8-18-1992

The 2/3 Bound for Rewritable n-tuples

Lawren Smithline
Harvard University

Catherine Sugar
Pomona College

Advisors:
Gary Sherman

MSTR 92-07

THE 2/3 BOUND FOR REWRITABLE n-TUPLES

L.M. Smithline and C.A. Sugar

MS TR 92-07

August 1992

Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute, IN 47803

FAX(812) 877-3198

Phone: (812) 877-8391

The 2/3 bound for rewritable n-tuples

Lawren M. Smithline and Catherine A. Sugar *

August 18, 1992

Let $\mathcal{X}$ be an n-tuple of finite group G and $r(\mathcal{X})$ be the number of rewritings of $\mathcal{X}$ (see [1]), including the trivial rewriting. Let $R(\mathcal{X}) = \dfrac{r(\mathcal{X})}{n!}$.

Theorem 1 $R(\mathcal{X}) \notin (\frac{2}{3}, 1)$.

Proof. For $n = 1$, $R(\mathcal{X}) = 1$.

Now suppose $n > 1$. Suppose for $(n-1)$-tuples $\mathcal{Y}$ of G, $R(\mathcal{Y}) \notin (\frac{2}{3}, 1)$. Consider a word $\mathcal{X}$ of length n, with $R(\mathcal{X}) > \frac{2}{3}$. Then for some letter a of $\mathcal{X}$, more than $\frac{2}{3}$ of the reorderings of $\mathcal{X}$ which put a in the first position are rewritings of $\mathcal{X}$. For convenience, let $\mathcal{X}^a$ be the word of length $n-1$ which is equivalent to $\mathcal{X}$, except with a deleted. By the inductive assumption, all of the letters of $\mathcal{X}^a$ are pairwise commutative. We now have two cases:

1. There is a letter b in $\mathcal{X}^a$ which commutes with a:
   In this case, b commutes with all letters of $\mathcal{X}$, so to rewrite $\mathcal{X}$, we can rewrite $\mathcal{X}^b$ and then insert b in any position. So we have reduced rewriting $\mathcal{X}$ to the case of a word of length $n-1$.

2. There is no b in $\mathcal{X}^a$ which commutes with a:
   Consider any permutation $\mathcal{P}$ of the letters of $\mathcal{X}^a$. Now create a list of reorderings of $\mathcal{X}$ by inserting a in successive positions in $\mathcal{P}$. Consecutive items in this list cannot be equal, since a does not commute with any of the letters of $\mathcal{P}$. The highest fraction of these reorderings which can be rewritings is $\dfrac{\left\lceil \frac{n}{2} \right\rceil}{n} \leq \frac{2}{3}$, so $R(\mathcal{X}) \leq \frac{2}{3}$. ♠

*Both authors supported by NSF Grant DMS-910059

Theorem 2 *If no n-tuple of a group G has exactly $q \cdot n!$ rewritings (counting itself) implies that G has some property P, then if no $(n+j)$-tuple of a group G has exactly $q \cdot (n+j)!$ rewritings, G has property P.*

Proof. Suppose no $(n+j)$-tuple of a group has $q \cdot (n+j)!$ rewritings. Then no $(n+j)$-tuple, with the last j elements equal to e has $q \cdot (n+j)!$ rewritings, so no n-tuple has $q \cdot n!$ rewritings. Hence G has property P. ♠

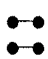
Conjecture *For any n-tuple $\mathcal{X}$ in G, if $R(\mathcal{X}) > \frac{1}{2}$, then $R(\mathcal{X}) = \frac{u+1}{2u+1}$ for some nonnegative integer u.*

Theorem 3 *For $\mathcal{X}$ a quadruple in group G, $r(\mathcal{X}) \notin \{13, 14, 15\}$.*

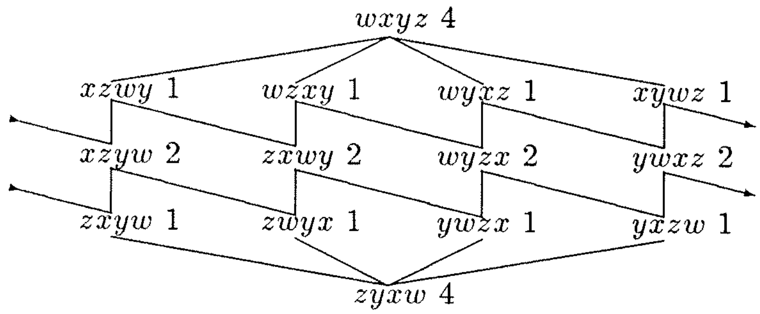
Proof. Let $\mathcal{X} = (w, x, y, z)$. If any of w, x, y, z commutes with all of the others, we can rewrite the word of the three others and insert the fourth at any position. So $4 | r(\mathcal{X})$, hence $r(\mathcal{X}) \notin \{13, 14, 15\}$.

Suppose one of w, x, y, z commutes with none of the others. The argument from case 2 of theorem 1 applies: $r(\mathcal{X}) < \frac{4!}{2}$.

So we are left with several cases which can be related to the isomorphism classes of connected graphs on four vertices whose complements are connected. We correspond each vertex to a letter, w, x, y, z, and and edge between letters indicates that they commute. There are three classes.

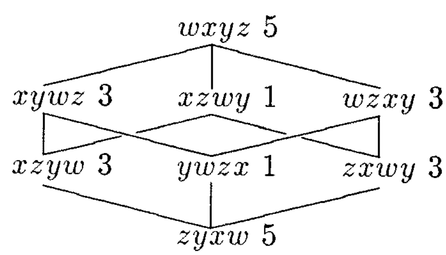
1.  WOLOG, let $[w, x] = [y, z]$.
   We can construct a graph where each vertex corresponds to a set of necessarily equal anagrams of $wxyz$, and adjacent vertices correspond to necessarily unequal anagrams. Each vertex here is shown with a

set representative and the size of the set.

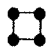


To start, we just need to consider the number of anagrams at each vertex, so we name the vertices merely by how many anagrams are there. If we choose no 4, choosing a 2 excludes four 1's and choosing a 1 excludes two 2's, making the number of possible rewritings twelve or less. So we must choose at least one 4 and exclude four 1's. If we choose one of the remaining 1's, then we exclude two 2's and a 4, pushing the maximum total to twelve or less. All of the 1's are thus excluded. We must have both 4's to have a chance of exceding twelve rewritings. In order to get fourteen, we must take three 2's. The pairs are symmetrically related, so WOLOG, we can exclude any one of them. Say we exclude $wzyx$ and $wyzx$. Since $xwzy = xyzw$, $[w, zy] = e$, so $zywx = wzyx$, hence we deduce the excluded pair as rewritings.

2.  WOLOG, let $[w, x] = [x, y] = [y, z]$.
We construct a graph as in case 1.



This graph is a cube. We can choose no more than four vertices. If we choose three, we must take both 5's. If we take four, we must take a 5, two 3's, and a 1. Clearly, neither case permits the selection of 13, 14, or 15 rewritings.

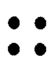
3.  WOLOG, let $[w,x] = [x,y] = [y,z] = [z,y]$.
The twentyfour anagrams split into four groups of six each.♠

Theorem 4 *There exist groups with quadruples $\mathcal{X}$ so that $r(\mathcal{X})$ assumes each integer value 1 to 12; 16; and 24.*

This theorem can be proven by example. The case $r(\mathcal{X}) = 11$ is interesting.

Proposition *Let $\mathcal{X}$ be a quadruple from G so that $r(\mathcal{X}) = 11$. Then there is a way to name the letters of $\mathcal{X}$ as w, x, y, z so that $[w,x] = [x,y] = [y,z] = e$ and $[w,y] = [z,w] = [x,z] \neq e$. Furthermore, if G is 4-rewritable, then $[w,y]$ has order 3.*

Proof. We prove this proposition by examining commutative pairs á la Theorem 2. The cases we need are indexed by isomorphism classes of graphs on four vertices with no vertex adjacent to all three others.

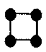
1.  No pairs from w, x, y, z commute.
The graph of necessarily equal and unequal anagrams is bipartite, with even and odd permutations determining the partition of the vertices. If we choose eleven permutations of one parity, say even, then we can deduce the twelve of the same parity. The result is computational and can be derived by conjugating and multiplying the different rewritings together to determine some commutator relations. Using graph theory, we can determine that no other division between even and odd permutations besides 11-0 is possible.

2.  WOLOG, [w,x]=e.
This case can be checked by constructing the graph of the twentyfour anagrams and observing that the total 11 is impossible to obtain except for the following list (up to symmetries): $wxyz$, $xwyz$, $ywxz$,

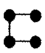
$yxwz$, $yzwx$, $yzxw$, $wyzx$, $xyzw$, $wzxy$, $zwyx$, and one of $zxyw$ and $xzwy$. Whichever choice is made between the last two, we can deduce the other.

3.  WOLOG, [w,x]=[x,y].
Again, the graph permits only one choice up to symmetries: $wxyz$, $xwyz$, $wyxz$, $zywx$, $zyxw$, $zxyw$, $xyzw$, $yxzw$, $wzyx$, $wzxy$, $ywzx$. These imply a twelfth rewriting, $xzwy$.

4.  This case is entirely resolved by the graph of equalities and inequalities.

5.  This is case 3 from Theoerem 2: four groups of six.

6.  All anagrams are necessarily equal in groups of even size.

7.  WOLOG, [w,x]=[x,y]=[y,z].
The only way to get 11 rewritings is to take $wxyz$, $xwzy$, $wxzy$, $xwzy$, $wyxz$, $xzyw$, $xyzw$, $yxzw$, $zxwy$, $zwxy$, $zwyx$ all equal. These imply that their reversals are all equal to each other, as well as the commutator equations as stated in the theorem. We get $ywzx = xzwy$ exactly when $[w, y]^3 = e$. Otherwise, each of these words has no rewritings.♠

References

[1] J. L. Leavitt, G. J. Sherman, and M. E. Walker, *Rewriteability in finite groups,* Amer. Math. Monthly **99**, no. 5, (1992) 446-452.

Acknowledgement. Group theoretic experimentation and computation were greatly facilitated by the computer algebra system CAYLEY.

Lawren M. Smithline, Harvard University
Catherine A. Sugar, Pomona College