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Kevin O'Bryant
Rose-Hulman Institute of Technology

D. Patrick
Carnegie Mellon University

Lawren Smithline
Harvard University

Eric Wepsic
Harvard University

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K. O'Bryant, D. Patrick, L. Smithline and E. Wepsic

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Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute, IN 47803

FAX(812) 877-3198 Phone: (812) 877-8391

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K. O'Bryant* D. Patrick* L. Smithline* E. Wepsic*

1 Definitions

Note: In this paper, all groups mentioned are finite. We denote the identity element of a group G by e, and $G \setminus \{e\}$ by $G^{\times}$. Also, $H \leq G$ denotes that H is a subgroup of G.

Recall that the centralizer of an element $x \in G$ can be defined by

$$C(x) = \{y \in G \mid \langle x, y \rangle \text{ is abelian}\}.$$

If, in the above definition, we replace the word "abelian" with the word "cyclic", we get a subset of the centralizer, called the *cyclicizer*. To be explicit, define the cyclicizer of an element $x \in G$, denoted $\mathrm{Cyc}(x)$, by

$$\mathrm{Cyc}(x) = \{y \in G \mid \langle x, y \rangle \text{ is cyclic}\}.$$

Some properties of cyclicizers are discussed in [1].

Also, just as the center of the group can be defined by

$$Z(G) = \bigcap_{x \in G} C(x),$$

we may define a similar construct, called the *cycel*[1], denoted $K(G)$, by

$$K(G) = \bigcap_{x \in G} \mathrm{Cyc}(x).$$

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[1]The notation for the cycel comes from the Hungarian word *kerek*, which means "round".

2 Properties of the cycel

2.1 Basic properties

It is clear from the definitions above that $Cyc(x) \subseteq C(x)$ for all $x \in G$, and thus $K(G) \subseteq Z(G)$. It is also clear that for all $g \in K(G)$ and all $x \in G$, we have $\langle x, g\rangle$ cyclic. This allows us to prove the following:

Theorem 1 *For all groups G, $K(G) \triangleleft G$.*

Proof: Let $g, h \in K(G)$ and $x \in G$ be given. Then $\langle g^{-1}, x\rangle = \langle g, x\rangle$, which is cyclic, so $g^{-1} \in K(G)$. Also, $\langle gh, x\rangle \leq \langle g, h, x\rangle$, which is cyclic, so $gh \in K(G)$. So $K(G) \leq G$. But $K(G) \leq Z(G)$ as well, so $K(G)$ is normal. ■

Recall that $Z(G)$ can also be defined as the intersection of the maximal abelian subgroups of G. We can characterize $K(G)$ similarly.

Theorem 2 *For all groups G, $K(G)$ is the intersection of the maximal cyclic subgroups of G.*

Proof: Let x be in the intersection of the maximal cyclic subgroups of G, and pick arbitrary $g \in G$. Then g is contained in some maximal cyclic subgroup $\langle h\rangle$ of G. But $x \in \langle h\rangle$ as well, so $\langle x, g\rangle \leq \langle h\rangle$. Hence g and x generate a cyclic subgroup. Since this is true for all $g \in G$, we have that $x \in K(G)$.

Conversely, Let $x \in K(G)$ and $g \in G$ be given such that $\langle g\rangle$ is a maximal cyclic subgroup of G. Then $\langle x, g\rangle$ is cyclic, but since $\langle g\rangle$ is maximal we must have $\langle x, g\rangle = \langle g\rangle$. Thus $x \in \langle g\rangle$. Since this is true for all maximal cyclic subgroups of G, we must have x in the intersection of the maximal cyclic subgroups of G. ■

An immediate corollary of Theorem 2 is

Corollary 3 *For all groups G, $K(G)$ is cyclic.* □

Let us summarize some of the similarities of the center and the cycel:

$Z(G)$	$K(G)$
Intersection of centralizers	Intersection of cyclicizers
Intersection of maximal abelian subgroups	Intersection of maximal cyclic subgroups
$\langle x, g\rangle$ abelian for $x \in Z(G)$, $g \in G$	$\langle x, g\rangle$ cyclic for $x \in K(G)$, $g \in G$
$Z(G)$ abelian	$K(G)$ cyclic

We may also wish to consider the cyclic analog of nilpotency, which could be termed "cycelpotency", by constructing the "ascending cycel series":

$$\langle e \rangle, K(G), K(G/K(G)), \ldots.$$

However, this concept proves to be trivial, as shown in the following theorem:

Theorem 4 *For all groups G, $K(G/K(G)) = \langle e \rangle$.*

Proof: Let $K = K(G) = \langle k \rangle$, by Corollary 3. Suppose $L \in K(G/K(G))$. For all $c \in G$, where $C = cK$, there is a $D \in G/K$ such that $\langle L, C \rangle = \langle D \rangle$. But $L = lK, D = dK$ for some $l, d \in G$, and hence

$$\langle l, c, k \rangle = \langle l, c \rangle K = \langle d \rangle K = \langle d, k \rangle = \langle d' \rangle$$

for some $d' \in G$. In particular $\langle l, c, k \rangle$ is cyclic, thus $\langle l, c \rangle \leq \langle l, c, k \rangle$ is cyclic. But $c \in G$ is arbitrary, so $l \in K(G)$ and hence $L = e$. ∎

2.2 Miscellaneous properties

The above theorem deals with the cycel of $G/K(G)$. We can also prove a similar result about $G/Z(G)$. First we need the following technical lemma.

Lemma 5 *For all groups G, if $gZ \in (G/Z(G))^{\times}$, then $\langle \mathrm{Cyc}(gZ) \rangle \neq G/Z(G)$.*

Proof: Suppose $hZ \in \mathrm{Cyc}(gZ)$. Then $\langle gZ, hZ \rangle$ is cyclic in $G/Z(G)$, so $\langle g, h, Z \rangle$ is abelian in G. Thus $h \in C(g)$. Thus, since $C(g) \leq G$, we have that if h is in the preimage of $\langle \mathrm{Cyc}(gZ) \rangle$, then $h \in C(g)$. Thus, since $gZ \neq e$, we have that $C(g) \neq G$, so $\langle \mathrm{Cyc}(gZ) \rangle \neq G/Z$. ∎

This immediately gives the following:

Theorem 6 *For all groups G, $K(G/Z(G)) = \langle e \rangle$.*

Proof: This follows immediately from Lemma 5, since if $\bar{g} \in K(G/Z(G))$, then $\mathrm{Cyc}(\bar{g}) = G/Z$, so by Lemma 5 we must have $\bar{g} = e$. ∎

Lemma 5 also gives us the following interesting result on the structure of G/Z. Recall that a non-abelian group is said to be *Dedekind* if all of its proper subgroups are normal.

Theorem 7 *For all groups G, G/Z is not a Dedekind group.*

Proof: Suppose G/Z is a Dedekind group. Then $G/Z = Q \times A$, where Q is the quaternion group and A is an abelian group. Consider the element $(-1,0)$ of G/Z. Then $\mathrm{Cyc}((-1,0)) = (Q \times \{0\}) \cup (Q \setminus \{-1,1\} \times A)$, and thus $\langle \mathrm{Cyc}((-1,0)) \rangle = G/Z$, a contradiction of Lemma 5. ∎

We can also characterize the p-groups with non-trivial cycels.

Theorem 8 *Suppose G is a p-group. Then $K(G) \neq \langle e \rangle$ if and only if G is cyclic or generalized quaternion. Moreover, if $K(G) \neq \langle e \rangle$, then $K(G) = Z(G)$.*

Proof: If G is cyclic, then $K(G) = Z(G) = G$. If G is generalized quaternion, then every maximal cyclic subgroup contains the center (see [2]), so by Theorem 2, $K(G) = Z(G)$.

Conversely, suppose that G is a p-group with non-trivial cycel. Then by Corollary 3, $K(G)$ is cyclic, so $K(G)$ has exactly $p-1$ elements of order p. Suppose there is another element $x \in G$, with $x \notin K(G)$, such that $|x| = p$. Consider H, the maximal cyclic subgroups containing x. Then by the previous lemma, $K(G) \leq H$. But H is cyclic, and contains p elements of order p, a contradiction. Hence, G has exactly $p-1$ elements of order p, and by a well-known result (in [2], among other sources), G must be either cyclic or generalized quaternion. ∎

Finally, we have the result which indicates when a non-cyclic group G is spanned by a minimal number of cyclic subgroups.

Theorem 9 *G is the union of three proper cyclic subgroups if and only if G is isomorphic to either $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{C}$ or $\mathbf{Q} \times \mathbf{C}$, where $\mathbf{C}$ is a cyclic group of odd order, and $\mathbf{Q}$ is the quaternion group of order 8.*

Proof: It is clear that both $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{C}$ and $\mathbf{Q} \times \mathbf{C}$ are the union of three proper cyclic subgroups. Bruckheimer, Bryan and Muir [3] have shown that a group is the union of three proper subgroups if and only if their intersection N is normal and $G/N \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. It follows from Theorem 1, Theorem 2, and [3] that the groups spanned by three cyclic subgroups are precisely those for which $G/K(G) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. We now classify such groups.

First, note that G is nilpotent. This can be seen by observing that

$$G/Z(G) \cong (G/K(G))/(Z(G)/K(G))$$

is a factor group of an abelian group, and so is abelian. Thus, we can write $G \cong Syl_2 \times Syl_3 \times Syl_5 \times \cdots$. Since $\mid G/K \mid = 2^2$, $K(G)$ must contain $Syl_3 \times Syl_5 \times \cdots$. It follows from Corollary 3 that $\mathrm{C} \cong Syl_3 \times Syl_5 \times \cdots$ is cyclic and of odd order. Note that we have also proven that $|K(G)| = 2^a|\mathrm{C}|$, where a is a nonnegative integer.

Suppose $K(G) \cap Syl_2 = \langle e \rangle$. Then $|Syl_2| = |G/K(G)| = 4$. If Syl_2 were cyclic, then G would be as well. In that case, however, G would not be the union of three proper cyclic subgroups. Hence $Syl_2 \cong \mathbf{Z}_2 \times \mathbf{Z}_2$, and $G \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathrm{C}$. Now, suppose that $K(G) \cap Syl_2 \neq \langle e \rangle$. It is clear that $K(G) \cap Syl_2 \leq K(Syl_2)$, from which we see that Syl_2 is a 2-group with a non-trivial cycel. It follows from Theorem 8 that Syl_2 is cyclic or generalized quaternion. If Syl_2 were cyclic, then G would also be cyclic and, consequently, not the union of three proper cyclic subgroups. If Syl_2 is generalized quaternion, then by Theorem 8, $|K(Syl_2)| = |Z(Syl_2)|$. But the center of a generalized quaternion is of order 2 (see [2]). We can now see that $|K(G) \cap Syl_2| \leq |K(Syl_2)| = 2$, and so $|K(G)| = 2^a|\mathrm{C}| = 2|\mathrm{C}|$. Thus, $|G| = |G/K(G)| \cdot |K(G)| = 4 \cdot 2|\mathrm{C}|$. Hence, $|Syl_2| = 8$ and Syl_2 is the quaternion group of order 8. ■

3 Tidy groups

Note that unlike its analog the centralizer, a cyclicizer is not necessarily a subgroup. For example, in the group $\mathbf{Z}_4 \times \mathbf{Z}_2$, the cyclicizer of the element $(2,0)$ is of order 6:

$$\mathrm{Cyc}((2,0)) = \{(0,0),(1,0),(1,1),(2,0),(3,0),(3,1)\}.$$

A natural question to ask is: what can be said about those groups in which all of the cyclicizers are actually subgroups?

Let G be a group. G is said to be *tidy* if $\mathrm{Cyc}(x)$ is a subgroup for all $x \in G$. We begin with the following lemma on direct products of tidy groups.

Lemma 10 *Let G and H be groups such that $|G|$ and $|H|$ are relatively prime. Then $G \times H$ is tidy if and only if G and H are tidy.*

Proof: Suppose G and H are such that $|G|$ and $|H|$ are relatively prime, and that G and H are both tidy. Then for all $(g,h) \in G \times H$, $\mathrm{Cyc}((g,h)) = \mathrm{Cyc}(g) \times \mathrm{Cyc}(h)$ (since for all $(g_1,h_1),(g_2,h_2) \in G \times H$, we have $\langle (g_1,h_1),(g_2,h_2) \rangle = \langle g_1, g_2 \rangle \times \langle h_1, h_2 \rangle$). Hence $\mathrm{Cyc}((g,h))$ is a subgroup.

Conversely, suppose that $G \times H$ is tidy. Then for all $g \in G$, we have that $\mathrm{Cyc}(g) = \pi_G(\mathrm{Cyc}((g,e)))$ (where π_G is the projection homomorphism from $G \times H$ to G). So $\mathrm{Cyc}(g)$ is a subgroup. ■

We now classify all the abelian tidy groups.

Theorem 11 *Let G be an abelian group. Then G is tidy if and only if every p-Sylow subgroup of G is cyclic or elementary abelian.*

Proof: By the previous lemma, G is tidy if and only if each p-Sylow subgroup of G is tidy. Let P be a p-Sylow subgroup of G. If P is cyclic, then for all $x \in P$, we have $\mathrm{Cyc}(x) = P$. If P is elementary abelian, then every $x \in P^{\times}$ is of order p, so $\mathrm{Cyc}(x) = \langle x \rangle$. Otherwise, P contains a subgroup isomorphic to $\mathbf{Z}_p \oplus \mathbf{Z}_{p^2}$, which is not tidy (since the element $(0,p)$ has a cyclicizer which is not a subgroup). ■

We also classify tidy p-groups. First we need the following lemma:

Lemma 12 *Suppose G is a tidy p-group. Let $x \in G$ be such that $|x| \neq p$, and $\langle x \rangle$ is a maximal cyclic subgroup. Then $Z(G) \leq \langle x \rangle$.*

Proof: Let $z \in Z(G)$ be given, and consider x as above. Then $\langle z, x \rangle$ is abelian. Furthermore $\langle z, x \rangle$ is not elementary abelian, since $|x| \neq p$. So by Theorem 11, $\langle z, x \rangle$ is cyclic. But $\langle x \rangle$ is maximal; hence $z \in \langle x \rangle$. Since $z \in Z(G)$ was arbitrary, we conclude that $Z(G) \leq \langle x \rangle$. ■

An immediate corollary is

Corollary 13 *Suppose G is a tidy p-group. Then $Z(G)$ is cyclic.* □

This allows us to prove the following:

Theorem 14 *Suppose G is a p-group. Then G is tidy if and only if there exists $H \triangleleft G$, where H is cyclic or generalized quaternion, and for all $x \in G \setminus H$, we have $|x| = p$.*

Proof: Suppose G is a tidy p-group. Let $H = \mathrm{Cyc}(z)$, where z is a non-identity element of $Z(G)$. First, consider any element $x \in G \setminus H$, with $|x| \neq p$. Then x is contained in some maximal cyclic subgroup $< y >$, with $|y| \neq p$, so by Lemma 12, $z \in \langle y \rangle$. But then $\langle x, z \rangle \in \langle y \rangle$, so $\langle x, z \rangle$ is cyclic, and hence $x \in \mathrm{Cyc}(z) = H$.

Next, we observe that $z \in K(H)$, since $\langle z, h \rangle$ is cyclic for any $h \in H$, so by Theorem 8, H is either cyclic or generalized quaternion. Finally, if $\langle h \rangle$ is a maximal cyclic subgroup of H, with $|h| > p$, then any conjugate of $\langle h \rangle$ is a cyclic subgroup of order greater than p, and hence must be in H. So $H \triangleleft G$.

For the converse, suppose G is a p-group with a normal subgroup H as above. Then if $x \in H$, $\mathrm{Cyc}(x)$ is a subgroup (since cyclic and generalized quaternion groups are themselves tidy), and if $x \notin H$, then $|x| = p$, so $\mathrm{Cyc}(x) = \langle x \rangle$. So $\mathrm{Cyc}(x)$ is a subgroup for all $x \in G$, and hence G is tidy. ■

4 References

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[3] M. Bruckheimer, A.C. Bryan, and A. Muir, "Groups Which Are the Union of Three Subgroups", American Mathematical Monthly 77 (1970).

David Patrick	Eric Wepsic & Lawren Smithline	Kevin O'Bryant
Carnegie Mellon Univ.	Harvard University	Rose-Hulman Inst. of Tech.
Pittsburgh, PA 15213	Cambridge, MA 02138	Terre Haute, IN 47803