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Calibrating the Complexity of Ternary Propositional Connectives

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Abstract. For each ternary propositional connective, we determine the minimum number of binary connectives needed to construct a logically equivalent formula. In order to reduce this problem to a computably feasible one, we prove a number of lemmas showing that every element of a large set of formulas is logically equivalent to a formula in a much smaller associated set.

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We will be working with formulas of propositional calculus. In usual treatments of propositional calculus [2], formulas are built from statement letters and binary (two variable) connectives. These connectives are usually written as infix operators, yielding formulas like $P \to Q$. We will also be using ternary (three variable) connectives, using prefix notation as shown below. We say two formulas are *logically equivalent* if the last columns of their truth tables agree.

Each ternary (three variable) connective has a truth table with eight rows. There are 256 such tables and so 256 ternary connectives. We can label a connective with the hexadecimal code for the last column of its table. The truth table for [E0](P,Q,R) is shown below.

P	Q	R	$[E0](P,Q,R)$	
1	1	1	1	
1	1	0	1	E
1	0	1	1	
1	0	0	0	
0	1	1	0	
0	1	0	0	0
0	0	1	0	
0	0	0	0	

[E0](P,Q,R) has exactly the same truth table as $P \wedge (Q \vee R)$. (If we like we can use single digit hexadecimal representations for the binary connectives, writing $P \wedge (Q \vee R)$ as P[8](Q[E]R).) Also, though this is not at all obvious, no formula with fewer than two binary connectives has the same truth table as [E0](P,Q,R). Since two is the smallest number n such that [E0](P,Q,R) is logically equivalent to a formula with n binary connectives, we say that the connective [E0] is 2-optimal. We generalize this notion in the following definition.

Definition. A ternary connective $[i](P,Q,R)$ is called *n-optimal* if it is logically equivalent to a formula containing n binary connectives, but not logically equivalent to any formula containing $n-1$ binary connectives.

It should be noted with our definition we consider only formulas with binary (two-place) connectives. Consequently, $\neg P$ can be written as $[3](P,P)$ using one binary connective. The 0-ary (zero place) connectives $\top$ and $\bot$ are logically equivalent to $[F](P,P)$ and $[0](P,P)$ respectively. For the cases described above there are no logically equivalent formulas that can be constructed (using only binary connectives and statement letters) with fewer than one connective, meaning they are 1-optimal.

Our goal is to find all the ternary connectives that are n-optimal for each value of n. For the initial case, we can easily find the 0-optimal connectives.

Lemma 1. *Every 0-optimal ternary connective $[i](P,Q,R)$ is logically equivalent to P, Q, or R.*

Proof. A ternary connective is 0-optimal if it is logically equivalent to a formula consisting of zero binary connectives and a statement letter selected from the set $\{P,Q,R\}$. The only formulas of this sort are the single letter formulas P, Q, and R. $\square$

Now we can take each of the three 0-optimal formulas and write out their truth table as a ternary connective. The table for the formula P is as follows.

P	Q	R	$P \equiv [\mathrm{F0}](P,Q,R)$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

The hexadecimal code for the last column is F0, so the formula P is logically equivalent to the ternary connective $[\mathrm{F0}](P,Q,R)$. Similar calculations for Q and R complete the proof of the following proposition.

Proposition 2. *There are exactly three ternary connectives that are* 0*-optimal. They are:* [AA]*,* [CC]*, and* [F0]*.*

The 1-optimal connectives are all those connectives that are not 0-optimal and are logically equivalent to a formula with one binary connective. Whenever two of these formulas are logically equivalent, they both correspond to the same ternary connective, so examining one of them is sufficient. The following lemma identifies many logically equivalent pairs.

Lemma 3. *Every formula of the form $P[i]Q$ (with one binary connective) is logically equivalent to a formula of the form $Q[j]P$.*

Proof. Suppose that the binary connective $[i]$ has a truth table whose last column contains the binary string *abcd*. If we like, we can write $[i]$ as $[abcd]$. Let $[j]$ be the connective $[acbd]$. Then $P[i]Q$ is logically equivalent to $Q[j]P$, as shown in the following truth table.

P	Q	$P[i]Q$	$Q[i]P$	$Q[j]P$
1	1	a	a	a
1	0	b	c	b
0	1	c	b	c
0	0	d	d	d

Informally, swapping the middle bits in the connective and swapping the order of the letters in the formula are inverse operations. □

Applying Lemma 3, we can find a smaller set of formulas that still includes at least one formula that is logically equivalent to each 1-optimal ternary connective.

Lemma 4. *Every* 1*-optimal ternary connective $[i](P,Q,R)$ is logically equivalent to a formula in the list*

$$P[j]Q \qquad P[j]R \qquad Q[j]R$$

where $0 \le j \le \mathrm{F}$. (Here F *is the hexadecimal code for 15.)*

Proof. A ternary connective is 1-optimal if it is logically equivalent to a formula with 1 binary connective. The entire list of formulas that contain letters taken from $\{P, Q, R\}$ and one binary connective is: $P[j]P$, $P[j]Q$, $P[j]R$, $Q[j]Q$, $Q[j]P$, $Q[j]R$, $R[j]R$, $R[j]P$, $R[j]Q$ where $0 \le j \le \mathrm{F}$. We need to show that all of these are logically equivalent to a formula of the form $P[j]Q$, $P[j]R$, or $Q[j]R$, where $0 \le j \le \mathrm{F}$.

$P[j]R$, $P[j]Q$, and $Q[j]R$ are already in our short list of formulas. A formula of the form $P[k]P$ must be always true, equivalent to P, equivalent to $\neg P$, or always false. The following table shows that $P[k]P$ is equivalent to a formula of the form $P[j]Q$ for an appropriate choice of j.

P	Q	$P[\mathrm{F}]Q$	$P[\mathrm{C}]Q$	$P[3]Q$	$P[0]Q$
1	1	1	1	0	0
1	0	1	1	0	0
0	1	1	0	1	0
0	0	1	0	1	0

Similarly, $Q[k]Q$ is equivalent to a formula of the form $Q[j]R$. Formulas of the form $R[k]R$ are equivalent to ones of the form $R[m]P$, which are in turn equivalent to ones of the form $P[j]R$ by Lemma 3. The connectives $Q[j]P$, $R[j]P$, and $R[j]Q$ can all be addressed by Lemma 3. □

In light of Lemma 4, we can discover all the 1-optimal ternary connectives by examining the truth tables of the 48 formulas listed in the lemma. The total number of formulas that contain statement letters from $\{P, Q, R\}$ and exactly one binary connective is much larger. Since each letter can be selected from a set of three letters and the connective is selected from a set of 16 connectives, the total number of such formulas is $3 \cdot 16 \cdot 3 = 144$. Thus Lemma 4 substantially reduces the number of truth tables that must be checked. Even so, many of the formulas listed in Lemma 4 are logically equivalent to each other or to a 0-optimal connective. Consequently, when all 48 truth tables are computed, only 35 different 1-optimal connectives are found. We used Maple software to calculate these truth tables, verifying the following proposition.

Proposition 5. *There are exactly* 35 *ternary connectives that are* 1*-optimal. They are:*

[00]	[03]	[05]	[0A]	[0C]	[0F]	[11]
[22]	[30]	[33]	[3C]	[3F]	[44]	[50]
[55]	[5A]	[5F]	[66]	[77]	[88]	[99]
[A0]	[A5]	[AF]	[BB]	[C0]	[C3]	[CF]
[DD]	[EE]	[F3]	[F5]	[FA]	[FC]	[FF]

For 2-optimal and 3-optimal connectives, the number of formulas to be searched increases very rapidly. The next lemma allows us to eliminate a number of connectives from our formulas, saving a substantial amount of computer time.

Lemma 6. *Suppose that $n \geq 2$ and φ is a formula containing n binary connectives. If φ includes the connective $[i]$ where $i \in \{0, 3, 5, [\mathrm{A}], [\mathrm{C}], [\mathrm{F}]\}$, then there is a formula with less than n connectives which is logically equivalent to φ.*

Proof. We will argue by contradiction. Suppose $n \geq 2$ and n is the least integer such that there is a formula φ containing exactly n connectives, at least one of which is in $\{0, 3, 5, [\mathrm{A}], [\mathrm{C}], [\mathrm{F}]\}$, and no formula with fewer connectives is logically equivalent to φ. We address two cases for each proscribed connective.

Case $[0]_A$: Suppose φ is $\alpha[0]\beta$. Then φ is logically equivalent to $P[0]P$, where P is a (actually any) statement letter. This contradicts the claim that φ is not equivalent to any formula with fewer connectives.

Case $[0]_B$: Suppose φ is $\alpha[i]\beta$ where β or α contains [0]. First, suppose that β contains [0]. If β contains two or more connectives, then by the minimality of n, β is logically equivalent to a formula with fewer connectives. Substituting this formula for β yields a formula with fewer than n connectives which is equivalent to φ, contradicting our hypothesis. Thus, β must be of the form $P[0]Q$ for some choice of statement letters P and Q. Consequently, φ is $\alpha[i](P[0]Q)$. Suppose that the binary representation of $[i]$ is $[abcd]$. Then $\varphi = \alpha[abcd](P[0]Q)$ is logically equivalent to $\alpha[bbdd]\gamma$ for any choice of γ, as shown by the following truth table.

α	γ	$\alpha[abcd]\gamma$	$P[0]Q$	$\alpha[abcd](P[0]Q)$	$\alpha[bbdd]\gamma$
1	1	a	0	b	b
1	0	b	0	b	b
0	1	c	0	d	d
0	0	d	0	d	d

In particular, φ is logically equivalent to $\alpha[bbdd]P$ which has fewer connectives than φ, yielding another contradiction and completing the case when β contains [0]. By Lemma 3, the situation where α contains [0] also yields a contradiction.

Case $[\mathrm{F}]_A$: Suppose φ is $\alpha[\mathrm{F}]\beta$. This case is similar to Case $[0]_A$, except φ is equivalent to $P[\mathrm{F}]P$.

Case $[\mathrm{F}]_B$: Suppose φ is $\alpha[i]\beta$ where β or α contains [F]. This case is similar to case $[0]_B$ except $\alpha[abcd]\beta$ is logically equivalent to $\alpha[aacc]P$.

Case $[3]_A$: Suppose φ is $\alpha[3]\beta$. Then φ is logically equivalent to $\neg\alpha$. If α is a statement letter (say P), then $\neg\alpha$ is equivalent to $P[3]P$. If α is not a statement letter, then α has the form $\gamma[i]\delta$ where $[i]$ is $[abcd]$. In this situation, $\neg\alpha$ is equivalent to $\gamma[\bar{a}\bar{b}\bar{c}\bar{d}]\delta$ where $\bar{s} = 1 - s$ for all $s \in \{a, b, c, d\}$. In either case, $\neg\alpha$ is logically equivalent to a formula with fewer than n connectives, yielding a contradiction.

Case $[3]_B$: Suppose φ is $\alpha[i]\beta$ where β or α contains [3]. First, suppose β contains [3]. If β contains two or more connectives, then by emulating Case $[0]_B$ and applying the minimality of n, we can derive a contradiction. Thus β is of the form $P[3]Q$ for some statement letters P and Q. Consequently, φ is logically equivalent to $\alpha[i]\neg P$. If $[i]$ is $[abcd]$, then we can use the following truth table to verify that φ is equivalent to $\alpha[badc]P$.

α	P	$\alpha[abcd]P$	$\alpha[abcd]\neg P$	$\alpha[badc]P$
1	1	a	b	b
1	0	b	a	a
0	1	c	d	d
0	0	d	c	c

Consequently, φ is logically equivalent to a formula with fewer than n connectives, yielding a contradiction and completing the case where β contains [3]. An application of Lemma 3 shows that a contradiction also arises when α contains [3].

Case $[5]_A$: Suppose φ is $\alpha[5]\beta$. Then φ is logically equivalent to $\neg\beta$. This case is very similar to Case $[3]_A$.

Case $[5]_B$: Suppose φ is $\alpha[i]\beta$ where β or α contains [5]. This case is very similar to Case $[3]_B$.

Case $[\mathrm{A}]_A$: Suppose φ is $\alpha[\mathrm{A}]\beta$. Then φ is logically equivalent to β. Since β contains fewer connectives than φ, we have an immediate contradiction.

Case $[\mathrm{A}]_B$: Suppose φ is $\alpha[i]\beta$ where β or α contains [A]. First, suppose β contains [A]. If β contains two or more connectives, by minimality of n we are done. Thus β is $P[\mathrm{A}]Q$ for some statement letters P and Q. So φ is logically equivalent to $\alpha[i]Q$, which contains fewer than n connectives and yields a contradiction. The situation where α contains [A] can be handled via Lemma 3.

Case $[\mathrm{C}]_A$: Suppose φ is $\alpha[\mathrm{C}]\beta$. Then φ is logically equivalent to α, yielding the desired contradiction.

Case $[\mathrm{C}]_B$: Suppose φ is $\alpha[i]\beta$ where β or α contains [C]. This is very similar to Case $[\mathrm{A}]_B$, except that $\alpha[i](P[\mathrm{C}]Q)$ is logically equivalent to $\alpha[\mathrm{i}]P$.

In light of the preceding cases, there is no least $n \geq 2$ for which the lemma fails, so the lemma must hold for all $n \geq 2$. □

Now we can apply Lemma 6 to prove the lemma that will help us determine the 2-optimal ternary connectives.

Lemma 7. *Every* 2*-optimal ternary connective* $[i](P, Q, R)$ *is logically equivalent to a formula in the list*

$$P[j](Q[k]R) \qquad Q[j](P[k]R) \qquad R[j](P[k]Q)$$

where j *and* k *are in* $\{1, 2, 4, 6, 7, 8, 9, \mathrm{B}, \mathrm{D}, \mathrm{E}\}$.

Proof. We will begin by eliminating formulas with repeated letters. Suppose φ is a formula that contains two connectives and at most two statement letters. For convenience, we may assume that the statement letters in φ are in $\{P, Q\}$. The formula φ has a truth table of the following form, where a, b, c, and d represent binary digits.

P	Q	φ
1	1	a
1	0	b
0	1	c
0	0	d

Thus φ is logically equivalent to $P[abcd]Q$, and is at most 1-optimal. Consequently, if φ is 2-optimal, it must contain all three statement letters.

Now suppose φ is 2-optimal and written using two connectives. The main connective of φ must connect a single statement letter to a formula containing one connective. By Lemma 3, φ is logically equivalent to a formula in which the single statement letter appears first. By another application of Lemma 3, we may assume that the statement letters in the remaining subformula occur in alphabetical order. By Lemma 6, we may assume that the connectives all lie in the restricted set, completing the proof. □

Lemma 7 vastly reduces the potential search space for 2-optimal connectives. The total number of two connective formulas containing two binary operators and three letters taken from $\{P, Q, R\}$, parenthesized with either a leading or trailing single statement letter is $16^2 \cdot 3^3 \cdot 2 = 13824$. By Lemma 7, we can restrict attention to the three given forms and ten listed connectives, and only examine $3 \cdot 10^2 = 300$ formulas. Among these formulas, we can find the 114 2-optimal connectives listed in the following proposition.

Proposition 8. *There are exactly* 114 *ternary connectives that are* 2*-optimal. They are:*

[01]	[02]	[04]	[06]	[07]	[08]	[09]	[0B]	[0D]	[0E]
[10]	[12]	[13]	[14]	[15]	[1E]	[1F]	[20]	[21]	[23]
[28]	[2A]	[2D]	[2F]	[31]	[32]	[36]	[37]	[39]	[3B]
[40]	[41]	[45]	[48]	[4B]	[4C]	[4F]	[51]	[54]	[56]
[57]	[59]	[5D]	[60]	[63]	[65]	[69]	[6A]	[6C]	[6F]
[70]	[73]	[75]	[78]	[7B]	[7D]	[7F]	[80]	[82]	[84]
[87]	[8A]	[8C]	[8F]	[90]	[93]	[95]	[96]	[9A]	[9C]
[9F]	[A2]	[A6]	[A8]	[A9]	[AB]	[AE]	[B0]	[B3]	[B4]
[B7]	[BA]	[BE]	[BF]	[C4]	[C6]	[C8]	[C9]	[CD]	[CE]
[D0]	[D2]	[D5]	[D7]	[DC]	[DE]	[DF]	[E0]	[E1]	[EA]
[EB]	[EC]	[ED]	[EF]	[F1]	[F2]	[F4]	[F6]	[F7]	[F8]
[F9]	[FB]	[FD]	[FE]						

Both Lemma 6 and Lemma 7 are very useful in reducing the number of potential 3-optimal formulas.

Lemma 9. *Every* 3*-optimal ternary connective* $[i](P, Q, R)$ *is logically equivalent to a formula in the list*

$$R[j](P[k](Q[m]R)) \qquad (P[j]Q)[k](P[m]R)$$

where j*,* k*, and* m *are in* $\{1, 2, 4, 6, 7, 8, 9, \mathrm{B}, \mathrm{D}, \mathrm{E}\}$*, or as a formula resulting from permuting the statement letters in a formula of this form.*

Proof. Suppose φ is a formula containing three binary connectives. The main connective of φ must either connect a single statement letter to a formula containing two connectives, or connect two formulas each containing exactly one connective. We will consider these cases separately.

Suppose the main connective of φ connects a single statement letter to a formula containing two connectives. By Lemma 3, we may assume the single statement letter appears first. If φ is 3-optimal, then the two connective subformula must be 2-optimal. By Lemma 7, we may assume this formula has the form $P[k](Q[m]R)$ up to permutation of the statement letters. Now any formula of the form $P[j](P[k]\alpha)$ is logically equivalent to a formula of the form $P[n]\alpha$. Replacing α with $Q[m]R$, we see that every formula of the form $P[j](P[k](Q[m]R))$ is logically equivalent to a formula of the form $P[n](Q[m]R)$, and therefore is not 3-optimal. Thus φ must be of the form (1) $R[j](P[k](Q[m]R))$ or (2) $Q[j](P[k](Q[m]R))$. By Lemma 3 every formula of type (2) is logically equivalent to a formula of type (1), modulo permutation of the statement letters. By Lemma 6, we may assume that $[j]$, $[k]$, and $[m]$ are taken from the restricted set of connectives. Summarizing, if φ is 3-optimal and the main connective of φ connects a single statement letter to a two connective formula, then φ is logically equivalent to a formula of the form $R[j](P[k](Q[m]R))$ where j, k, and m are in $\{1, 2, 4, 6, 7, 8, 9, [\mathrm{B}], [\mathrm{D}], [\mathrm{E}]\}$, or to a formula resulting from permuting the statement letters in such a formula.

Now suppose that the main connective of φ connects two subformulas, each containing a single connective. If two statement letters are repeated in φ, then φ contains only two statement letters and so is logically equivalent to a formula with a single connective. Thus, if φ is 3-optimal, it must contain exactly one repeated statement letter. Since we will eventually allow permutations of the statement letters, for the moment we may assume that P is the repeated statement letter. Any formula of the form $(P[j]P)[k](Q[m]R)$ is at most 2-optimal. Applying Lemma 3, if φ is 3-optimal, it must be logically equivalent to a formula of the form $(P[j]Q)[k](P[m]R)$, or to a formula resulting from permuting the statement letters in this formula. By Lemma 6, we may assume that $[j]$, $[k]$, and $[m]$ are in the restricted set. Summarizing, if φ is 3-optimal and its main connective connects two one connective formulas, then modulo permutation of the statement letters, φ is logically equivalent to $(P[j]Q)[k](P[m]R)$, where j, k, and m are in $\{1, 2, 4, 6, 7, 8, 9, [\mathrm{B}], [\mathrm{D}], [\mathrm{E}]\}$. This case completes the proof of the lemma. □

We can easily compare the number of potential 3-optimal connectives described by Lemma 9 to the total number of formulas with three binary connectives. In Lemma 9 we have two forms, six permutations of letters, and three connectives selected from the reduced collection of ten connectives. This yields a list of $2 \cdot 6 \cdot 10^3 = 12,000$ formulas to examine. As large as this might seem, the total number of formulas with 3 binary connectives and letters from $\{P, Q, R\}$ is much larger. There are five ways to parenthesize such a formula. (Note that $5 = C_3$, the third Catalan number [1]. In general, the nth Catalan number is the number of ways to write a string of n pairs of correctly matched parentheses.) Each of the four letters is selected from the set of three, and each of the three connectives is selected from a set of sixteen. Thus, the set of formulas with three binary connectives and letters selected from $\{P, Q, R\}$ is $5 \cdot 3^4 \cdot 16^3 = 1,658,880$. Rather than generating truth tables for all of these, we looked at the 12,000 formulas described in Lemma 9, verifying that 80 of them correspond to 3-optimal connectives as described in the following proposition.

Proposition 10. *There are exactly* 80 *ternary connectives that are* 3*-optimal. They are:*

[18]	[19]	[1A]	[1B]	[1C]	[1D]	[24]	[25]	[26]	[27]
[2C]	[2E]	[34]	[35]	[38]	[3A]	[3D]	[3E]	[42]	[43]
[46]	[47]	[4A]	[4E]	[52]	[53]	[58]	[5B]	[5C]	[5E]
[62]	[64]	[67]	[6E]	[72]	[74]	[76]	[7A]	[7C]	[7E]
[81]	[83]	[85]	[89]	[8B]	[8D]	[91]	[98]	[9B]	[9D]
[A1]	[A3]	[A4]	[A7]	[AC]	[AD]	[B1]	[B5]	[B8]	[B9]
[BC]	[BD]	[C1]	[C2]	[C5]	[C7]	[CA]	[CB]	[D1]	[D3]
[D8]	[D9]	[DA]	[DB]	[E2]	[E3]	[E4]	[E5]	[E6]	[E7]

Combining the results of the preceding propositions, we know that there are $3+35+114+80 = 232$ ternary connectives that are n-optimal for $n \leq 3$. Only 24 ternary connectives have not been listed. The number of formulas with letters taken from $\{P, Q, R\}$ and four binary connectives is $14 \cdot 3^5 \cdot 16^4 = 222,953,472$. (The fourth Catalan number is $C_4 = 14$.) Assuming that we can examine a thousand truth tables every second (which is not unreasonable using a desktop computer), it would take almost 62 hours to examine all of these formulas. However, a bit of experimentation yields a much shorter list of just over a thousand formulas that includes formulas equivalent to each of the remaining 24 ternary connectives.

Lemma 11. *Every* 4*-optimal ternary connective* $[i](P, Q, R)$ *is logically equivalent to a formula of the form*

$$Q[j]((P[k]Q)[m](P[n]R))$$

where j*,* k*,* m*, and* n *are in* $\{1, 2, 4, 6, 8, 9\}$*.*

Proof. Reasonable collections of formulas to search over for 4-optimal ternary connectives include formulas created by appending one statement letter and one connective to formulas described in Lemma 9. When this list is restricted to just those formulas described in the statement of Lemma 11, logically equivalent forms are found for all ternary connectives which are not already known to be n-optimal for $n \leq 3$. □

Proposition 12. *There are exactly* 24 *ternary connectives that are* 4*-optimal. They are:*

[16]	[17]	[29]	[2B]	[49]	[4D]	[61]	[68]
[6B]	[6D]	[71]	[79]	[86]	[8E]	[92]	[94]
[97]	[9E]	[B2]	[B6]	[D4]	[D6]	[E8]	[E9]

In the course of verifying the preceding propositions, for each n-optimal ternary connective we found a logically equivalent formula with n binary connectives. For each ternary connective, the following table indicates the value of n for which it is n-optimal. Each n-optimal connective is included in the lists following the table, together with a logically equivalent formula containing n binary connectives. The shaded example shows that [02] is 2-optimal and $[02](P,Q,R) \equiv P[1](Q[\mathrm{D}]R)$.

n-optimal Ternary Connectives

	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	1	2	2	1	2	1	2	2	2	2	1	2	1	2	2	1
1	2	1	2	2	2	2	4	4	3	3	3	3	3	3	2	2
2	2	2	1	2	3	3	3	3	2	4	2	4	3	2	3	2
3	1	2	2	1	3	3	2	2	3	2	3	2	1	3	3	1
4	2	2	3	3	1	2	3	3	2	4	3	2	2	4	3	2
5	1	2	3	3	2	1	2	2	3	2	1	3	3	2	3	1
6	2	4	3	2	3	2	1	3	4	2	2	4	2	4	3	2
7	2	4	3	2	3	2	3	1	2	4	3	2	3	2	3	2
8	2	3	2	3	2	3	4	2	1	3	2	3	2	3	4	2
9	2	3	4	2	4	2	2	4	3	1	2	3	2	3	4	2
A	1	3	2	3	3	1	2	3	2	2	0	2	3	3	2	1
B	2	3	4	2	2	3	4	2	3	3	2	1	3	3	2	2
C	1	3	3	1	2	3	2	3	2	2	3	3	0	2	2	1
D	2	3	2	3	4	2	4	2	3	3	3	3	2	1	2	2
E	2	2	3	3	3	3	3	3	4	4	2	2	2	2	1	2
F	0	2	2	1	2	1	2	2	2	2	1	2	1	2	2	1

1-optimal Ternary Connectives

$[00](P,Q,R) \equiv P[0]P$	$[03](P,Q,R) \equiv P[1]Q$	$[05](P,Q,R) \equiv P[1]R$
$[0A](P,Q,R) \equiv P[2]R$	$[0C](P,Q,R) \equiv P[2]Q$	$[0F](P,Q,R) \equiv P[1]P$
$[11](P,Q,R) \equiv Q[1]R$	$[22](P,Q,R) \equiv Q[2]R$	$[30](P,Q,R) \equiv P[4]Q$
$[33](P,Q,R) \equiv P[5]Q$	$[3C](P,Q,R) \equiv P[6]Q$	$[3F](P,Q,R) \equiv P[7]Q$
$[44](P,Q,R) \equiv Q[4]R$	$[50](P,Q,R) \equiv P[4]R$	$[55](P,Q,R) \equiv P[5]R$
$[5A](P,Q,R) \equiv P[6]R$	$[5F](P,Q,R) \equiv P[7]R$	$[66](P,Q,R) \equiv Q[6]R$
$[77](P,Q,R) \equiv Q[7]R$	$[88](P,Q,R) \equiv Q[8]R$	$[99](P,Q,R) \equiv Q[9]R$
$[A0](P,Q,R) \equiv P[8]R$	$[A5](P,Q,R) \equiv P[9]R$	$[AF](P,Q,R) \equiv P[B]R$
$[BB](P,Q,R) \equiv Q[B]R$	$[C0](P,Q,R) \equiv P[8]Q$	$[C3](P,Q,R) \equiv P[9]Q$
$[CF](P,Q,R) \equiv P[B]Q$	$[DD](P,Q,R) \equiv Q[D]R$	$[EE](P,Q,R) \equiv Q[E]R$
$[F3](P,Q,R) \equiv P[D]Q$	$[F5](P,Q,R) \equiv P[D]R$	$[FA](P,Q,R) \equiv P[E]R$
$[FC](P,Q,R) \equiv P[E]Q$	$[FF](P,Q,R) \equiv P[9]P$	

2-optimal Ternary Connectives

$[01](P,Q,R) \equiv P[1](Q[E]R)$	$[02](P,Q,R) \equiv P[1](Q[D]R)$
$[04](P,Q,R) \equiv P[1](Q[B]R)$	$[06](P,Q,R) \equiv P[1](Q[9]R)$
$[07](P,Q,R) \equiv P[1](Q[8]R)$	$[08](P,Q,R) \equiv P[1](Q[7]R)$
$[09](P,Q,R) \equiv P[1](Q[6]R)$	$[0B](P,Q,R) \equiv P[1](Q[4]R)$
$[0D](P,Q,R) \equiv P[1](Q[2]R)$	$[0E](P,Q,R) \equiv P[1](Q[1]R)$
$[10](P,Q,R) \equiv P[4](Q[E]R)$	$[12](P,Q,R) \equiv Q[1](P[9]R)$
$[13](P,Q,R) \equiv Q[1](P[8]R)$	$[14](P,Q,R) \equiv R[1](P[9]Q)$
$[15](P,Q,R) \equiv R[1](P[8]Q)$	$[1E](P,Q,R) \equiv P[6](Q[E]R)$
$[1F](P,Q,R) \equiv P[7](Q[E]R)$	$[20](P,Q,R) \equiv P[4](Q[D]R)$
$[21](P,Q,R) \equiv Q[1](P[6]R)$	$[23](P,Q,R) \equiv Q[1](P[4]R)$
$[28](P,Q,R) \equiv R[4](P[9]Q)$	$[2A](P,Q,R) \equiv R[4](P[8]Q)$
$[2D](P,Q,R) \equiv P[6](Q[D]R)$	$[2F](P,Q,R) \equiv P[7](Q[D]R)$
$[31](P,Q,R) \equiv Q[1](P[2]R)$	$[32](P,Q,R) \equiv Q[1](P[1]R)$
$[36](P,Q,R) \equiv Q[6](P[E]R)$	$[37](P,Q,R) \equiv Q[7](P[E]R)$
$[39](P,Q,R) \equiv Q[6](P[D]R)$	$[3B](P,Q,R) \equiv Q[7](P[D]R)$
$[40](P,Q,R) \equiv P[4](Q[B]R)$	$[41](P,Q,R) \equiv R[1](P[6]Q)$
$[45](P,Q,R) \equiv R[1](P[4]Q)$	$[48](P,Q,R) \equiv Q[4](P[9]R)$
$[4B](P,Q,R) \equiv P[6](Q[B]R)$	$[4C](P,Q,R) \equiv Q[4](P[8]R)$
$[4F](P,Q,R) \equiv P[7](Q[B]R)$	$[51](P,Q,R) \equiv R[1](P[2]Q)$
$[54](P,Q,R) \equiv R[1](P[1]Q)$	$[56](P,Q,R) \equiv R[6](P[E]Q)$
$[57](P,Q,R) \equiv R[7](P[E]Q)$	$[59](P,Q,R) \equiv R[6](P[D]Q)$
$[5D](P,Q,R) \equiv R[7](P[D]Q)$	$[60](P,Q,R) \equiv P[4](Q[9]R)$

2-optimal Ternary Connectives, continued

$[63](P,Q,R) \equiv Q[6](P[\mathrm{B}]R)$	$[65](P,Q,R) \equiv R[6](P[\mathrm{B}]Q)$
$[69](P,Q,R) \equiv P[6](Q[9]R)$	$[6\mathrm{A}](P,Q,R) \equiv R[6](P[8]Q)$
$[6\mathrm{C}](P,Q,R) \equiv Q[6](P[8]R)$	$[6\mathrm{F}](P,Q,R) \equiv P[7](Q[9]R)$
$[70](P,Q,R) \equiv P[4](Q[8]R)$	$[73](P,Q,R) \equiv Q[7](P[\mathrm{B}]R)$
$[75](P,Q,R) \equiv R[7](P[\mathrm{B}]Q)$	$[78](P,Q,R) \equiv P[6](Q[8]R)$
$[7\mathrm{B}](P,Q,R) \equiv Q[7](P[9]R)$	$[7\mathrm{D}](P,Q,R) \equiv R[7](P[9]Q)$
$[7\mathrm{F}](P,Q,R) \equiv P[7](Q[8]R)$	$[80](P,Q,R) \equiv P[4](Q[7]R)$
$[82](P,Q,R) \equiv R[4](P[6]Q)$	$[84](P,Q,R) \equiv Q[4](P[6]R)$
$[87](P,Q,R) \equiv P[6](Q[7]R)$	$[8\mathrm{A}](P,Q,R) \equiv R[4](P[4]Q)$
$[8\mathrm{C}](P,Q,R) \equiv Q[4](P[4]R)$	$[8\mathrm{F}](P,Q,R) \equiv P[7](Q[7]R)$
$[90](P,Q,R) \equiv P[4](Q[6]R)$	$[93](P,Q,R) \equiv Q[6](P[7]R)$
$[95](P,Q,R) \equiv R[6](P[7]Q)$	$[96](P,Q,R) \equiv P[6](Q[6]R)$
$[9\mathrm{A}](P,Q,R) \equiv R[6](P[4]Q)$	$[9\mathrm{C}](P,Q,R) \equiv Q[6](P[4]R)$
$[9\mathrm{F}](P,Q,R) \equiv P[7](Q[6]R)$	$[\mathrm{A}2](P,Q,R) \equiv R[4](P[2]Q)$
$[\mathrm{A}6](P,Q,R) \equiv R[6](P[2]Q)$	$[\mathrm{A}8](P,Q,R) \equiv R[4](P[1]Q)$
$[\mathrm{A}9](P,Q,R) \equiv R[6](P[1]Q)$	$[\mathrm{AB}](P,Q,R) \equiv R[\mathrm{D}](P[\mathrm{E}]Q)$
$[\mathrm{AE}](P,Q,R) \equiv R[\mathrm{D}](P[\mathrm{D}]Q)$	$[\mathrm{B}0](P,Q,R) \equiv P[4](Q[4]R)$
$[\mathrm{B}3](P,Q,R) \equiv Q[7](P[7]R)$	$[\mathrm{B}4](P,Q,R) \equiv P[6](Q[4]R)$
$[\mathrm{B}7](P,Q,R) \equiv Q[7](P[6]R)$	$[\mathrm{BA}](P,Q,R) \equiv R[\mathrm{D}](P[\mathrm{B}]Q)$
$[\mathrm{BE}](P,Q,R) \equiv R[\mathrm{D}](P[9]Q)$	$[\mathrm{BF}](P,Q,R) \equiv P[7](Q[4]R)$
$[\mathrm{C}4](P,Q,R) \equiv Q[4](P[2]R)$	$[\mathrm{C}6](P,Q,R) \equiv Q[6](P[2]R)$
$[\mathrm{C}8](P,Q,R) \equiv Q[4](P[1]R)$	$[\mathrm{C}9](P,Q,R) \equiv Q[6](P[1]R)$
$[\mathrm{CD}](P,Q,R) \equiv Q[\mathrm{D}](P[\mathrm{E}]R)$	$[\mathrm{CE}](P,Q,R) \equiv Q[\mathrm{D}](P[\mathrm{D}]R)$
$[\mathrm{D}0](P,Q,R) \equiv P[4](Q[2]R)$	$[\mathrm{D}2](P,Q,R) \equiv P[6](Q[2]R)$
$[\mathrm{D}5](P,Q,R) \equiv R[7](P[7]Q)$	$[\mathrm{D}7](P,Q,R) \equiv R[7](P[6]Q)$
$[\mathrm{DC}](P,Q,R) \equiv Q[\mathrm{D}](P[\mathrm{B}]R)$	$[\mathrm{DE}](P,Q,R) \equiv Q[\mathrm{D}](P[9]R)$
$[\mathrm{DF}](P,Q,R) \equiv P[7](Q[2]R)$	$[\mathrm{E}0](P,Q,R) \equiv P[4](Q[1]R)$
$[\mathrm{E}1](P,Q,R) \equiv P[6](Q[1]R)$	$[\mathrm{EA}](P,Q,R) \equiv R[\mathrm{D}](P[7]Q)$
$[\mathrm{EB}](P,Q,R) \equiv R[\mathrm{D}](P[6]Q)$	$[\mathrm{EC}](P,Q,R) \equiv Q[\mathrm{D}](P[7]R)$
$[\mathrm{ED}](P,Q,R) \equiv Q[\mathrm{D}](P[6]R)$	$[\mathrm{EF}](P,Q,R) \equiv P[7](Q[1]R)$
$[\mathrm{F}1](P,Q,R) \equiv P[\mathrm{D}](Q[\mathrm{E}]R)$	$[\mathrm{F}2](P,Q,R) \equiv P[\mathrm{D}](Q[\mathrm{D}]R)$
$[\mathrm{F}4](P,Q,R) \equiv P[\mathrm{D}](Q[\mathrm{B}]R)$	$[\mathrm{F}6](P,Q,R) \equiv P[\mathrm{D}](Q[9]R)$
$[\mathrm{F}7](P,Q,R) \equiv P[\mathrm{D}](Q[8]R)$	$[\mathrm{F}8](P,Q,R) \equiv P[\mathrm{D}](Q[7]R)$
$[\mathrm{F}9](P,Q,R) \equiv P[\mathrm{D}](Q[6]R)$	$[\mathrm{FB}](P,Q,R) \equiv P[\mathrm{D}](Q[4]R)$
$[\mathrm{FD}](P,Q,R) \equiv P[\mathrm{D}](Q[2]R)$	$[\mathrm{FE}](P,Q,R) \equiv P[\mathrm{D}](Q[1]R)$

3-optimal Ternary Connectives

$[18](P,Q,R) \equiv (P[9]Q)[1](P[9]R)$
$[1A](P,Q,R) \equiv (P[8]Q)[1](P[9]R)$
$[1C](P,Q,R) \equiv (P[9]Q)[1](P[8]R)$
$[24](P,Q,R) \equiv (P[9]Q)[1](P[6]R)$
$[26](P,Q,R) \equiv (Q[8]P)[1](Q[9]R)$
$[2C](P,Q,R) \equiv (P[9]Q)[1](P[4]R)$
$[34](P,Q,R) \equiv (P[9]Q)[1](P[2]R)$
$[38](P,Q,R) \equiv (P[9]Q)[1](P[1]R)$
$[3D](P,Q,R) \equiv P[6](Q[\mathrm{D}](R[\mathrm{E}]P))$
$[42](P,Q,R) \equiv (P[6]Q)[1](P[9]R)$
$[46](P,Q,R) \equiv (Q[2]P)[1](Q[9]R)$
$[4A](P,Q,R) \equiv (P[4]Q)[1](P[9]R)$
$[52](P,Q,R) \equiv (P[2]Q)[1](P[9]R)$
$[58](P,Q,R) \equiv (P[1]Q)[1](P[9]R)$
$[5C](P,Q,R) \equiv (P[1]Q)[1](P[8]R)$
$[62](P,Q,R) \equiv (Q[4]P)[1](Q[9]R)$
$[67](P,Q,R) \equiv (Q[\mathrm{E}]P)[7](Q[9]R)$
$[72](P,Q,R) \equiv (R[1]P)[1](R[8]Q)$
$[76](P,Q,R) \equiv (Q[\mathrm{D}]P)[7](Q[9]R)$
$[7C](P,Q,R) \equiv P[6](Q[4](R[2]P))$
$[81](P,Q,R) \equiv (P[6]Q)[1](P[6]R)$
$[85](P,Q,R) \equiv (P[4]Q)[1](P[6]R)$
$[8B](P,Q,R) \equiv P[6](Q[7](R[9]P))$
$[91](P,Q,R) \equiv (Q[4]P)[1](Q[6]R)$
$[9B](P,Q,R) \equiv (Q[\mathrm{E}]P)[7](Q[6]R)$
$[A1](P,Q,R) \equiv (P[2]Q)[1](P[6]R)$
$[A4](P,Q,R) \equiv (P[1]Q)[1](P[6]R)$
$[AC](P,Q,R) \equiv (P[1]Q)[1](P[4]R)$
$[B1](P,Q,R) \equiv (R[4]P)[1](R[2]Q)$
$[B8](P,Q,R) \equiv P[6](Q[4](R[9]P))$
$[BC](P,Q,R) \equiv P[6](Q[4](R[8]P))$
$[C1](P,Q,R) \equiv (P[6]Q)[1](P[2]R)$
$[C5](P,Q,R) \equiv (P[4]Q)[1](P[2]R)$
$[CA](P,Q,R) \equiv (P[4]Q)[1](P[1]R)$
$[D1](P,Q,R) \equiv P[6](Q[1](R[6]P))$
$[D8](P,Q,R) \equiv (R[1]P)[1](R[4]Q)$
$[DA](P,Q,R) \equiv (P[7]Q)[7](P[9]R)$
$[E2](P,Q,R) \equiv P[6](Q[1](R[9]P))$
$[E4](P,Q,R) \equiv (R[4]P)[1](R[1]Q)$
$[E6](P,Q,R) \equiv (Q[7]P)[7](Q[9]R)$

$[19](P,Q,R) \equiv (Q[8]P)[1](Q[6]R)$
$[1B](P,Q,R) \equiv (R[8]P)[1](R[2]Q)$
$[1D](P,Q,R) \equiv P[6](Q[\mathrm{D}](R[6]P))$
$[25](P,Q,R) \equiv (P[8]Q)[1](P[6]R)$
$[27](P,Q,R) \equiv (R[2]P)[1](R[8]Q)$
$[2E](P,Q,R) \equiv P[6](Q[\mathrm{D}](R[9]P))$
$[35](P,Q,R) \equiv (P[8]Q)[1](P[2]R)$
$[3A](P,Q,R) \equiv (P[8]Q)[1](P[1]R)$
$[3E](P,Q,R) \equiv P[6](Q[\mathrm{D}](R[\mathrm{B}]P))$
$[43](P,Q,R) \equiv (P[6]Q)[1](P[8]R)$
$[47](P,Q,R) \equiv P[6](Q[7](R[6]P))$
$[4E](P,Q,R) \equiv (R[8]P)[1](R[1]Q)$
$[53](P,Q,R) \equiv (P[2]Q)[1](P[8]R)$
$[5B](P,Q,R) \equiv (P[\mathrm{E}]Q)[7](P[9]R)$
$[5E](P,Q,R) \equiv (P[\mathrm{D}]Q)[7](P[9]R)$
$[64](P,Q,R) \equiv (Q[1]P)[1](Q[9]R)$
$[6E](P,Q,R) \equiv (Q[\mathrm{B}]P)[7](Q[9]R)$
$[74](P,Q,R) \equiv P[6](Q[4](R[6]P))$
$[7A](P,Q,R) \equiv (P[\mathrm{B}]Q)[7](P[9]R)$
$[7E](P,Q,R) \equiv (P[9]Q)[7](P[9]R)$
$[83](P,Q,R) \equiv (P[6]Q)[1](P[4]R)$
$[89](P,Q,R) \equiv (Q[2]P)[1](Q[6]R)$
$[8D](P,Q,R) \equiv (R[2]P)[1](R[4]Q)$
$[98](P,Q,R) \equiv (Q[1]P)[1](Q[6]R)$
$[9D](P,Q,R) \equiv (Q[\mathrm{B}]P)[7](Q[6]R)$
$[A3](P,Q,R) \equiv (P[2]Q)[1](P[4]R)$
$[A7](P,Q,R) \equiv (P[\mathrm{E}]Q)[7](P[6]R)$
$[AD](P,Q,R) \equiv (P[\mathrm{D}]Q)[7](P[6]R)$
$[B5](P,Q,R) \equiv (P[\mathrm{B}]Q)[7](P[6]R)$
$[B9](P,Q,R) \equiv (Q[\mathrm{D}]P)[7](Q[6]R)$
$[BD](P,Q,R) \equiv (P[9]Q)[7](P[6]R)$
$[C2](P,Q,R) \equiv (P[6]Q)[1](P[1]R)$
$[C7](P,Q,R) \equiv P[6](Q[7](R[\mathrm{E}]P))$
$[CB](P,Q,R) \equiv P[6](Q[7](R[\mathrm{B}]P))$
$[D3](P,Q,R) \equiv P[6](Q[1](R[2]P))$
$[D9](P,Q,R) \equiv (Q[7]P)[7](Q[6]R)$
$[DB](P,Q,R) \equiv (P[6]Q)[7](P[9]R)$
$[E3](P,Q,R) \equiv P[6](Q[1](R[8]P))$
$[E5](P,Q,R) \equiv (P[7]Q)[7](P[6]R)$
$[E7](P,Q,R) \equiv (P[6]Q)[7](P[6]R)$

4-optimal Ternary Connectives

$[16](P,Q,R) \equiv Q[9]((P[8]Q)[1](P[6]R))$	$[17](P,Q,R) \equiv Q[9]((P[9]Q)[1](P[6]R))$
$[29](P,Q,R) \equiv Q[9]((P[8]Q)[1](P[9]R))$	$[2B](P,Q,R) \equiv P[9]((P[9]Q)[1](P[6]R))$
$[49](P,Q,R) \equiv Q[6]((P[4]Q)[1](P[6]R))$	$[4D](P,Q,R) \equiv Q[6]((P[6]Q)[1](P[6]R))$
$[61](P,Q,R) \equiv Q[9]((P[2]Q)[1](P[9]R))$	$[68](P,Q,R) \equiv Q[6]((P[1]Q)[1](P[6]R))$
$[6B](P,Q,R) \equiv Q[9]((P[1]Q)[1](P[9]R))$	$[6D](P,Q,R) \equiv Q[6]((P[2]Q)[1](P[6]R))$
$[71](P,Q,R) \equiv P[6]((P[6]Q)[1](P[6]R))$	$[79](P,Q,R) \equiv Q[9]((P[4]Q)[1](P[9]R))$
$[86](P,Q,R) \equiv Q[6]((P[4]Q)[1](P[9]R))$	$[8E](P,Q,R) \equiv P[9]((P[6]Q)[1](P[6]R))$
$[92](P,Q,R) \equiv Q[9]((P[2]Q)[1](P[6]R))$	$[94](P,Q,R) \equiv Q[6]((P[1]Q)[1](P[9]R))$
$[97](P,Q,R) \equiv Q[9]((P[1]Q)[1](P[6]R))$	$[9E](P,Q,R) \equiv Q[6]((P[2]Q)[1](P[9]R))$
$[B2](P,Q,R) \equiv Q[9]((P[6]Q)[1](P[6]R))$	$[B6](P,Q,R) \equiv Q[9]((P[4]Q)[1](P[6]R))$
$[D4](P,Q,R) \equiv P[6]((P[9]Q)[1](P[6]R))$	$[D6](P,Q,R) \equiv Q[6]((P[8]Q)[1](P[9]R))$
$[E8](P,Q,R) \equiv Q[6]((P[9]Q)[1](P[6]R))$	$[E9](P,Q,R) \equiv Q[6]((P[8]Q)[1](P[6]R))$

A number of natural questions arise from our work:

1. Given a ternary connective, is there a fast way to directly calculate the n such that it is n-optimal?

2. What effect would restricting the allowable binary connectives have on the calculation of n? Which sets of connectives give the highest possible values of n? What are the smallest sets of connectives that give the same values that we found? What happens if we use only NAND and NOR?

3. What happens with n-ary connectives for $n \geq 3$? We conjecture that every quaternary connective is logically equivalent to a formula containing at most three ternary connectives. In general, we conjecture that if $n \geq 4$, then every n-ary connective is logically equivalent to a formula containing at most three $(n-1)$-ary connectives. Is there a value of n such that no n-ary connective is 3-optimal?

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