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Abstract

In this paper, we consider the probability that two elements chosen at random from a finite group G generate a subgroup of a given nilpotency class. It is shown that in solvable non-nilpotent groups, the probability that two elements generate a nilpotent subgroup is $\leq 1/p_s$, where p_s is the smallest prime dividing the order of the group, and it is also shown that there exist groups such that the probability of two elements generating a subgroup of class i approaches one (and other groups for which it approaches zero) for all $i \geq 2$. It is also shown that the number of pairs which generate a subgroup of a given class is always a multiple of the order of the group. Some preliminary results on the analogous problem for solvability are also given.

1 Introduction

A problem in finite group theory (all groups referred to in this paper will be finite) which has received a great deal of attention is that of determining the probability that two elements in a given group commute. It is known that there is an "commutativity gap," that is, if a group is abelian, the probability that two elements commute is 1, while the highest proportion of commuting pairs possible in a non-abelian group is 5/8 [5]. It is also known [2] that the number of commuting pairs equals the product of the number of conjugacy classes and the order of the group.

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In this paper we study a natural extension of this problem. Bearing in mind that two elements commute if, and only if, they generate an abelian group, we consider instead the probabilty that two elements in a given group generate a nilpotent subgroup. Specifically, we look for analogs to the known results for commutativity. Are there limits on the fraction of pairs generating a subgroup of nilpotency class i? Is there a limit to the fraction of pairs in a non-nilpotent group that can generate nilpotent groups? Is the number of pairs which generate a nilpotent subgroup of class i a multiple of the order of the group?

We now introduce some notation in order to phrase these questions quantitatively. We define $p_i(G)$ to be the probability that two elements chosen from a group G generate a subgroup of G with nilpotency class i (if a subgroup is non-nilpotent, then it has nilpotency class zero). In formal terms,

$$p_i(G) = \frac{|\{(x,y) : x,y \in G \text{ and } \langle x,y\rangle \text{ nilpotent of class } i\}|}{|G|^2}.$$

In this paper we will show the following results:

1. For any integer $k \neq 1$, there exists a sequence of groups $\{G_n\}$ such that $p_k(G_n)$ approaches 1, and for any integer $k \neq 0$, there exists another sequence of groups $\{H_n\}$ such that $p_k(H_n)$ approaches zero. This will show that the only possible "nilpotency gaps" are the already known 5/8 upper bound for p_1 in non-abelian groups, and possibly a lower bound for p_0 in non-nilpotent groups.

2. The lower bound for p_0 does exist for solvable groups, and in fact equals 1/2.

3. The number of pairs which generate a subgroup of nilpotency class i is a multiple of the order of the group.

In the last section, we will briefly investigate the analogous problem on solvability.

2 Limiting Values of p_i

We wish to prove upper and lower bounds on the various p_i's. To do this, we will repeatedly take advantage of the following lemma.

Lemma 1 *For all groups G, H and all $m \geq 1$,*

$$\sum_{i=1}^{m} p_i(G \times H) = \left(\sum_{i=1}^{m} p_i(G)\right)\left(\sum_{i=1}^{m} p_i(H)\right).$$

PROOF: It suffices to show that

$$|G \times H|^2 \sum_{i=1}^{m} p_i(G \times H) = |G|^2 \left(\sum_{i=1}^{m} p_i(G)\right) |H|^2 \left(\sum_{i=1}^{m} p_i(H)\right).$$

To prove this we show that $\langle x_G, y_G\rangle \times \langle x_H, y_H\rangle$ and $\langle x, y\rangle$, have equal nilpotency class, where x_G and x_H denote the projection of x onto G and H, respectively. Since the nilpotency class of a direct product is the maximum of the nilpotency classes of its factors and because both $\langle x_G, y_G\rangle$ and $\langle x_H, y_H\rangle$ are quotient groups of $\langle x, y\rangle$, it follows that $\langle x, y\rangle$ has nilpotency class greater than or equal to $\langle x_G, y_G\rangle \times \langle x_H, y_H\rangle$. The opposite inequality follows since $\langle x, y\rangle$ is a subgroup of $\langle x_G, y_G\rangle \times \langle x_H, y_H\rangle$. ♣

Corollary 1 *For each non-negative integer m except for $m = 1$, there exists a sequence $\{G_n\}$ of groups such that $p_m(G_n)$ approaches 1 as n approaches infinity.*

PROOF: It is known [5] that p_1, the probability of two elements commuting, is either 1 or less than or equal to 5/8. For the other values of m, we will define a sequence of groups $\{G_n\}$ in which $G_n = G_{n-1} \times G_1$. First we consider the case $m = 0$. We let $G_1 = S_3$, the symmetric group on 3 symbols. It may be easily checked that $p_0(G_1) = 1/2 > 0$ and $p_1(G_1) = 1/2$. But then by Lemma 1, $p_1(G_n) = (1/2)^n$, which approaches zero, so $p_0(G_n)$ must approach 1. Now we consider the case where $m \geq 2$. We define G_1 to be the dihedral group on 2^m symbols. G_1 has nilpotency class m

and is 2-generated, so $p_m(G_i) > 0$ and $\sum_{i=1}^{m-1} p_i(G_1) < 1$. By the previous theorem, this summation is multiplicative, so

$$\lim_{n\to\infty} \sum_{i=1}^{m-1} p_i(G_n) = \lim n \to \infty (\sum_{i=1}^{m-1} p_i(G_1))^n = 0,$$

meaning again that $p_m(G_n)$ approaches 1. ♣

The following corollary shows that for all integers $m \geq 1$, some sequence of groups has p_m value approaching 0. This is not the case for $m = 0$, as the next section will show.

Corollary 2 *For all integers $m \geq 1$, there exists a sequence $\{G_n\}$ of groups such that $p_m(G_n) > 0$ for all n, but $\lim_{n\to\infty} p_m(G_n) = 0$.*

PROOF: Let G_1 be the dihedral group on 2^{m+1} symbols. Note that G_1 is two generated and has nilpotency class m, so $p_m(G_1) > 0$. As in the proof of Corollary 1, we will define a sequence of groups $\{G_n\}$ in which $G_n = G_{n-1} \times G_1$. Since G_1 contains a subgroup isomorphic to the dihedral group on 2^{m-1} symbols, all G_n contain such a subgroup, so $p_m(G_n) > 0$ for all n. By the proof of Corollary 1, we know that $p_{m+1}(G_n)$ approaches 1, so it follows that $\{G_n\}$ is a sequence of groups for which p_m approaches 0. ♣

3 A Lower Bound on p_0 for Non-nilpotent, Solvable Groups

This section will be entirely devoted to the proof of the following theorem.

Theorem 1 *If G is a solvable group, then either*

1. *G is nilpotent, in which case $p_0(G) = 0$, or*
2. *G is non-nilpotent, in which case $p_0(G) \geq (p_s - 1)/p_s$, where p_s is the smallest prime dividing $|G|$.*

Equality in this second case occurs if, and only if, $G/Z^{(n)}(G) \cong S_3$, *where* $Z^{(n)}(G)$ *is the largest subgroup in the ascending central series of* G, *and* S_3 *is the symmetric group on* 3 *symbols.*

The proof of this theorem is quite long, so first we will prove some more elementary results which will be used in the course of its proof. By using the various definitions of nilpotency, we may prove the inequality in Theorem 1 for a large class of non-nilpotent groups without any of the more intricate methods that we will call upon later in this section.

Lemma 2 *If* G *is non-nilpotent, and* p_s *is the smallest prime dividing* $|G|$, *then* $p_1(G) \leq 1/p_s$.

PROOF: We note that $p_1(G) \leq p_1(G/Z(G))$, since if two elements commute in G, their cosets commute in $G/Z(G)$. Thus it suffices to prove the lemma for groups with trivial center. Now by Erdös [2], we know that we may write $p_1(G) = k/|G|$, where k is the number of distinct conjugacy classes of G. In order to prove the lemma, we assume that $k/|G| > 1/p_s$ and derive a contradiction. The assumed inequality implies that $k \geq |G|/p_s + 1$, since p_s divides the order of G. But then we may use the class equation as follows ($cl(G)$ refers to the conjugacy classes of G):

$$\begin{aligned} |G| &= |Z(G)| + \sum_{cl(G)} \frac{|G|}{|C(x)|} \\ &\geq 1 + p_s(k-1) \\ &= 1 + |G|, \end{aligned}$$

a contradiction. ♣

Lemma 3 *If all Sylow subgroups of a group* G *are abelian, then* $p_i(G) = 0$ *for all* $i \geq 2$, *and either the group is abelian or* $p_0(G) \geq (p_s - 1)/p_s$, *where* p_s *is the smallest prime dividing the order of* G.

PROOF: We will show that in such a group G, either two elements commute or they generate a non-nilpotent subgroup. Combining this with Lemma 2 gives the desired result, because if $p_i(G) = 0$ for all $i \geq 2$, then $p_0(G) + p_1(G) = 1$.

Consider two elements $x, y \in G$ for which $\langle x, y \rangle$ is nilpotent. This means that $\langle x, y \rangle$ can be written as a direct product of its Sylow subgroups, each of which is a subgroup of a Sylow subgroup of G. Thus $\langle x, y \rangle$ can be written as a direct product of abelian groups, so that $\langle x, y \rangle$ is abelian, so x and y must commute. ♣

Corollary 3 *If $|G|$ is not divisible by the cube of any prime, then $p_0(G) \geq (p_s - 1)/p_s$.*

PROOF: If $|G|$ is cube-free, then all Sylow subgroups of G have order p or p^2. This means that they are all abelian, so we may use Lemma 3 again. ♣

To proceed further with our proof of Theorem 1, we first require several preliminary results on the value of p_0 in quotient groups, and on conditions for two elements to generate a nilpotent group.

Theorem 2 *The group $\langle x, y \rangle$ is nilpotent if, and only if, the following two conditions hold:*

1. *For any positive m, n, if x^m and y^n have relatively prime orders, then they commute.*
2. *For any positive m, n, if x^m and y^n have orders which are powers of the same prime p, then $\langle x^m, y^n \rangle$ is a p-group.*

PROOF: The forward implication is immediate, due to the fact that a nilpotent group is the direct product of its Sylow subgroups. To prove the converse, we will show that the two conditions imply that $\langle x, y \rangle = H$ is a direct product of its Sylow subgroups. Let $|x| = p_1^{a_1} \cdots p_k^{a_k}$ and $|y| = p_1^{b_1} \cdots p_k^{b_k}$, where some of the a_i's and b_i's may be zero. Then there exist $x_1, \ldots, x_k$ which are powers of x such that $|x_i| = p_i^{a_i}$ (we let $x_i = x^{|x|/p_i^{a_i}}$). Since $\gcd(|x|/p_1^{a_1}, \ldots, |x|/p_k^{a_k}) = 1$, we know that $\langle x \rangle = \langle x_1, \ldots, x_k \rangle$. Similarly, there exist $y_1, \ldots, y_k$ which are all powers of y such that $|y_i| = p_i^{b_i}$ and $\langle y \rangle = \langle y_1, \ldots, y_k \rangle$, so we may write $H = \langle x_1, \ldots, x_k, y_1, \ldots y_k \rangle$. Since x_i and x_j are both powers of x, they must commute for all i, j. Also, due to the first condition in the theorem, if $i \neq j$, then x_i

and y_j must commute, since they have relatively prime order. The second condition in the theorem implies that $\langle x_i, y_i \rangle$ is a p_i-group for all i, and since all other generators of H commute with both x_i and y_i, $\langle x_i, y_i \rangle$ is in fact the normal p_i-Sylow subgroup of H. We thus have k normal Sylow subgroups in H. But since all Sylow subgroups of H are normal, H must in fact be a direct product of its Sylow subgroups, and thus is nilpotent. This completes the proof of the theorem. ♣

This theorem allows us to prove Theorem 1 for nilpotent groups immediately.

Corollary 4 *G is nilpotent if, and only if, $p_0(G) = 0$.*

PROOF: If G is nilpotent, then all subgroups of G are nilpotent, so $p_0(G) = 0$. If G is non-nilpotent, then it is not the direct product of its Sylow subgroups. Therefore, there exist x and y in G of relatively prime order such that x and y do not commute. By Theorem 2, these generate a non-nilpotent group. ♣

Now we prove several important results dealing with how $p_0(G)$ is affected by taking quotients of G.

Lemma 4 *For any group G, $p_0(G) = p_0(G/Z(G))$.*

PROOF: It is clear from the commutator definition of nilpotency that if $\langle x, y \rangle$ is nilpotent, then so is $\langle z_1 x, z_2 y \rangle$, where z_1 and z_2 denote any two elements in $Z(G)$. Since cosets of $Z(G)$ all have the same order, to prove this lemma it suffices to show that $\langle x, y \rangle$ is nilpotent in G if, and only if, $\langle xZ(G), yZ(G) \rangle$ is nilpotent in $G/Z(G)$.

We assume that $\langle x, y \rangle$ is nilpotent in G. It is well known that homomorphisms preserve nilpotency. Thus $\langle xZ(G), yZ(G) \rangle$ is nilpotent in $G/Z(G)$. This reasoning also shows that $p_0(G) \geq p_0(G/N)$ for any $N \trianglelefteq G$.

Next we assume that $\langle x, y\rangle$ is non-nilpotent in G. We must show that $\langle xZ(G), yZ(G)\rangle$ is non-nilpotent in $G/Z(G)$. Since $H = \langle x, y, Z(G)\rangle$ has a non-nilpotent subgroup, it is clearly non-nilpotent. This easily implies that $H/Z(H)$ is non-nilpotent. By a basic isomorphism theorem and the fact that the center of H contains the center of G, $H/Z(H)$ is isomorphic to a quotient group of $H/Z(G)$. Therefore, $H/Z(G)$ cannot be nilpotent. So $\langle xZ(G), yZ(G)\rangle \cong H/Z(G)$ is also non-nilpotent, completing the proof. ♣

Theorem 3 *For any group* G, $p_0(G) = p_0(G/Z(G)) = p_0(G/Z^{(2)}(G)) = \ldots = p_0(G/Z^{(n)}(G))$.

PROOF: It suffices to prove that for any $i, p_0(G/Z^{(i)}(G)) = p_0(G/Z^{(i-1)}(G))$. Let H_i denote $G/Z^{(i-1)}$. It follows from the construction of the ascending central series that

$$Z^{(i)}(G)/Z^{(i-1)}(G) \cong Z(H_i).$$

By a fundamental isomorphism theorem,

$$G/Z^{(i)}(G) \cong (G/Z^{(i-1)}(G))/(Z^{(i)}(G)/Z^{(i-1)}(G)) \cong H_i/Z(H_i).$$

By Lemma 4, $p_0(H_i) = p_0(H_i/Z(H_i))$, so $p_0(G/Z^{(i)}(G)) = p_0(G/Z^{(i-1)}(G))$. ♣

Corollary 5 *If* $G/Z^{(n)} \cong S_3$ *then* $p_0(G) = \frac{1}{2}$.

PROOF: This follows immediately from Theorem 3 and the fact that $p_0(S_3) = 1/2$. ♣

Corollary 6 *If* N *is a normal subgroup of* G *and is contained in* $Z^{(n)}$, *then* $p_0(G) = p_0(G/N)$.

PROOF: As noted in the proof of Lemma 4, $p_0(G) \geq p_0(G/N)$. Since N is contained in $Z^{(n)}, G/Z^{(n)}$ is a quotient group of G/N, so that $p_0(G/N) \geq p_0(G/Z^{(n)}) = p_0(G)$. Combining the two inequalities yields the desired result. ♣

Lemma 5 *If G has trivial center, then $p_0(G) > p_0(G/N)$ for all non-trivial normal subgroups N.*

PROOF: Since $\langle x, y\rangle$ nilpotent in G implies $\langle xN, yN\rangle$ nilpotent in G/N, it suffices to show that some subgroup $\langle x, y\rangle$ is non-nilpotent and its image $\langle xN, yN\rangle$ is nilpotent in G/N. If N is non-nilpotent, we are done, because by Corollary 4, $\langle x, y\rangle$ is a non-nilpotent subgroup, and since both x and y map to the identity in G/N, $\langle xN, yN\rangle$ is nilpotent.

Now we consider the case in which N is nilpotent and $p_0(G) = p_0(G/N)$. First we show that we can assume N to be a p-group. N is the direct product of its Sylow subgroups $P_1 \times P_2 \cdots \times P_n$. Since N is normal in G, P_1 is normal in G. Since p_0 is non-increasing over quotients, $p_0(G) \geq p_0(G/P_1) \geq p_0((G/P_1)/(N/P_1)) = p_0(G/N) = p_0(G)$, so $p_0(G) = p_0(G/P_1)$. If N is not a p-group, we replace N by P_1. Thus we may assume that N is a p-group. It suffices to show that some element in N generates a non-nilpotent with some element outside of N, because the image of the element in N gets mapped to the identity in G/N. Suppose instead that every pair of elements containing at least one element of N generates a nilpotent group. By Theorem 5.8 in Rose [6], some element n in N must be in the center of a p-Sylow subgroup of G. Also, by Theorem 2, n must commute with all elements of order relatively prime to p, since it generates a nilpotent group with all such elements. Writing G as a product (not necessarily direct) of its Sylow subgroups , we see that n commutes with all of G, contradicting $Z(G) = e$. ♣

We wish to prove that there is no non-nilpotent group G for which $p_0(G) < (p_s - 1)/p_s$, so we will instead assume that at least one such G exists and derive a contradiction. If there are any such G, there is one of minimal order. The rest of this section will be devoted to proving that no such minimally-ordered group exists. First we prove a result about the minimal G which enables us to examine its structure.

Lemma 6 *If G is the minimal order non-nilpotent solvable group for which $p_0(G) < (p_s - 1)/p_s$, then all proper quotients of G are nilpotent.*

PROOF: Suppose instead that G has a non-trivial normal subgroup N such that G/N is non-nilpotent. Then G/N is solvable and non-nilpotent, the smallest prime dividing $|G/N|$, p'_s, is at least as large as the smallest prime dividing $|G|$, and $p_0(G/N) \leq p_0(G) < (p_s - 1)/p_s \leq (p'_s - 1)/p'_s$, contradicting the minimality of the order of G. Thus all proper quotients of G must be nilpotent.

♣

Next we define a term from a paper by Franciosi and de Giovanni [3] which, along with a result from that same paper, will be of great use to us in our proof.

Definition 1 *A solvable, non-nilpotent group G is just-non-nilpotent (JNN) if all the proper quotients of G are nilpotent.*

We note that all JNN groups must have trivial center, since otherwise $G/Z(G)$ is a proper non-nilpotent quotient.

Theorem 4 (S. Franciosi and F. de Giovanni [3]) *A finite group G is JNN if, and only if, $G = L \propto A$ (by which we mean the semi-direct product of A by L), where A is an elementary abelian p-group, L is a finite nilpotent group whose order is not divisible by p, and the action of L on A is faithful and irreducible.*

Thus we see that the minimal group G which is a counterexample to Theorem 1 can be written as a semidirect product of a nilpotent group L and an elementary abelian group A. We note that since L is nilpotent, we may write $L \cong P_1 \times \cdots \times P_k$, where the P_i's are the unique p_i-Sylow subgroups of L. But then we may write $G = L \propto A$ as $G = P_k \propto (P_{k-1} \propto \cdots \propto (P_1 \propto A))$. We will take advantage of this way of expressing G later on in the course of this paper.

Due to Theorem 2 and the above comment, we see that the number of p-Sylow subgroups containing a given element in G will play an important role in the proof of Theorem 1. We introduce the following notation. Given $\{x_1, \ldots, x_k\}$, we define $M_P(x_1, \ldots, x_k)$ as the number of p-Sylow subgroups containing the entire set.

Lemma 7 *If x and y are in a common p-Sylow subgroup of $P \propto N$, where P is a p-group and p does not divide $|N|$, then*

$$M_P(x, y) = \frac{|C(x) \cap C(y) \cap N|}{|C(P) \cap N|}.$$

PROOF: We may assume without loss of generality that $x, y \in P$, since the group may be written as the semi-direct product of any of its p-Sylow subgroups with N. Since $G = PN$, we may write any other p-Sylow subgroup as $P' = (x_p x_n)^{-1} P (x_p x_n) = x_n^{-1}(x_p^{-1} P x_p) x_n = x_n^{-1} P x_n$, where $x_n \in N$, so all p-Sylow subgroups are conjugate to P, and thus to each other, by elements in N. Now each p-Sylow subgroup contains exactly one element from each coset of N, and conjugation by an element of N preserves cosets of N, so conjugating by $z_n \in N$ will yield a coset containing x and y if, and only if, $z_n \in C(x) \cap C(y) \cap N$. For the same reasons, conjugation by z_n fixes P if, and only if, n commutes with all of P, so we must divide by $|C(P) \cap N|$. This completes the proof. ♣

Corollary 7 *For a group of the form $P \propto N$, as in Lemma 7, $M_P(x) = |C(x) \cap N|/|C(P) \cap N|$ and $M_P(e) = |N|/|C(P) \cap N|$.*

PROOF: We may set $y = e$ in Lemma 7 to find that

$$M_P(x) = \frac{|C(x) \cap N|}{|C(P) \cap N|},$$

since $M_P(x) = M_P(x, e)$. Similarly, we may also set $x = e$ to find that

$$M_P(e) = \frac{|N|}{|C(P) \cap N|}.$$

Note that $M_P(e)$ is just the number of p-Sylow subgroups in the group. Hereafter, we will denote this number by M_P. ♣

Lemma 8 *If x and y are in p-Sylow subgroups of $P \propto N$ and in the same coset of N, then $M_P(x) = M_P(y)$.*

PROOF: Since all p-Sylow subgroups are conjugate by an element in N, and conjugation by N preserves cosets of N, there is a group automorphism (conjugation by some element of N) that sends x to y. ♣

Corollary 8 *If x is in a p-Sylow subgroup of $P \propto N$, then $M_P(x)$ divides the number of p-Sylow subgroups of $P \propto N$.*

PROOF: This follows immediately by taking the ratio $M_P/M_P(x) = |N|/|C(X) \cap N|$. ♣

It is worth noting at this point that our calculations with CAYLEY lead us to believe that Corollary 8 is actually true for any group, not just those of the particular form $P \propto N$.

Lemma 9 *If $x \in G - N$, where $G = P \propto N$, then x has order divisible by p.*

PROOF: Assume to the contrary that p does not divide the order of x. Then $x^{|N|} = e$. Thus the coset xN has order a divisor of N in G/N. This is impossible, since G/N is a p-group, and N has order relatively prime to p. ♣

Lemma 10 *If $\langle x, y \rangle$ is nilpotent in $G = P \propto N$, then there exists a p-Sylow subgroup $P_{x,y} \in G$ and unique elements x_p, y_p, x_N, y_N such that*

1. $x = x_p x_N$, $y = y_p y_N$,
2. $\langle x \rangle = \langle x_p, x_N \rangle$, $\langle y \rangle = \langle y_p, y_N \rangle$,

3. $x_p, y_p \in P_{x,y}$,

4. $x_N, y_N \in C(x_p) \cap C(y_p) \cap N$, *and*

5. $\langle x_N, y_N \rangle$ *is nilpotent.*

Note that this means that $x_p \equiv x \pmod{N}$ and $y_p \equiv y \pmod{N}$.

PROOF: We pick $x_p = x^{h_1 N}$ and $x_N = x^{h_2 p^k}$, where $|P| = p^k$, and assign h_1 and h_2 by the equation

$$h_1 N + h_2 p^k \equiv 1 \pmod{p^k N}.$$

By the Chinese Remainder Theorem, this equation has a solution (mod $p^k N$), since p^k and N are relatively prime. That solution is in fact unique in the context of the group, because if

$$h_1' N + h_2' p^k \equiv 1 \pmod{p^k N},$$

then we may subtract the two equivalences and get

$$(h_1' - h_1)N + (h_2' - h_2)p^k \equiv 0 \pmod{p^k N}.$$

But then $(h_1' - h_1)$ must be divisible by p^k, so $x^{h_1 N} = x^{h_1' N}$, and similarly for h_2. Thus we have $x_p x_N = x^{h_1 N + h_2 p^k} = x$, since $|x|$ is a divisor of $p^k N$. We choose y_p and y_N in a similar fashion, so part (1) is satisfied. Clearly $\langle x \rangle = \langle x_p, x_N \rangle$ and $\langle y \rangle = \langle y_p, y_N \rangle$, satisfying part (2), and we may write $\langle x, y \rangle = \langle x_p, y_p, x_N, y_N \rangle$. Now since $|x_p|$ and $|y_p|$ are both powers of p and $\langle x, y \rangle$ is nilpotent, Theorem 2 implies that $\langle x_p, y_p \rangle$ is a p-group, so there is some p-Sylow subgroup $P_{x,y}$ which contains them both. This proves part (3). Again using Theorem 2, we note that:

- Since x_p and y_N have relatively prime order, they must commute, so $y_N \in C(x_p)$.
- Since y_p and y_N are both powers of y, they must commute, so $y_N \in C(y_p)$.
- Since the order of y_N is relatively prime to p, by Lemma 9, $y_N \in N$.

The same argument about x_N completes the proof of part (4). Finally, since $\langle x_p, y_p, x_N, y_N \rangle$ is nilpotent, its subgroup $\langle x_N, y_N \rangle$ is nilpotent as well, so part (5) is also satisfied. Now we show that the elements x_p, y_p, x_N, y_N are the only ones which satisfy all the conditions of the lemma. By part (2), x_p and x_N had to be powers of x, and by the way we chose h_1 and h_2 in the proof of part (1), we know that there is a unique way to write x as a product of two powers of x, one with order divisible by p and the other with order dividing $|N|$. Likewise, the selection of y_p, y_N is unique. ♣

We are now able to prove Theorem 1 for non-nilpotent groups.

PROOF OF THEOREM 1: We will show that if G is a JNN group, then $p_0(G) \geq (p_s - 1)/p_s$, with equality only if $G \cong S_3$. This is sufficient to prove the theorem, because we have seen that the counterexample of minimal order is a JNN group. To do this, we write $G = P_k \propto (P_{k-1} \propto \cdots \propto (P_1 \propto A))$, with A an elementary abelian q-group and P_k the unique p_k-Sylow subgroup of L. We now induct on k. We will show that if $N = P_{i-1} \propto \cdots \propto (P_1 \propto A)$ and $p_0(N) \geq (p_s - 1)/p_s$, then $p_0(P_i \propto N) \geq (p_s - 1)/p_s$. After that, we will show that $p_0(P_1 \propto A) \geq (p_s - 1)/p_s$. These two clearly suffice to prove the inequality for all JNN groups. After that, we will show what the equality conditions must be.

First we look at $P_i \propto N$, and consider how to count the number of pairs (x, y) such that x is in one fixed coset of N, y is in another fixed coset of N, and $\langle x, y \rangle$ is nilpotent. We will refer throughout this part of the proof to p_i as p. First we fix a p-Sylow subgroup P of $P_i \propto N$ and ask how many ordered pairs of elements (x, y) are in the fixed ordered pair of cosets $(x_p N, y_p N)$, with $x_p, y_p \in P$ such that we may represent $x = x_p x_N$ and $y = y_p y_N$ with all of the conditions in Lemma 10 holding for x_N, y_N (x_p and y_p are fixed). We denote this number by $N_0(x_p, y_p)$. We may find an upper bound for $N_0(x_p, y_p)$ by noting that x_N and y_N both satisfy condition (4) of Lemma 10, so there are no more than $|C(x_p) \cap C(y_p) \cap N|$ choices for x_N, and likewise for y_N. Thus the total number of pairs can be bounded as $N_0(x_p, y_p) \leq |C(x_p) \cap C(y_p) \cap N|^2$. We note that this

need not be an exact count of the number of nilpotent pairs, just an upper bound, since we have not included the condition that $\langle x_N, y_N \rangle$ is nilpotent.

Any other two elements x'_p, y'_p which are in some other p-Sylow subgroup P' and the same cosets of N as x_p, y_p, respectively, satisfy $N_0(x'_p, y'_p) = N_0(x_p, y_p)$, since there is a conjugation (which is an automorphism of the group) which sends x_p, y_p to x'_p, y'_p. The number of such x'_p, y'_p is just equal to the number of distinct p-Sylow subgroups in the group divided by the number which contain both x_p and y_p, that is, $M_P/M_P(x_p, y_p)$. But every pair of elements (x, y) with $x \in x_p N$ and $y \in y_p N$ and $\langle x, y \rangle$ nilpotent must yield exactly one of the x'_p, y'_p's, due to Lemma 10, so the total number of nilpotent pairs (x, y) with $x \in x_p N$, $y \in y_p N$ (denoted $N_T(x_p, y_p)$) can be expressed as follows:

$$\begin{aligned} N_T(x_p, y_p) &= N_0(x_p, y_p) \left(\frac{M_P}{M_P(x_p, y_p)} \right) \\ &\leq |C(x_p) \cap C(y_p) \cap N|^2 \left(\frac{|N|}{|C(P) \cap N|} \right) / \left(\frac{|C(x_p) \cap C(y_p) \cap N|}{|C(P) \cap N|)} \right) \\ &= |C(x_p) \cap C(y_p) \cap N||N|. \end{aligned}$$

But the total number of pairs (x, y) with x and y in the appropriate cosets is just $|N|^2$, so the probability that a pair (x, y) chosen from the coset pair $(x_p N, y_p N)$ generates a nilpotent subgroup is bounded by $|C(x_p) \cap C(y_p) \cap N|/|N|$. By Theorem 4, the action of P on A (which is a subgroup of N) is faithful, so unless both x_p and y_p are the identity, either x_p or y_p (or both) commutes with no more than $\frac{1}{q}$ of the elements A. This in turn means that at least one of x_p or y_p commutes with no more than $\frac{1}{q}$ of the elements of N. Thus unless both x_p and y_p are the identity, the probability that a pair of elements (x, y), chosen from the cosets $x_p N, y_p N$ respectively, generates a nilpotent group is bounded by $\frac{1}{q} \leq \frac{1}{p_s}$, as desired. But if the probability that two elements both chosen from N generate a nilpotent group is also less than or equal to $\frac{1}{p_s}$, then the probability that two elements generate a nilpotent group is less than or equal to $\frac{1}{p_s}$ for any coset pair. Thus given that $p_0(N) \geq (p_s - 1)/p_s$, we have shown that $p_0(P_i \propto N) \geq (p_s - 1)/p_s$, and the induction step is

complete.

Now we proceed with the base case of the induction. We need to show that $p_0(P \propto A) \geq (p_s - 1)/p_s$ for $A = (\mathbf{Z}_q)^n$ and P a p-Sylow subgroup, $p \neq q$. Using the argument made in the induction step, we know that if the two elements in a pair are not both in A, then the probability that the pair generates a nilpotent subgroup is less than or equal to $\frac{1}{q}$. The probability that two elements chosen at random from the group generate a non-nilpotent group is thus at least $\frac{p^{2m}-1}{p^{2m}}\frac{q-1}{q}$. We consider two cases, remembering that the choice of which Sylow subgroup of L would serve as P_1 was arbitrary, since L was just the direct product of the P_i's.

- If q is not the largest prime which divides $|G|$, then we choose some Sylow subgroup P of L, where $p > q$, and act first with it. Let $|P| = p^m$. We will first show that not all of the values of $|C(x_p) \cap C(y_p) \cap A|$ that were used in the induction proof are actually equal to q^{n-1}. Suppose instead that they were. This implies that $C(x_p) \cap A$ and $C(y_p) \cap A$ have order q^{n-1} for any choice of $x_p, y_p \in P$ (they cannot have order q^n, because then the action of P on A would not be faithful) But for any x_p not equal to the identity, $|C(x_p) \cap A| \leq q^{n-1}$, since P acts faithfully on A. Thus every element in P must commute with exactly the same q^{n-1} elements in A, so $|C(P) \cap A| = q^{n-1}$. But then the number of p-Sylow subgroups of $P \propto A$ is equal to $|A|/|C(P) \cap A| = q$. Since no non-identity element of P is in all of the p-Sylow subgroups (since the action is faithful, no non-identity element commutes with all of A), and the number of Sylow p-groups an element is in must divide the total number of p-Sylow subgroups (due to Lemma 8), they must all be in exactly one p-Sylow subgroup, namely P. Thus the total number of elements in p-Sylow subgroups is just $q(p^m - 1) + 1 = qp^m - q + 1$. By Frobenius [4], this number must be divisible by p^m, so $q \equiv 1 \pmod{p^m}$. This is impossible, however, since $q < p$, so not all of the $|C(x_p) \cap C(y_p) \cap A|$ are equal to q^{n-1}.

Now if $|C(x_p) \cap C(y_p) \cap A| \leq q^{n-2}$, then there are at least $p-1$ elements of P, namely $y_p, y_p^2, \ldots, y_p^{p-1}$, all of which are in different cosets of N and whose centralizers intersect $C(x_p) \cap A$ in no more than q^{n-2} elements. We will show that this is in fact enough to get the total probability over $\frac{q-1}{q}$. Given this set of $2(p-1)$ ordered pairs in P with sufficiently small centralizer intersections (x_p can be either the first or last element in the pair, so there is a 2 in the expression), the probability that two elements in $P \propto A$ generate a non-nilpotent group can be bounded as follows:

$$\begin{aligned} p_0(P \propto A) &\geq \left(\frac{p^{2m}-2p+1}{p^{2m}}\right)\left(\frac{q-1}{q}\right) + \left(\frac{2p-2}{p^{2m}}\right)\left(\frac{q^2-1}{q^2}\right) \\ &= \left(\frac{q-1}{q}\right)\left(\frac{q(p^{2m}-2p+1)+(q+1)(2p-2)}{qp^{2m}}\right) \\ &= \left(\frac{q-1}{q}\right)\left(\frac{qp^{2m}-2pq+q+2pq-2q+2p-2}{qp^{2m}}\right) \\ &= \left(\frac{q-1}{q}\right)\left(\frac{qp^{2m}-q+2p-2}{qp^{2m}}\right) \\ &> \frac{q-1}{q}, \end{aligned}$$

as desired. We note that equality cannot hold for this case, since $p > q \geq 2$ implies that $2p > q+2$.

- The only case left to consider is if q is the largest prime dividing the order of G. In such a case, we act first with the Sylow subgroup of L corresponding to the largest prime which divides L, which we call p. Note that $p < q$. But then $q \geq p+1$, so $(q-1)/q \geq p/(p+1)$. In such case we can write

$$\begin{aligned} p_0(P \propto A) &\geq \left(\frac{p^{2m}-1}{p^{2m}}\right)\left(\frac{p}{p+1}\right) \\ &= \frac{p(p^2-1)(p^{2m-2}+\ldots+1)}{p^k(p+1)} \\ &\geq \frac{p^{2m-1}(p^2-1)}{p^{2m}(p+1)} \text{ (equality only if } m=1) \end{aligned}$$

$$= \frac{p-1}{p}$$
$$\geq \frac{p_s - 1}{p_s}.$$

As a result, we see that we have equality only if $m = 1$ and $q = p + 1$, i.e., if $p^m = 2$ and $q = 3$. But since p was the largest prime dividing $|L|$, this means that for equality to occur, $L \cong \mathbf{Z}_2$ and $A \cong (\mathbf{Z}_3)^n$.

Thus the base case of our induction is complete, and so is our proof that, for all solvable groups G, $p_0(G) \geq (p_s - 1)/p_s$.

Now we prove the equality condition of Theorem 1. From our analysis of the base case of the induction, we know that the only way that $p_0(G)$ can actually equal $(p_s - 1)/p_s$ (for a JNN group G) is if $G \cong \mathbf{Z}_2 \propto (\mathbf{Z}_3)^n$. If G is of such type, then all Sylow subgroups of G are abelian, so by Lemma 3 the only way for $p_0(G) = (p_s - 1)/p_s = \frac{1}{2}$ is if $p_1(G) = \frac{1}{2}$ as well. But it is known [7] that the only groups in which the probability of two elements commuting is exactly one half are those H such that $H/Z(H) \cong S_3$, the symmetric group on 3 symbols. Therefore, the only JNN group G for which $p_0(G) = (p_s - 1)/p_s$ is $G \cong S_3$. As an immediate result, if a group G is solvable (but not JNN), $p_0(G) = (p_s - 1)/p_s$ only if S_3 is a quotient group of G, and $p_0(G) = p_0(S_3)$. By Lemma 6 and Corollary 5, this requires that $G/N \cong S_3$, where $N \subseteq Z^{(n)}(G)$. If N is not equal to $Z^{(n)}(G)$, then $G/Z^{(n)}(G)$ must be a proper quotient group of S_3. But all proper quotients of S_3 are abelian, which contradicts the fact that G must be non-nilpotent, so $N \cong Z^{(n)}(G)$. Thus $p_0(G) = (p_s - 1)/p_s$ for a solvable group G if, and only if, $G/Z^{(n)}(G) \cong S_3$, and the proof of Theorem 1 is complete. ♣

4 Divisibility by $|G|$

We will now show that the number of n-tuples which generate a subgroup of nilpotency class i is a multiple of the order of the group for all n and i. The case $n = 2$ corresponds to the values

considered thus far in the paper. We inductively define the concept of a commutator of length i by saying that a commutator of the form $[x, y]$ has length 2, while a commutator of length i is of the form $[x, C_{i-1}]$, where C_{i-1} is a commutator of length $i-1$. Thus a commutator of length 3 all of whose elements are x_k's is of the form $[x_{k_3}, [x_{k_2}, x_{k_1}]]$. We begin with a result which characterizes the nilpotency class of groups with a fixed number of generators.

Lemma 11 *The group $G = \langle x_1, \ldots, x_n \rangle$ is nilpotent of class less than or equal to i if, and only if, all commutators of length $i+1$ with only the x_k's as entries are equal to the identity.*

PROOF: Assume that G is nilpotent of class less than, or equal to, i. By the commutator definition of nilpotency, $G^{(i)} = [G^{(i-1)}, G] = e$, so in particular the commutators of length $i+1$ with x_k's as entries must equal the identity.

For the converse, we proceed by induction on i. The case $i = 1$ is trivial. Now suppose that all commutators of length $i+1$ with the x_k's as entries equal the identity. Then all commutators of length i are contained in $Z(G)$. Thus in $G/Z(G)$, all commutators of length i are trivial. By the induction hypothesis, $G/Z(G)$ has nilpotency class less than or equal to $i-1$, and since G has always has nilpotency class exactly one greater than $G/Z(G)$, the theorem follows. ♣

Theorem 5 *The number of ordered n-tuples, $(x_1, x_2, \ldots, x_n)$, such that $\langle x_1, \ldots, x_n \rangle$ has nilpotency class i is a multiple of $|G|$, for all $i \geq 1$.*

PROOF: In this argument, all the entries in the commutators are the x_k's. It clearly suffices to show that the number of n-tuples generating a subgroup of nilpotency class less than or equal to k is a multiple of $|G|$. By Lemma 11, being nilpotent of class less than or equal to i is equivalent to having all commutators of length $i+1$ equal the identity. Now we define a sequence $\mathcal{C} = \{c_k\}$ of commutators (of the x_k's) and all the x_k's except for x_n. This sequence begins with all commutators

of length i, is followed by all commutators of length $i-1$, and so on until commutators of length two are reached which are followed by the x_k's. For example, if $i = 2$ and $n = 2$, then the sequence would be

$$\mathcal{C} = \{[x_1, x_1], [x_1, x_2], [x_2, x_1], [x_2, x_2], x_1\}.$$

For simplicity of notation, let K denote the intersection of the centralizers of all elements of $\mathcal{C}$.

Let us say that x_n "works" with $\mathcal{C}$ if $x_1, \ldots, x_n$ yield $\mathcal{C}$ and if all commutators of the x_k's of length $i+1$ are the identity. Let $N(\mathcal{C})$ denote the number of x_n working with $\mathcal{C}$. We claim that $N(\mathcal{C})$ is either 0 or $|K|$. To prove this, we show that if s works with $\mathcal{C}$ then t works with $\mathcal{C}$ if, and only if, $t^{-1}s$ is in K. First, let t be some other element of the group which works with $\mathcal{C}$. Since $[t, c_k] = [s, c_k]$, $t^{-1}s \in C(c_k)$. Since this is true for all c_k's, $t^{-1}s$ must be in K. The converse is immediate using the same reasoning.

Now let $g^{-1}\mathcal{C}g$ denote the sequence obtained by conjugating each component of $\mathcal{C}$ by g. Observe that $g^{-1}x_n g$ works with $g^{-1}\mathcal{C}g$ if, and only if, x_n works with X. Thus, $N(\mathcal{C}) = N(g^{-1}\mathcal{C}g)$ for any $g \in G$. It is easy to see that the number of distinct sequences obtained by conjugating $\mathcal{C}$ by an element in G is $|G|/|K|$. Thus,

$$\frac{|G|}{|K|} N(\mathcal{C}) = \sum_{g^{-1}\mathcal{C}g} N(g^{-1}\mathcal{C}g) = |G| \text{ or } 0.$$

Thus the sum over all possible $\mathcal{C}$ can be expressed as

$$\sum \sum_{g^{-1}\mathcal{C}g} N(g^{-1}\mathcal{C}g) = \sum |G|$$

which is also a multiple of $|G|$. This completes the proof. ♣

Corollary 9 *The number of n-tuples,* $(x_1, x_2, \ldots, x_n)$, *that generate a non-nilpotent subgroup is also a multiple of* $|G|$.

PROOF: Since the number of n-tuples which generate a nilpotent group must be multiples of the order of the group and the total number of n-tuples is $|G|^n$, the number of n-tuples which generate a non-nilpotent group must also be a multiple of $|G|$. ♣

Corollary 10 *The number of pairs that generate a non-nilpotent subgroup is a multiple of* $|G||Z^{(n)}|$.

PROOF: By Corollary 9, the number of pairs that generate a non-nilpotent subgroup in $G/Z^{(n)}$ is a multiple of $|G|/|Z^{(n)}|$. By Theorem 3, $p_0(G) = p_0(G/Z^{(n)})$, so the number of pairs in G is a multiple of $|Z^{(n)}|^2|G|/|Z^{(n)}| = |G||Z^{(n)}|$. ♣

5 Solvable Pairs

Another natural extension of the study of commuting pairs is the study of pairs producing groups that are solvable of length i. As with $p_i(G)$ in the preceeding sections, we define $s_i(G)$ to be the probability that $\langle x, y\rangle$ is solvable of length i for a randomly chosen ordered pair of elements from G. A non-solvable group is considered to be solvable of length 0.

In this section we prove various facts about the s_i's that correspond to results proved earlier in the paper about the p_i's.

Lemma 12 *If* $G = \langle x_1, x_2, \ldots, x_n\rangle$ *then* G' *is generated by commutators of finite length, where the entries are the* x_i*'s and their inverses.*

PROOF: An example illustrates the truth of this lemma. Consider the commutator $[x_1x_2, x_3]$. We have:

$$\begin{aligned}(x_1x_2)(x_3)(x_2^{-1}x_1^{-1})(x_3^{-1}) &= x_1x_2[x_3, x_2^{-1}]x_2^{-1}x_3x_1^{-1}x_3^{-1}\\ &= x_1x_2[x_3, x_2^{-1}]x_2^{-1}[x_3, x_1^{-1}]x_1^{-1}\end{aligned}$$

$$= \quad x_1[x_2,[x_3,x_2^{-1}]][x_3,x_2^{-1}][x_3,x_1^{-1}]x_1^{-1}.$$

Clearly the x_1 can be moved over in a similar way, and we are done. ♣

Theorem 6 *The number of n-tuples,* $(x_1, x_2, \ldots, x_n)$, *that generate a subgroup of solvable class* 2 *is a multiple of* $|G|$.

PROOF: In this argument, all the entries in the commutators will be the x_k's and their inverses. By the Lemma 13, given an n-tuple $(x_1, \ldots, x_n)$, there exists some smallest h, in general dependent upon the x_k's, such that only the commutators of length less than or equal to h are required to generate G'. Let i be the maximum value of h over all choices of $(x_1, \ldots, x_n)$. As in the previous section, we define a sequence of commutators and elements, $\mathcal{C} = \{c_k\}$. This sequence begins with all commutators of length i, followed by all commutators of length $i-1$, and so on until commutators of length two are reached, after which appear all of the x_k's except x_n. Note that $\langle x_1, \ldots, x_n \rangle$ is solvable of length 2 if, and only if, all of the commutators in $\mathcal{C}$ commute.

For simplicity of notation, let K_i denote the intersection of the centralizers of all of the elements in $\mathcal{C}$, and let K_{i-1} denote the intersection of the centralizers of all members in the sequence except for the commutators of length i. We note that K_i is a subgroup of K_{i-1}, so $|K_i|$ divides $|K_{i-1}$. As before, we say that x_n "works" with $\mathcal{C}$ if $x_1, \ldots, x_n$ give rise to $\mathcal{C}$, and we let $N(\mathcal{C})$ denote the number of x_n working with $\mathcal{C}$. We claim that $N(\mathcal{C})$ is either 0 or $|K_{i-1}|$. To prove this, we show that if s works with $\mathcal{C}$ then t works with $\mathcal{C}$ if, and only if, $t^{-1}s$ is in K_{i-1}. First, let t work with $\mathcal{C}$. Since $[t, c_k] = [s, c_k]$ for all $c_k \in \mathcal{C}$ except for the c_k of length i, $t^{-1}s$ is in K_{i-1}. Conversely, note that if $t^{-1}s$ is in K_{i-1}, all commutators will have the same value. Since $N(\mathcal{C})$ is always either $|K_{i-1}|$ or zero, it must always be a multiple of $|K_i|$. We say that $N(\mathcal{C}) = k_C|K_i|$.

Now let $g^{-1}\mathcal{C}g$ denote the sequence obtained by conjugating each component of $\mathcal{C}$ by g. Observe

that $g^{-1}x_n g$ works with $g^{-1}\mathcal{C}g$ if, and only if, x_n works with $\mathcal{C}$. Thus, $N(\mathcal{C}) = N(g^{-1}\mathcal{C}g)$ for any $g \in G$. It is easy to see that the number of distinct sequences obtained by conjugating $\mathcal{C}$ by an element in G is exactly equal to $|G|/|K_i|$. Thus,

$$\sum_{g^{-1}\mathcal{C}g} N(g^{-1}\mathcal{C}g) = \frac{|G|}{|K_i|} k_C |K_i| = k_C |G|.$$

Thus if we add over all possible $\mathcal{C}$,

$$\sum \sum_{g^{-1}\mathcal{C}g} N(g^{-1}\mathcal{C}g) = \sum k_C |G|$$

is also a multiple of $|G|$. This completes the proof. ♣

We note that the behavior of s_0 under quotients is simpler to express than it was for p_0.

Lemma 13 *$s_0(G) = s_0(G/S)$, where S is any solvable, normal subgroup of G.*

PROOF: Since cosets of S have the same size, it suffices to show $\langle x, y\rangle$ is solvable in G if, and only if, $\langle xS, yS\rangle$ is solvable in G/S. Assume that $\langle x, y\rangle$ is solvable in G. Since homomorphisms preserve solvability, $\langle xS, yS\rangle$ is solvable in G/S. Now suppose, suppose that $\langle x, y\rangle$ is not solvable in G. Since all subgroups of a solvable group are solvable, $\langle x, y, S\rangle$ cannot be solvable in G. It is known that if N and G/N are solvable groups, then so is G, so if S is solvable, $\langle x, y, S\rangle/S = \langle xS, yS\rangle$ cannot also be solvable. This completes the proof. ♣

We also conjecture the existence of a "5/8-like" bound for solvability.

Conjecture 1 *If G is not solvable, then $s_0 \geq 19/30$, with equality if, and only if, the process of taking successive derived groups eventually reaches a group isomorphic to A_5.*

We note that $s_0(G) \leq p_0(G)$, since if two elements generate a non-solvable group, they generate a non-nilpotent group as well. Since all non-solvable groups have even order, this conjecture implies that Theorem 1 is true for non-solvable groups as well, because $p_0(G) \geq s_0(G) \geq 19/30 > 1/2 = (p_s - 1)/p_s$.

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