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**DIVISIBILITY BY $|G|$ FOR POWERS
OF ORDERED k -SETS**

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Divisibility by $|G|$ for powers of ordered k -sets

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Abstract

It is shown that the number of ordered k -sets of a group G whose n th power contains exactly i elements is always a multiple of $|G|$. An elementary proof of the fact that the number of ordered pairs (x, y) such that $x^2 = y^2$ is equal to $k_r|G|$ is also given.

1 Motivation

We define the n th power of an ordered k -set $X = (x_1, \dots, x_k)$ to be the set $Y = \{x_{i_1} \cdots x_{i_n} \mid x_{i_1}, \dots, x_{i_n} \in X\}$, where the x_i 's need not be distinct. We say that the n th power of X has i elements if $|Y| = i$. While working with cubing ordered pairs and squaring ordered triples, it became apparent that the number of pairs/triples/etc. whose n th power had exactly i elements was always a multiple of the order of the group. It is known [1] that the number of ordered pairs whose square has i elements is always a multiple of the order of the group, but the proof of this uses character theory to show that the number of pairs (x, y) for which $x^2 = y^2$ is a multiple of the order of the group, and does not generalize to larger k -sets or higher powers in any simple fashion. This paper contains an elementary proof of the result that had required character theory as well as a generalization of the entire result to n th powers of ordered k -sets.

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2 Counting elements with equal squares

Theorem 1 *The number of ordered pairs (x, y) with $x, y \in G$ and $x^2 = y^2$ is equal to $k_r|G|$, where k_r is the number of real conjugacy classes in G .*

PROOF: For notational purposes, we say that a pair (x, y) “works” for q if $q = y^{-1}x$ and $x^2 = y^2$. The strategy for this proof will be to fix an element q , and show that the number of ordered pairs (x, y) which work for q equals $|C(q)|$ if, and only if, q is in a real conjugacy class, and otherwise it must equal zero. Any q for which there is at least one such ordered pair (x, y) must be in a real conjugacy class, because $x^2 = y^2$ implies $q = y^{-1}x = yx^{-1} = yx^{-1}yy^{-1} = y(y^{-1}x)^{-1}y^{-1} = yq^{-1}y^{-1}$. Once this is proven, the proof will be complete, because then the total number of pairs (x, y) such that $x^2 = y^2$ will be equal to

$$\begin{aligned}
 \sum_{q \in G} \text{number of pairs working for } q &= \sum_{q \in RC} |C(q)| \\
 &= \sum_{RC} |C(q)|(|G|/|C(q)|) \\
 &= \sum_{RC} |G| \\
 &= k_r|G|,
 \end{aligned}$$

where RC denotes the set of real conjugacy classes, $q \in RC$ denotes those elements which are in a real conjugacy class, and $k_r = |RC|$. The proof has three steps:

1. There exists at least one ordered pair (x, y) for a fixed q in a real conjugacy class.
2. If (x, y) works for q , then so does (cx, cy) , where $c \in C(q)$.
3. If (x, y) and (a, b) both work for q , then $a = kx$ and $b = ky$, where $k \in C(q)$.

Now we prove them, in the same order as proposed.

1. Since q is in a real conjugacy class, there exists a y such that $q = yq^{-1}y^{-1}$. We pick $x = yq$.

Then $y^{-1}x = y^{-1}yq = q$, and

$$\begin{aligned} x^2 &= yqyq \\ &= yyq^{-1}y^{-1}yq \\ &= y^2 \end{aligned}$$

so the pair (x, y) works for q .

2. Say (x, y) works for q , and $c \in C(q)$. Then $(cy)^{-1}(cx) = y^{-1}c^{-1}cx = y^{-1}x = q$, and $cxcx = cyy^{-1}xcx = cycy^{-1}x^2 = cycy^{-1}y^2 = cycy$, as desired. Thus (cx, cy) works for q as well.

3. Say (x, y) and (a, b) both work for q . Certainly there exists a k such that $a = kx$, and certainly there exists a z such that $b = kz$. First we show that $z = y$, then we show that $k \in C(q)$. Now $y^{-1}x = q = b^{-1}a = (kz)^{-1}(kx) = z^{-1}x$, so we have $z = y$. Now we also know that $a^2 = b^2$, so we may write $kxkx = kyky \Rightarrow xkx = yky \Rightarrow y^{-1}xk = kyx^{-1} \Rightarrow y^{-1}x = k(yx^{-1})k^{-1}$. But since $y^2 = x^2$, we know that $yx^{-1} = y^{-1}x$, so we may rewrite this last equation as $y^{-1}x = k(y^{-1}x)k^{-1}$, implying that k commutes with $y^{-1}x$, so $k \in C(q)$.

Since we have shown that the number of pairs (x, y) which work for q is a multiple of $|C(q)|$, the proof is complete. •

3 Divisibility by $|G|$ for all powers

Theorem 2 *The number of ordered k -sets of a group whose n^{th} power has exactly i elements is divisible by the order of the group.*

PROOF: For the sake of notation, we refer to the ordered k -set as a vector with k components: $X = (x_1, \dots, x_k)$. The elements of the n th power of the k -set must all have the form $x_{i_1} x_{i_2} \dots x_{i_n}$, where the i 's are all integers between 1 and k (not necessarily distinct). We refer to the elements of the n th power of the k -set as an ordered tuple $Y = (y_1, \dots, y_{k^n})$, rather than as a set, with the understanding that some of the y_i 's may also not be distinct. To keep a fixed order on the y_i 's, we assign them in lexicographic order by subscripts, that is, $y_1 = x_1^n$, $y_2 = x_1^{n-1} x_2$, and so on up to $y_{k^n} = x_k^n$. As an example, if we are squaring the ordered pair (x_1, x_2) , then $(y_1, y_2, y_3, y_4) = (x_1^2, x_1 x_2, x_2 x_1, x_2^2)$, in that order. We consider the $(k^n(k^n - 1)/2)$ -tuple $S = (y_i y_j^{-1} | j > i)$, where again we put the elements in lexicographical order by subscripts: the first element is $y_1 y_2^{-1}$, and so on. In the same example, we would then have $S = (x_1^2(x_1 x_2)^{-1}, x_1^2(x_2 x_1)^{-1}, x_1^2(x_2^2)^{-1}, x_1 x_2(x_2 x_1)^{-1}, x_1 x_2(x_2^2)^{-1}, x_2 x_1(x_2^2)^{-1})$. Clearly, if all the y_i 's are distinct, no element in S is equal to the identity. In addition, the location and frequency of the identity element in S completely determine which elements among the y_i 's are equal (and thus, how many distinct elements there are in the n th power of the ordered k -set). We introduce a new array of elements $q_{s,t} = x_s^t x_{s+1}^{-t}$, and another array $r_{s,t} = x_s^{-t} x_{s+1}^t$, for s ranging from 1 to $(k - 1)$ and t ranging from 1 to v , where v is to be assigned. The purpose of these is as follows:

Lemma 1 *If $c \in C(q_{s,t})$ for all s and all $t \leq v$, then c commutes with any product of the form $x_{i_1} x_{i_2} \dots x_{i_w} x_{i_w+1}^{-1} \dots x_{i_{2w}}^{-1}$, where $w \leq v$.*

PROOF: First we note that c must in fact commute with $x_i^t x_j^{-t}$ for any i, j , for any $t \leq v$ since if $i < j$, then $x_i^t x_j^{-t} = (x_i^t x_{i+1}^{-t})(x_{i+1}^t x_{i+2}^{-t}) \dots (x_{j-1}^t x_j^{-t})$, all of which commute with c , and if $i > j$, then the same argument shows that c commutes with $x_j^t x_i^{-t}$, and thus with $x_i^t x_j^{-t}$. Thus it suffices to write the product in the lemma as

$$x_{i_1} \dots x_{i_{2w}}^{-1} = (x_{i_1} x_{i_2}^{-1})(x_{i_2}^2 x_{i_3}^{-2}) \dots (x_{i_w}^w x_{i_{w+1}}^{-w})(x_{i_{w+1}}^{w-1} x_{i_{w+2}}^{1-w}) \dots (x_{i_{2w-1}} x_{i_{2w}}^{-1}),$$

a product which commutes with c . •

Similarly, we also have

Lemma 2 *If $c \in C(r_{s,t})$ for all s and all $t \leq v$, then c commutes with any product of the form $x_{i_1}^{-1} x_{i_2}^{-1} \cdots x_{i_w}^{-1} x_{i_{w+1}} \cdots x_{i_{2w}}$, with $w \leq v$.*

We also define \overline{C} to be the intersection of all of the $C(q_{s,t})$'s and the $C(r_{s,t})$'s, and note that if $c \in \overline{C}$, then c satisfies the conditions of Lemmas 1 and 2. With the notation out of the way, we may proceed with the proof. We wish to show that the number of ordered k -sets which yield a given subsequence of identity elements in S is a multiple of the order of the group. This is actually a more general result than that stated in Theorem 2, since two different sequences of identity elements could easily yield the same number of distinct y_i 's. To prove this more general result, we will show that the number of ordered k -sets which yield a given value for $q_{1,1}$ as well as a fixed sequence of identity elements in S is always a multiple of $|C(q_{1,1})|$. This will complete the proof, since any conjugate of the given value of $q_{1,1}$ will have the same number of ordered k -sets that yield the given sequence of identity elements in S (for all i , just conjugate x_i by the same element $q_{1,1}$ had been conjugated by), so the total number of ordered k -sets yielding that sequence and with a $q_{1,1}$ conjugate to the original $q_{1,1}$ is a multiple of $|C(q_{1,1})|[G : C(q_{1,1})] = |G|$. Summing over all conjugacy classes then shows that the total number of ordered k -sets yielding that sequence of identity elements in S is also a multiple of the order of the group, as desired.

With all this in mind, we fix a sequence of identity elements in S and fix values not just for $q_{1,1}$ but also for $q_{s,t}$ and $r_{s,t}$ for all s and all t up to v , where $n = 2v$ or $n = 2v + 1$ (depending on the parity of n). Our goal will be first to show that the number of ordered k -sets X which yield the given values for all the q 's and r 's as well as the given sequence of identity elements in S is a multiple of $|\overline{C}|$. We will prove this by showing that, if we choose any $c \in \overline{C}$, then

1. the q 's and the r 's are fixed when we send X to cX , and
2. the ordered set X yields the identity for a specific element in S if, and only if, cX does as well.

First we prove part (1). This, just like everything else in this proof, will consist only of eliminating a c and a c^{-1} every time they appear on opposite sides of an expression of the form given in the first Lemma.

$$\begin{aligned}
(cx_s)^t(cx_{s+1})^{-t} &= cx_s \cdots cx_s(cx_{s+1})^{-1} \cdots (cx_{s+1})^{-1} \\
&= cx_s \cdots cx_s x_{s+1}^{-1} c^{-1} \cdots x_{s+1}^{-1} c^{-1} \\
&= x_s^t x_{s+1}^{-t}.
\end{aligned}$$

We could make this last step because conjugating $x_s^m x_{s+1}^{-m}$ with c does not change its value for any $m \leq v$, so that none of the c 's changed the value of the expression. The argument that the $r_{s,t}$ values are also fixed proceeds in exactly the same manner.

Now we prove part (2). First we must show that if an element in the S created by X is the identity, then that same element in the S created by cX is also the identity. Let that element be written as $x_{i_1} \cdots x_{i_n} x_{i_{n+1}}^{-1} \cdots x_{i_{2n}}^{-1}$. The element we wish to show equal to the identity can then be written as $(cx_{i_1}) \cdots (cx_{i_n})(cx_{i_{n+1}})^{-1} \cdots (cx_{i_{2n}})^{-1}$. Again we may make repeated use of Lemma 1, since in the middle of this expression we find the product $cx_{i_n} x_{i_{n+1}}^{-1} c^{-1}$, which we know to equal $x_{i_n} x_{i_{n+1}}^{-1}$. But upon eliminating this innermost c and c^{-1} , we find that the new innermost c and c^{-1} are in the product $cx_{i_{n-1}} x_{i_n} x_{i_{n+1}}^{-1} x_{i_{n+2}}^{-1} c^{-1}$. Again using Lemma 1, we see that we may cancel the c and the c^{-1} . We may repeat this process until there are $2v + 2$ elements between the innermost c and c^{-1} , at which point we can no longer use Lemma 1 to cancel the c and the c^{-1} . At this point the element which we wish to show equals the identity can be written as

$$cx_{i_1} \cdots c(x_{i_{n-v}} x_{i_{n-v+1}} \cdots x_{i_n} x_{i_{n+1}}^{-1} \cdots x_{i_{n+v}}^{-1} x_{i_{n+v+1}}^{-1}) c^{-1} \cdots x_{i_{2n}}^{-1} c^{-1}.$$

But we may replace the product between the innermost c and c^{-1} with a shorter product, since

$$x_{i_1} \cdots x_{i_n} x_{i_{n+1}}^{-1} \cdots x_{i_{2n}}^{-1} = e$$

implies that

$$x_{i_{n-v}} \cdots x_{i_n} x_{i_{n+1}}^{-1} \cdots x_{i_{n+v+1}}^{-1} = x_{i_{n-v-1}}^{-1} \cdots x_{i_1}^{-1} x_{i_{2n}} \cdots x_{i_{n+v+2}}.$$

As a result, we may write the element of S as

$$c x_{i_1} \cdots c x_{i_{n-v-1}} c (x_{i_{n-v-1}}^{-1} \cdots x_{i_1}^{-1} x_{i_{2n}} \cdots x_{i_{n+v+2}}) c^{-1} x_{i_{n+v+2}}^{-1} c^{-1} \cdots x_{i_{2n}}^{-1} c^{-1}.$$

But now we again have either a product of $2v$ elements or $(2v - 2)$ elements (depending on the parity of n) in the middle of the expression. Since this product is of the form of the product in Lemma 2, c commutes with it, so we may cancel the innermost c and c^{-1} . But then we are also able to cancel both $x_{i_{n-v-1}}$ and $x_{i_{n+v+2}}$ with their inverses, meaning that we now have a $(2v - 2)$ -tuple or a $(2v - 4)$ -tuple between the innermost c and c^{-1} . Repeating the process, we see that we may continue eliminating c 's and cancelling until we reduce all the way to the identity, which was what we wanted.

This argument also implies that if the ordered k -set cX yields the identity for a specific element in S , then so does the ordered k -set X , since we can just consider the ordered k -set $c^{-1}cX$. Thus we have proven that the number of ordered k -sets that work for the given q 's and r 's is in fact a multiple of $|\overline{C}|$.

To finish the proof, we note that the number of ordered k -sets that work for a given set of q 's and r 's is equal to the number that work for the same $q_{1,1}$ but with all the other q 's and r 's conjugated by any element in the centralizer of $q_{1,1}$, since we may just conjugate each x_i by the same element. The number of different sets of q 's and r 's we may get in this fashion is just equal to the number of elements in the centralizer of $q_{1,1}$ divided by $|\overline{C}|$ (since the number of elements in $C(q_{1,1})$ that fix all of the q 's and r 's upon conjugation is equal to $|\overline{C}|$). But the number of ordered

k-sets that work for any given collection of q 's and r 's is just a multiple of $|\overline{C}|$, and we may take advantage of as follows. We count the number of ordered k-sets X which

- yield the fixed sequence of identity elements in S , and
- either yield the original q 's and r 's or yield instead $aq_{s,t}a^{-1}$ and $ar_{s,t}a^{-1}$, where a is an element in $C(q_{1,1})$.

This total is just (number of X for original q 's and r 's)(number of distinct conjugates over $C(q_{1,1})$ of the original $q_{s,t}$'s and $r_{s,t}$'s), which we may write as $(k_q|\overline{C}|)(|C(q_{1,1})|/|\overline{C}|) = k_q|C(q_{1,1})|$, where k_q is an integer. The total number of ordered k-sets which fix the sequence of identities in S and the value of $q_{1,1}$ can be counted by totalling over conjugacy classes of possible q 's and r 's, and as we have just shown, the count for any one of these conjugacy classes is a multiple of $|C(q_{1,1})|$. The total number of ordered k-sets that have a given sequence of identity elements in S and a given value for $q_{1,1}$ is thus equal to a multiple of $|C(q_{1,1})|$ as well, and the proof is complete. •

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