9-1990

Fibonacci Sequences in Finite Groups

Steven W. Knox
College of Wooster

FIBONACCI SEQUENCES

IN FINITE GROUPS

Steven W. Knox

MS TR 90-03

September 1990

Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute, IN 47803

FAX(812) 877-3198

Phone: (812) 877-8391

FIBONACCI SEQUENCES IN FINITE GROUPS

Steven W. Knox*
Department of Mathematical Sciences,
The College of Wooster, Wooster, OH 44691
(Submitted May, 1990)

0. Introduction

The Fibonacci sequence and its related higher-order sequences ("tribonacci", "quatranacci", "k-nacci") are generally viewed as sequences of integers. In 1960, Wall [4] considered Fibonacci sequences modulo some fixed integer m; i.e., Fibonacci sequences of elements of $\mathbb{Z}_m$. He proved that these sequences were periodic for any m. Shah [3] partially determined for which integers the Fibonacci sequence modulo m contained the complete residue system, $\mathbb{Z}_m$. The papers of Wall [4] and Shah [3] provided the motivation for Wilcox's [5] study of the Fibonacci sequence in finite abelian groups.

This paper is in the spirit of [3], [4], and [5]. It addresses not only the traditional Fibonacci (2-nacci) sequence, but also the k-nacci sequence, and does so for finite (not necessarily abelian) groups.

1. Definitions and Notation

A *k-nacci sequence in a finite group* is a sequence of group elements $x_0, x_1, x_2, x_3, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, \ldots, x_{j-1}$ each element is defined by

$$x_n = \begin{vmatrix} x_0x_1 \cdots x_{n-1} \text{ for } j \le n < k \\ x_{n-k}x_{n-k+1} \cdots x_{n-1} \text{ for } n \ge k \end{vmatrix}.$$

* Work supported by NSF grant DMS - 8900507

We also require that the initial elements of the sequence, $x_0, \ldots, x_{j-1}$, generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group G seeded by $x_0, \ldots, x_{j-1}$ is denoted by $F_k(G; x_0, \ldots, x_{j-1})$.

The classic Fibonacci sequence in the integers modulo m can be written $F_2(\mathbb{Z}_m; 0, 1)$. We call a 2-nacci sequence of group elements a *Fibonacci sequence of a finite group*.

A finite group G is *k-nacci sequencable* if there exists a k-nacci sequence of G such that every element of the group appears in the sequence.

A sequence of group elements is *periodic* if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the *period* of the sequence. The sequence a, b, c, d, b, c, d, b, c, d, ... is periodic after the initial element a and has period 3. We denote the period of a k-nacci sequence $F_k(G; x_0, \ldots, x_{j-1})$ by $P_k(G; x_0, \ldots, x_{j-1})$. A sequence is *simply periodic* with period k if the first k elements in the sequence form a repeating subsequence. For example, a, b, c, d, e, a, b, c, d, e, ... is simply periodic with period 5.

Finally, given a group element g, let o(g) denote the order of g.

2. Theorems

Theorem 1: A k-nacci sequence in a finite group is simply periodic.

Proof: Let n be the order of G. Since there are n^k distinct k-tuples of elements of G, at least one of the k-tuples appears twice in a k-nacci sequence of G. Therefore the subsequence following this k-tuple repeats, hence the k-nacci sequence is periodic.

Since the sequence is periodic, there exist natural numbers i and j, with i > j, such that $x_{i+1} = x_{j+1}$, $x_{i+2} = x_{j+2}$, $x_{i+3} = x_{j+3}$, ... , $x_{i+k} = x_{j+k}$. By the defining relation of a k-nacci sequence we know that $x_i = x_{i+k}(x_{i+(k-1)})^{-1}(x_{i+(k-2)})^{-1} \cdots (x_{i+1})^{-1}$ and $x_j = x_{j+k}(x_{j+(k-1)})^{-1}(x_{j+(k-2)})^{-1} \cdots (x_{j+1})^{-1}$. Hence $x_i = x_j$ and it then follows that $x_{i-1} = x_{j-1}$, $x_{i-2} = x_{j-2}$, ... , $x_{i-j} = x_{j-j} = x_0$. Therefore the sequence is simply periodic.

This is a generalization of a theorem of Wall [4], which states that $F(\mathbb{Z}_m;\ 0,\ 1)$, the classically seeded Fibonacci sequence of the integers modulo m, is simply periodic. From the proof of Theorem 1 we have $|G|^k$ as an upper bound for the period of any k-nacci sequence in a group G.

We will now address the periods of k-nacci sequences in specific classes of groups. A group D_n is *dihedral* if $D_n = \langle a,\ b:\ a^n = b^2 = e \text{ and } ba = a^{-1}b\rangle$. The order of the group D_n

is $2n$. Note that in a dihedral group generated by a and b, $(ab)^2 = abab = aa^{-1}b^2 = e$ and $(ba)^2 = baba = baa^{-1}b = e$.

Theorem 2: Consider the dihedral group D_n for some $n \geq 3$ with generators a, b. Then $P_k(D_n;\ a,\ b) = P_k(D_n;\ b,\ a) = 2k + 2$.

Proof: Let the orders of a and b be n and 2, respectively. If $k = 2$, the possible sequences are a, b, ab, a^{-1}, a^2b, ab, a, b, ... and b, a, $a^{-1}b$, b, a^{-1}, ab, b, a, ..., both of which have period 6. If $k \geq 3$, the first k elements of $F_k(D_n;\ a,\ b)$ are $x_0 = a$, $x_1 = b$, $x_2 = ab$, $x_3 = (ab)^2$, ... , $x_{k-1} = (ab)^{2^{k-3}}$. This sequence reduces to a, b, ab, e, e, ... , e, e where $x_j = e$ for $3 \leq j \leq k-1$. Thus

$$x_k = \prod_{i=0}^{k-1} x_i = abab = e,$$

$$x_{k+1} = \prod_{i=1}^{k} x_i = bab = a^{-1},$$

$$x_{k+2} = \prod_{i=2}^{k+1} x_i = aba^{-1} = a^2b,$$

$$x_{k+3} = \prod_{i=3}^{k+2} x_i = a^{-1}a^2b = ab,$$

$$x_{k+4} = \prod_{i=4}^{k+3} x_i = a^{-1}a^2bab = e.$$

It follows that $x_{k+j} = e$ for $4 \le j \le k$. We also have

$$x_{k+k+1} = \prod_{i=k+1}^{k+k} x_i = a^{-1}a^2bab = e,$$

$$x_{k+k+2} = \prod_{i=k+2}^{k+k+1} x_i = a^2bab = a,$$

$$x_{k+k+3} = \prod_{i=k+3}^{k+k+2} x_i = aba = b,$$

$$x_{k+k+4} = \prod_{i=k+4}^{k+k+3} x_i = ab.$$

Since the elements succeeding x_{2k+2}, x_{2k+3}, x_{2k+4}, depend on a, b, and ab, for their values, the cycle begins again with the 2k+2nd element; i.e., $x_0 = x_{2k+2}$. Thus period of $F_k(D_n;\ a,\ b)$ is 2k+2.
If we choose to seed the sequence with the generators in the other order, we see that the sequence b, a, ba, $(ba)^2$, $(ba)^4$, $(ba)^8$, . . . , $(ba)^{2^{k-3}}$ reduces to b, a, ba, e, e, . . . , e, e and the proof works similarly.

If a group is minimally generated by i elements, then it is said to be an *i-generated group*.

Theorem 3: If G is a 2-generated group with generators a and b, and the identity element appears in $F_2(G;\ a,\ b)$ or $F_2(G;\ b,\ a)$, a Fibonacci sequence of G, then G is abelian.

Proof: Without loss of generality consider the sequence $F_2(G;\ a,\ b)$ and suppose the identity, e, is the n+1st element of this Fibonacci sequence for some natural number n. The nth element of the sequence may be any element of the group. Thus we have a sequence

$a,\ b,\ \ldots,\ s,\ e,\ \ldots$

What precedes s? Only s^{-1} could satisfy the defining relation for the n-1st position. Similarly s^2 must be in the n-2nd sequence position, s^{-3} in the n-3rd, and so on, forming the sequence

$a,\ b,\ \ldots,\ s^{-8},\ s^5,\ s^{-3},\ s^2,\ s^{-1},\ s^1,\ e,\ \ldots$

Since these elements have exponents generated using the relation $u_{i-2} = -u_{i-1} + u_i$, which is equivalent to $u_i = u_{i-1} + u_{i-2}$, we find the Fibonacci sequence of integers occuring in the exponents of s, with alternating signs. Hence a Fibonacci sequence of the group has one of two forms:

(i) n odd: The sequence is $s^{u_n},\ s^{-u_{n-1}},\ s^{u_{n-2}},\ \ldots,\ s^5,\ s^{-3},\ s^2,\ s^{-1},\ s^1,\ e$. In this case we have $s^{u_n} = a$, $s^{-u_{n-1}} = b$ (which implies $s^{u_{n-1}} = b^{-1}$), and $s^{u_{n-2}} = ab$.

Since $s^{u_{n-1}}s^{u_{n-2}} = s^{u_{n-1}+u_{n-2}} = s^{u_n}$, we have $b^{-1}ab = a$, or $ab=ba$. Therefore the group is abelian.

(ii) n even: The sequence is s^{-u_n}, $s^{u_{n-1}}$, $s^{-u_{n-2}}$, . . . , s^5, s^{-3}, s^2, s^{-1}, s^1, e. In this case we have $s^{-u_n} = a$, $s^{u_{n-1}} = b$ (which implies $s^{-u_{n-1}} = b^{-1}$), and $s^{-u_{n-2}} = ab$. Since $s^{-u_{n-1}}s^{-u_{n-2}} = s^{-(u_{n-1}+u_{n-2})} = s^{-u_n}$, we have $b^{-1}ab = a$, or $ab=ba$. Therefore the group is abelian.

The converse of theorem 3 does not hold. Consider the abelian group $A = \langle a, b: a^9 = b^2 = e \text{ and } ba = ab\rangle$. The Fibonacci sequences of this group are:

a, b, ab, a, a^2b, a^3b, a^5, a^8b, a^4b, a^3, a^7b, ab,
a^8, b, a^8b, a^8, a^7b, a^6b, a^4, ab, a^5b, a^6, a^2b, a^8b,
a, b, ab, . . .

and

b, a, ab, a^2b, a^3, a^5b, a^8b, a^4, a^3b, a^7b, a, a^8b,
b, a^8, a^8b, a^7b, a^6, a^4b, ab, a^5, a^6b, a^2b, a^8, ab,
b, a, ab, ...

The elements e, a^2, and a^7 do not appear in either sequence.

Corollary: A 2-nacci sequencable group s cyclic.

Proof: Let G be a 2-nacci sequencable group. Then G is either 1- or 2-generated. If G is 2-generated, then since e appears in the 2-nacci sequence of G, we can construct the sequence in terms of an element $s \in G$ as in the proof of Theorem 3. Every element of G appears in its 2-nacci sequence, and therefore all the elements of G may be represented in terms of a single element, s. Hence G is 1-generated, or cyclic.

For $k \geq 3$, k-nacci sequencable groups are not, in general, abelian. The dihedral group of six elements is 3-nacci sequencable.

Theorem 4: If the identity element appears in a Fibonacci sequence of a 2-generated group then the collection of subscripts of the sequence elements x_i for which $x_i = e$ contains a sequence which has an arithmetic progression.

Proof: By Theorem 3 the group $G = \langle a, b \rangle$ is abelian. Hence the nth term of the sequence has the form $a^{u_{n-1}}b^{u_n}$. By a theorem of Wall [4] we know that the terms where $u_n \equiv 0 \pmod m$ have subscripts that form a simple arithmetic progression. Thus the sequences of elements $a, a, a^2, \ldots, a^{u_n}$ and $b, b, b^2, b^3, \ldots, b^{u_n}$ both have e occurring in positions whose subscripts form arithmetic progressions, with the period of the occurrence of e depending on the order of a and b. The period of this induced occurrence of e in $a, b, ab, ab^2, a^2b^3, \ldots$ will be the least common multiple of the period of e in $a, a, a^2, \ldots$ and the period of e in $b, b, b^2, b^3, \ldots$. Hence the positions of e in $a, b, ab, ab^2, a^2b^3, \ldots$ will have subscripts which contain an arithmetic progression.

Corollary: If N is a normal subgroup of G, then the terms x_i of a Fibonacci sequence of G for which $x_i \in N$ have subscripts which contain an arithmetic progression.

Proof: Consider $F_2(G; a, b)$. Whenever $x_i \in N$, $x_iN = N$, the identity in G/N. By Theorem 4, the indices for which an element of $F_2(G/N; aN, bN)$ is the identity contains an arithmetic progression. Therefore the indices for which $x_i \in N$ in $F_2(G; a, b)$ contain an arithmetic progression.

3. An Open Question

It is clear that a homomorphic image of a k-nacci sequencable group is k-nacci sequencable. The extension of a k-nacci sequencable group by a k-nacci sequencable group is not necessarily k-nacci sequencable. In fact, the direct product of k-nacci sequencable groups is not necessarily k-nacci sequencable.

We refer to the abelian group $A = \langle a, b: a^9 = b^2 = e$ and $ba = ab\rangle$. The group $\langle b\rangle$ has a Fibonacci sequence $F_2(\langle b\rangle; e, b) = e, b, b, e, \ldots$, and hence is 2-nacci sequencable. The group $\langle a\rangle$ has a sequence $F_2(\langle a\rangle; e, a) = e, a, a, a^2, a^3, a^5, a^8, a^4, a^3, a^7, a, a^8, a^8, a^7, a^6, a^4, a, a^5, a^6, a^2, a^8, a, e, a, a, \ldots$ and hence is 2-nacci sequencable. We have already seen that A, the direct product of <a> and <b>, is not 2-nacci sequencable.

Question: Are all non-simple k-nacci sequencable groups non-trivial extensions of a k-nacci sequencable group by a k-nacci sequencable group? That is, does a non-simple k-nacci sequencable group have a k-nacci sequencable normal subgroup?

References

1. Derek K. Chang. "Higher-Order Fibonacci Sequences Modulo m." *The Fibonacci Quarterly* **24** (1986):138-139.
2. Mark Feinberg. "Fibonacci-Tribonacci." *The Fibonacci Quarterly* **1** (1963):71-74.
3. A. P. Shah. "Fibonacci Sequence Modulo m." *The Fibonacci Quarterly* **6** (1968):139-141.
4. D. D. Wall. "Fibonacci Series Modulo m." *American Math Monthly* **67** (1960):525-532.
5. Howard J. Wilcox. "Fibonacci Sequences of Period n in Groups." *The Fibonacci Quarterly* **24** (1986):356-361.