$. □

## 9. $L(d+2, d+1, d)$ Labeling of Paths

In this section we will find $k(P_n)$ for all paths of length $n$ using $L(d+2, d+1, d)$ labeling where $d \geq 2$. A summary of the results in this section can be found in Theorem 9.2.

**Lemma 9.1.** *For a path on $n$ vertices, $P_n$, with $n \geq 5$ and $d \geq 2$, $k(P_n) \geq 3d+5$ using $L(d+2, d+1, d)$ labeling.*

*Proof.* Let $f$ be a minimal $L(d+2, d+1, d)$ labeling for a path on $n$ vertices, $P_n$. Consider vertex $v_i$ with label 1. There is an induced subpath of at least 3 vertices with $v_i$ as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}\}$ be this subpath. Now we can consider the possibilities for $f(v_{i+1})$.

Case I: $d+3 \leq f(v_{i+1}) \leq 2d+1$.
Then $2d+5 \leq f(v_{i+2}) \leq 3d+4$. There are at least two vertices not yet labeled so we know that either $v_{i-1}$ or $v_{i+3}$ exists. If $v_{i+3}$ exists, then $f(v_{i+3}) \geq 3d+7$. If $v_{i-1}$ exists, then $f(v_{i-1}) \geq 3d+5$.

Case II: $f(v_{i+1}) = 2d+2$.
Then $f(v_{i+2}) = 3d+4$. There are at least two vertices not yet labeled so we know that either $v_{i-1}$ or $v_{i+3}$ exists. If $v_{i+3}$ exists and $v_{i-1}$ does not, then consequently $v_{i+4}$ exists. In this case $f(v_{i+3}) = d+1$ and $f(v_{i+4}) \geq 4d+5$. If $v_{i-1}$ exists, then $f(v_{i-1}) \geq 4d+4$, which is greater than $3d+5$ when $d > 1$.

Case III: $f(v_{i+1}) = 2d+3$.
Then $f(v_{i+2}) \geq 3d+5$.

Case IV: $2d+4 \leq f(v_{i+1}) \leq 3d+2$.
Then $d+2 \leq f(v_{i+2}) \leq 2d$. There are at least two vertices not yet labeled so we know that either $v_{i-1}$ or $v_{i+3}$ exists. If $v_{i+3}$ exists, then $f(v_{i+3}) \geq 3d+5$. If $v_{i-1}$ exists, then $f(v_{i-1}) \geq 3d+5$.

Case V: $3d+3 \leq f(v_{i+1}) \leq 3d+4$.
Then $d+2 \leq f(v_{i+2}) \leq 2d+2$. There are at least two vertices not yet labeled so we know that either $v_{i-1}$ or $v_{i+3}$ exists. If $v_{i+3}$ exists, then $f(v_{i+3}) \geq 4d+4$ which is greater than $3d+5$ when $d > 1$. If $v_{i-1}$ exists and $v_{i+3}$ does not, then consequently $v_{i-2}$ exists. If $d+2 \leq f(v_{i+2}) \leq d+3$, then $2d+2 \leq f(v_{i-1}) \leq 2d+3$. This forces $f(v_{i-2}) \geq 4d+3$ which is greater than or equal to $3d+5$ when $d \geq 2$. If $d+4 \leq f(v_{i+2}) \leq 2d+2$, then $f(v_{i-1}) \geq 4d+4$.

Therefore, we can conclude that $k(P_n) \geq 3d+5$ when $n \geq 5$ and $d \geq 2$ using $L(d+2, d+1, d)$ labeling. □

**Theorem 9.2.** *For any path, $P_n$, when $d \geq 2$*

$$k(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ d+3, & \text{if } n = 2; \\ 2d+4, & \text{if } n = 3; \\ 3d+3, & \text{if } n = 4; \\ 3d+5, & \text{if } n \geq 5; \end{cases}$$

*using $L(d+2, d+1, d)$ labeling.*

*Proof.* Let $V = \{v_1, v_2, \dots, v_n\}$ be the set of vertices on $P_n$, with $v_i$ adjacent to $v_{i+1}$ for $1 \leq i \leq n-1$. For each $P_n$ we proceed with the following cases.

Case I: $n = 1$.
This is evidently true.

Case II: $n = 2$.
The labeling pattern $\{1, d+3\}$ shows that $k(P_2) = d + 3$.

Case III: $n = 3$.
Consider vertex $v_i$ where $f(v_i) = 1$. If $v_i$ is of degree 2, then we know that vertices $v_{i+1}$ and $v_{i-1}$ exist such that $f(v_{i+1}) \geq d+3$ and $f(v_{i-1}) \geq 2d+4$. If $v_i$ is of degree 1, then we know that either vertices $v_{i+1}$ and $v_{i+2}$ or $v_{i-1}$ and $v_{i-2}$ exist. Assume without the loss of generality, that $v_{i+1}$ and $v_{i+2}$ exist. Then $d+3 \leq f(v_{i+1}) \leq 2d+3$, which forces $f(v_{i+2}) \geq 2d+5$. Thus, the labeling pattern $\{2d+4, 1, d+3\}$ shows that $k(P_3) = 2d+4$.

Case IV: $n = 4$
Assume that $k(P_4) = 3d + 2$. Consider vertex $v_i$ where $f(v_i) = 3d + 2$. If $v_i$ is of degree 1, then we know that the either vertices $v_{i+1}$, $v_{i+2}$, and $v_{i+3}$ or $v_{i-1}$, $v_{i-2}$, and $v_{i-3}$ exist. Assume without the loss of generality, that $v_{i+1}$, $v_{i+2}$, and $v_{i+3}$ exist. Then, given the labeling restrictions in regards to only $f(v_i) = 3d+2$, $1 \leq f(v_{i+1}) \leq 2d$, $1 \leq f(v_{i+2}) \leq 2d+1$, and $1 \leq f(v_{i+3}) \leq 2d+2$. However, we know that $k(P_3) = 2d + 4$. Therefore, a path of three vertices requires a label of at least $2d + 4$, which in turn leads to a contradiction. Thus $k(P_4) \neq 3d + 2$ if the vertex labeled $3d + 2$ is of degree 1.

Now consider the case where the vertex labeled $3d + 2$ is of degree 2. Assume without the loss of generality, that $v_2$ is labeled $3d + 2$. We know that the label of 1 must be present on our graph. If vertex $v_1$ is labeled 1, then $d+2 \leq f(v_3) \leq 2d$. This forces $f(v_4) \geq 4d+3$. If $f(v_3) = 1$, then $d+3 \leq f(v_4) \leq 2d+1$. This forces $f(v_1) \geq 4d+4$. If $f(v_4) = 1$, then $d+1 \leq f(v_1) \leq 2d$. This forces $f(v_3) \geq 4d+4$. Thus $k(P_4) \neq 3d+2$ if the vertex labeled $3d+2$ is of degree 2.

Therefore, $k(C_4) \geq 3d + 3$. The labeling pattern $\{2d+2, 1, 3d+3, d+2\}$ shows that $k(P_4) = 3d + 3$.

Case V: $n \geq 5$
Let $f$ be defined as $f(\{v_1, v_2, v_3, v_4\}) = \{1, 2d+4, d+2, 3d+5\}$ and $f(v_i) = f(v_j)$ if $i \equiv j \pmod 4$. Therefore, we can conclude by the definition of $f$ that $k(P_n) \leq 3d + 5$ for $n \geq 5$. By combining this result with the results of Lemma 9.1, we obtain $k(P_n) = 3d + 5$ for $n \geq 5$. □

## 10. $L(d, j, s)$ Labeling of Complete and Complete Bipartite Graphs

In this section we will find the $L(d, j, s)$ labeling number for complete graphs and complete bipartite graphs.

**Theorem 10.1.** *For any complete graph on $n$ vertices, $k(K_n) = dn - d + 1$.*

*Proof.* In a complete graph every vertex is adjacent to every other vertex. Thus all labels must differ by $d$ or more. We know that 1 must be a labeling because it is a minimum $L(d, j, s)$ labeling. Thus, $k(K_n) = 1 + d(n-1) = dn - d + 1$. □

**Theorem 10.2.** *For any complete bipartite graph,* $k(K_{m,n}) = 1 + j(m-1) + d + j(n-1)$.

*Proof.* Let $K_{m,n}$ be a complete bipartite graph with partition sets $A$ and $B$. Each vertex in set $A$ is a distance of two from every other vertex in set $A$. The same is true for any two vertices in set $B$. So in a minimal $L(d, j, s)$ labeling of graph $K_{m,n}$ each vertex label in partition set $A$ must differ by $j$ or more and each vertex label in partition set $B$ must differ by $j$ or more. Also, there must be a difference of at least $d$ between the largest labeling in one partition and the smallest labeling in the other partition. Therefore, we have the formula $k(K_{m,n}) = 1+j(m-1)+d+j(n-1)$. □

## 11. $L(d, j, s)$ Surjective Labeling of Paths

Using the computer program described in Section 3, we compiled Table 4. The table shows the lengths of the shortest path that can be surjectively labeled by $d$ value, $j$ value, and $s$ value. By careful observation, one can notice that there are patterns that appear to be forming for the changing values of $d$, $j$ and $s$. Interesting trends appear in bold faced text. The explanations of said patterns will be left to further research or study of the material. Theorem 11.1 shows that if a path of length $n$ can be surjectively labeled with a give $d, j$ and $s$ then any longer path can also be surjectively labeled.

**Theorem 11.1.** *If there exists a surjective* $L(d, j, s)$ *labeling of path* $P_k$ *for some positive integer* $k$*, then path* $P_n$*, with* $n > k$*, can also be surjectively labeled.*

*Proof.* Assume the path $P_{n-1}$ can be surjectively labeled. Call the vertex labeled $n - d$ in $P_{n-1}$, $v_i$. Then if vertices $v_{i-1}$ and $v_{i+1}$ exist, they must be labeled less than $n - 2d$. Also, if vertices $v_{i-2}$ and $v_{i+2}$ exist, they must be labeled less than $n - 3d$ or greater than $n - d + j$. If vertex $v_i$ is of degree 1 in $P_{n-1}$, then append an additional vertex to $v_i$ and label this new vertex $n$. If vertex $v_i$ is of degree 2 in $P_{n-1}$ and $v_{i+2}$ is not labeled $n - 1$, then add an additional vertex between $v_i$ and $v_{i+1}$. Label this new vertex $n$. This creates a surjective labeling of $P_n$. If vertex $v_i$ is of degree 2 in $P_{n-1}$ and $v_{i+2}$ is labeled $n - 1$, then add an additional vertex between $v_i$ and $v_{i-1}$. Label this new vertex $n$. This also creates a surjective labeling of $P_n$. □

## References

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| | $s=1$ | | | | $s=2$ | | | | $s=3$ | | | | $s=4$ | | | | $s=5$ | | | | $s=6$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d$ | $j$ | $s$ | $n$ | $d$ | $j$ | $s$ | $n$ | $d$ | $j$ | $s$ | $n$ | $d$ | $j$ | $s$ | $n$ | $d$ | $j$ | $s$ | $n$ | $d$ | $j$ | $s$ | $n$ |
| | 3 | 2 | 1 | **7** | | | | | | | | | | | | | | | | | | | | |
| | 4 | 2 | 1 | 10 | | | | | | | | | | | | | | | | | | | | |
| | 5 | 2 | 1 | 11 | | | | | | | | | | | | | | | | | | | | |
| | 6 | 2 | 1 | 14 | | | | | | | | | | | | | | | | | | | | |
| $j=2$ | 7 | 2 | 1 | 15 | | | | | | | | | | | | | | | | | | | | |
| | 8 | 2 | 1 | 18 | | | | | | | | | | | | | | | | | | | | |
| | 9 | 2 | 1 | 19 | | | | | | | | | | | | | | | | | | | | |
| | 10 | 2 | 1 | 22 | | | | | | | | | | | | | | | | | | | | |
| | 11 | 2 | 1 | 23 | | | | | | | | | | | | | | | | | | | | |
| | 12 | 2 | 1 | | | | | | | | | | | | | | | | | | | | | |
| | 13 | 2 | 1 | 29 | | | | | | | | | | | | | | | | | | | | |
| | 4 | 3 | 1 | **10** | 4 | 3 | 2 | **12** | | | | | | | | | | | | | | | | |
| | 5 | 3 | 1 | 12 | 5 | 3 | 2 | **12** | | | | | | | | | | | | | | | | |
| | 6 | 3 | 1 | 15 | 6 | 3 | 2 | 15 | | | | | | | | | | | | | | | | |
| $j=3$ | 7 | 3 | 1 | 16 | 7 | 3 | 2 | 16 | | | | | | | | | | | | | | | | |
| | 8 | 3 | 1 | 18 | 8 | 3 | 2 | 18 | | | | | | | | | | | | | | | | |
| | 9 | 3 | 1 | 21 | 9 | 3 | 2 | 21 | | | | | | | | | | | | | | | | |
| | 10 | 3 | 1 | 22 | 10 | 3 | 2 | 22 | | | | | | | | | | | | | | | | |
| | 11 | 3 | 1 | 24 | 11 | 3 | 2 | 24 | | | | | | | | | | | | | | | | |
| | 5 | 4 | 1 | **13** | 5 | 4 | 2 | **16** | 5 | 4 | 3 | **17** | | | | | | | | | | | | |
| | 6 | 4 | 1 | 16 | 6 | 4 | 2 | **16** | 6 | 4 | 3 | **17** | | | | | | | | | | | | |
| $j=4$ | 7 | 4 | 1 | 17 | 7 | 4 | 2 | 17 | 7 | 4 | 3 | **17** | | | | | | | | | | | | |
| | 8 | 4 | 1 | 20 | 8 | 4 | 2 | 20 | 8 | 4 | 3 | 20 | | | | | | | | | | | | |
| | 9 | 4 | 1 | 21 | 9 | 4 | 2 | 21 | 9 | 4 | 3 | 21 | | | | | | | | | | | | |
| | 10 | 4 | 1 | 24 | 10 | 4 | 2 | 24 | 10 | 4 | 3 | 24 | | | | | | | | | | | | |
| | | | | | 11 | 4 | 2 | 25 | 11 | 4 | 3 | 25 | | | | | | | | | | | | |
| | 6 | 5 | 1 | **16** | 6 | 5 | 2 | **18** | 6 | 5 | 3 | **20** | 6 | 5 | 4 | **22** | | | | | | | | |
| | 7 | 5 | 1 | 18 | 7 | 5 | 2 | **18** | 7 | 5 | 3 | **20** | 7 | 5 | 4 | **22** | | | | | | | | |
| $j=5$ | 8 | 5 | 1 | 20 | 8 | 5 | 2 | 20 | 8 | 5 | 3 | **20** | 8 | 5 | 4 | **22** | | | | | | | | |
| | 9 | 5 | 1 | 22 | 9 | 5 | 2 | 22 | 9 | 5 | 3 | 22 | 9 | 5 | 4 | **22** | | | | | | | | |
| | 10 | 5 | 1 | 25 | 10 | 5 | 2 | 25 | 10 | 5 | 3 | 25 | 10 | 5 | 4 | 25 | | | | | | | | |
| | 11 | 5 | 1 | 26 | 11 | 5 | 2 | 26 | 11 | 5 | 3 | 26 | 11 | 5 | 4 | 26 | | | | | | | | |
| | 7 | 6 | 1 | **19** | 7 | 6 | 2 | **22** | 7 | 6 | 3 | 23 | 7 | 6 | 4 | 24 | 7 | 6 | 5 | 25 | | | | |
| $j=6$ | 8 | 6 | 1 | 22 | 8 | 6 | 2 | **22** | 8 | 6 | 3 | **25** | 8 | 6 | 4 | **26** | 8 | 6 | 5 | **27** | | | | |
| | 9 | 6 | 1 | 24 | 9 | 6 | 2 | 24 | 9 | 6 | 3 | **25** | 9 | 6 | 4 | **26** | 9 | 6 | 5 | **27** | | | | |
| | 10 | 6 | 1 | 26 | 10 | 6 | 2 | 26 | 10 | 6 | 3 | 26 | 10 | 6 | 4 | **26** | 10 | 6 | 5 | **27** | | | | |
| | 11 | 6 | 1 | 27 | 11 | 6 | 2 | 27 | 11 | 6 | 3 | 27 | 11 | 6 | 4 | 27 | 11 | 6 | 5 | **27** | | | | |
| | 12 | 6 | 1 | 30 | 12 | 6 | 2 | 30 | 12 | 6 | 3 | 30 | 12 | 6 | 4 | 30 | 12 | 6 | 5 | 30 | | | | |
| | 13 | 6 | 1 | 31 | 13 | 6 | 2 | 31 | 13 | 6 | 3 | 31 | 13 | 6 | 4 | 31 | 13 | 6 | 5 | 31 | | | | |
| | 14 | 7 | 1 | 34 | 14 | 7 | 2 | | 14 | 7 | 3 | | 14 | 7 | 4 | | 14 | 7 | 5 | 34 | | | | |
| | 8 | 7 | 1 | **22** | 8 | 7 | 2 | **24** | 8 | 7 | 3 | 26 | 8 | 7 | 4 | 28 | 8 | 7 | 5 | 28 | 8 | 7 | 6 | 30 |
| $j=7$ | 9 | 7 | 1 | 24 | 9 | 7 | 2 | **24** | 9 | 7 | 3 | **26** | 9 | 7 | 4 | **28** | 9 | 7 | 5 | **30** | 9 | 7 | 6 | **32** |
| | 10 | 7 | 1 | 26 | 10 | 7 | 2 | 26 | 10 | 7 | 3 | **26** | 10 | 7 | 4 | **28** | 10 | 7 | 5 | **30** | 10 | 7 | 6 | **32** |
| | 11 | 7 | 1 | 28 | 11 | 7 | 2 | 28 | 11 | 7 | 3 | 28 | 11 | 7 | 4 | **28** | 11 | 7 | 5 | **30** | 11 | 7 | 6 | **32** |
| | 12 | 7 | 1 | 30 | 12 | 7 | 2 | 30 | 12 | 7 | 3 | 30 | 12 | 7 | 4 | 30 | 12 | 7 | 5 | **30** | 12 | 7 | 6 | **32** |
| | 13 | 7 | 1 | 32 | 13 | 7 | 2 | 32 | 13 | 7 | 3 | 32 | 13 | 7 | 4 | 32 | 13 | 7 | 5 | 32 | 13 | 7 | 6 | **32** |
| | 14 | 7 | 1 | | 14 | 7 | 2 | | 14 | 7 | 3 | | 14 | 7 | 4 | | 14 | 7 | 5 | | 14 | 7 | 6 | 35 |

TABLE 4. This table shows the length of the shortest path, $P_n$ that can be surjectively labeled for the various parameters of $L(d, j, s)$ labeling.