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Abstract

Let G be a finite group and let $\Delta_i(G)$ be the proportion of triples (x, y, z) of elements in G such that the cardinality of $\{xyz, xzy, yxz, yzx, zxy, zyx\}$ is i. In this paper we show that:

- The average value of i is either 1 or at least 53/32.
- $\Delta_2 = 0 \Rightarrow \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 0$;
- $\Delta_3 = 0 \Rightarrow \Delta_4 = \Delta_5 = 0$.

1 Notation and Classification of Triples

Given a triple of elements (x, y, z) from a finite group G, the set $\{xyz, xzy, yxz, yzx, zxy, zyx\}$ of formal products can have cardinality ranging from 1 to 6. This report investigates the probability that a triple from a group yields a given number of distinct products. This problem was originally suggested as a research topic by Annin, Sherman, and Ziebarth [2].

We define $\Delta_i(G)$ to be the probability that a triple chosen at random from the elements of G yields precisely i distinct products. If G is finite and N_i is the number of triples in G that yield i distinct products, then $\Delta_i = N_i/|G|^3$. Clearly, $\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6 = 1$.

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It is important to note that Abelian groups are essentially uninteresting from the point of view of this study: If a group is Abelian, then all its elements commute and every triple yields exactly one distinct product. So if A is an Abelian group, $\Delta_1(A) = 1$ and $\Delta_2(A) = \Delta_3(A) = \Delta_4(A) = \Delta_5(A) = \Delta_6(A) = 0$. Further, taking the direct product of any group with an Abelian group produces no change in the probabilities of triples of various types:

Fact 1 *If G is a group and A is an Abelian group, then for all $1 \leq i \leq 6$, $\Delta_i(G) = \Delta_i(G \otimes A)$.*

PROOF: Let (x, y, z) be a triple in G, and suppose that (x, y, z) yields i distinct products. Consider any of the $|A|^3$ triples $((x, a_1), (y, a_2), (z, a_3))$, where a_1, a_2, and a_3 are elements of A. Since the a_i all commute, two products of these three elements are equal if and only if the corresponding products of x, y, and z are equal. Thus, the triple yields exactly as many distinct products, i, as (x, y, z). So, if N_i is the number of triples in G yielding i products, then

$$\Delta_i(G \otimes A) = \frac{N_i |A|^3}{|G \otimes A|^3} = \frac{N_i}{|G|^3} = \Delta_i(G).$$

□

If a triple (x, y, z) yields less than 6 distinct products, this indicates that one or more of the six formal products are equal. For our purposes, it will be helpful to divide the fifteen possible equalities into three classifications. The first classification of equalities, called **commutes**, occur when two elements of the triple (x, y, z) commute. These equalities always come in pairs; for example, x and y commute $\Leftrightarrow xyz = yxz \Leftrightarrow zxy = zyx$.

The second classification of equalities, **conjugate equalities**, involve an equality between two elements which are automatically conjugate. These equalities can also be viewed as equivalent to an element of the triple (x, y, z) commuting with a product of the other two. The two conjugacy classes possible divide these equalities into two sets (one set contains $xyz = yzx$, $yzx = zxy$, and $zxy = xyz$, and the other set contains $xzy = yxz$, $yxz = zyx$, and $zyx = xzy$). Two conjugate equalities are called **corresponding** if they involve the same triple element commuting with both the possible products of the other two. For example, $xyz = yzx$ and $xzy = zyx$ are corresponding conjugate equalities.

The last set of equalities are called **flip equalities**. These three equalities, $xyz = zyx$, $xzy = yzx$, and $yxz = zxy$, do not involve any group elements commuting.

A triple can also be classified by the distribution of equal formal products. We call a triple a $[n_1, n_2, ..., n_i]$ triple (where $n_1 \geq n_2 \geq \cdots \geq n_i$) if it yields i distinct products, and n_1 of the formal products are equal to a given value, n_2 of the formal products are equal to a second (distinct) value, and so forth. For example, if all six formal products are equal, the triple is a [6]-triple. If three are equal to one value, two are equal to a second value, and one is equal to a third value, the triple is a $[3, 2, 1]$-triple.

Appendix A contains a table showing the possible triple configurations. This information can be extremely useful for proving results about triples.

2 Formulae for Specific Triple Types

Here we follow the method originally used by J. Ellenberg [4] to express the number of triples 3-rewritable in a given number of ways in terms of three group constants. Though expressing the number of triples of a given type in terms of these constants is not generally possible, they do allow us to establish some equations which will lead up to upper bounds on the quantity $\Delta_1(G) + \Delta_2(G) + \Delta_3(G) + \Delta_4(G) + \Delta_5(G)$. Ellenberg's three constants are k, the number of conjugacy classes in G; s, the sum of the reciprocals of the conjugacy class sizes in G; and t, the number of mutually commuting triples in G divided by the $|G|^2$.

Fact 2 $\Delta_1 = t/|G|$.

PROOF: If a triple is mutually commutative, then clearly it yields only one product. Conversely, if a triple yields only one product, then $xyz = yxz$, implying that x and y commute; $yxz = yzx$, implying that x and z commute; and $xyz = xzy$, implying that y and z commute. Thus, the triple is mutually commutative. There are $t|G|^2$ mutually commutative triples, so there are $t|G|^2$ triples that yield one distinct product. Thus $\Delta_1(G) = t|G|^2/|G|^3 = t/|G|$. □

Fact 3 *The proportion of triples which have exactly two commutes (and thus yield two distinct values) is* $3(s-t)/|G|$.

PROOF: Consider the probability that x and y commute, x and z commute, but y and z do not commute. The number of triples such that x commutes with both y and z is given by the sum over all x in G of $|C(x)|^2$, where $C(x)$ is the centralizer of x. But as Ellenberg ([4]) points out, this sum is just $s|G|^2$. From this number must be subtracted the number of triples such that y and z commute as well, namely $t|G|^2$. So the total number of triples of this type is $(s-t)|G|^2$. By symmetry, there are equal number of triples such that the other two ways to have exactly two commutes occur. So the total number of such triples is $3(s-t)|G|^2$ and the proportion is $3(s-t)/|G|$. □

Fact 4 *The sums of the proportions of triples of type* $[4,2]$ *and of type* $[4,1,1]$ *is* $3(s-t)/|G|$.

PROOF: While this fact can be obtained from a combinatorial argument similar to the above, it is easier to appeal to the results on 3-rewritability. In a triple of type [4,1,1] or [4,2], the ordered triple (x,y,z) is 3-rewritable in precisely three ways (which is to say, xyz is equal to exactly 3 elements of the set $\{xzy, yxz, yzx, zxy, zyx\}$) exactly 2/3 of the time. Further, these are the only triples in which any ordered triple is 3-rewritable in precisely three ways (see the table in the Appendix). So the number of such triples is 3/2 times the number of 3-rewritable triples rewritable in precisely three ways (which is $2(s-t)|G|^2$, see [5]). It follows that the proportion is $3(s-t)/|G|^2$. □

Fact 5 *The probability of any given single equality occuring in a random triple in a group G is $k/|G|$.*

PROOF: The probability of a commute is well-known to be $k/|G|$. The probability of a conjugate equality is equal to the probability that one element commutes with the product of the other two, which is equal to the probability that two elements commute; i.e., $k/|G|$. Finally, by symmetry, the probability of a flip equality is also $k/|G|$. □

Theorem 1 *Each of the following is an upper bound on the sum $\Delta_1(G)+\Delta_2(G)+\Delta_3(G)+\Delta_4(G)+\Delta_5(G)$:*

- $12k/|G|$.
- $(12k - 11tS)/|G|$.
- $(12k - 21s + 10t)/|G|$.

PROOF: The probability of a single equality is $k/|G|$ by Fact 5. If there were no triples with more than one equality, the total proportion of triples with at least one equality would be $12k/|G|$ (since there are 15 equalities and commutes always come in pairs). Now count through each triple with more than one equality. Each of these will increase the number of triples with at least one equality by 1, but will reduce the number of such triples by a number greater than 1 (since it is one of the k triples containing two or more different equalities). Thus the total number of triples yielding at least one equality will be bounded above by $12k/|G|$.

The other two bounds come from refinements of the above procedure. In the first case, observe that $t/|G|$ of the triples are mutually commuting. This reduces the maximum proportion of each equality, occuring alone, to $(k - t)/|G|$ and thus the sum to $(12k - 11t)/|G|$.

Finally, take into account the number of triples containing two commutes, and the number of triples of the forms [4,1,1] and [4,2], as given in Facts 3 and 4. To obtain the highest possible number of remaining triples, we assume that all triples of the forms [4,1,1] and [4,2] are of the form [4,1,1] (which contains one fewer equality). Each of the $3(s-t)|G|^2$ triples containing two commutes contains a total of two commutes and two conjugate equalities. Each of the $3(s-t)|G|^2$ triples of type [4,1,1] contain a commute, two flips, and two conjugate equalities. Thus the proportion of triples is bounded above by

$$\frac{12k}{|G|} + \frac{3(s-t) - 12(s-t) + 3(s-t) - 15(s-t)}{|G|} = \frac{12k - 21s + 10t}{|G|}$$

□

3 Bounds on the Mean

The expression $M(G) = \Delta_1(G) + 2\Delta_2(G) + 3\Delta_3(G) + 4\Delta_4(G) + 5\Delta_5(G) + 6\Delta_6(G)$ represents the average number of distinct products yielded by a triple chosen randomly from G. This expression can be bounded both below and above using results from other publications.

Theorem 2 *G is Abelian, if and only if, $M(G) = 1$. If G is not Abelian, then $53/32 \leq M(G) < 6$ and 6 is the least upper bound.*

The proof of this theorem will be broken up into a series of lemmas.

Lemma 1 *G is Abelian, if and only if, $M(G) = 1$.*

PROOF: If G is Abelian, then every triple yields only one distinct product: $\Delta_1 = 1$, $\Delta_2 = \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 0$ and $M(G)$=1. If $M(G) = 1$, then $\Delta_1 = 1$. Thus every triple is mutually commutative which implies G is Abelian. □

Lemma 2 *If G is non-Abelian, then $M(G) \geq 53/32$.*

PROOF: The probability that a randomly chosen n-tuple in G is mutually commutative is bounded above by $3/2^n - 1/2^{2n-1}$ (see [6]). Thus, in the case of a 3-tuple, the probability that the triple is mutually commutative is less than or equal to $11/32$. Thus $\Delta_1(G) \leq 11/32$ for any non-Abelian group G. The minimal value of $M(G)$ will occur if $\Delta_1(G) = 11/32$ and no triple yields more than two distinct products. In this case $\Delta_2(G) = 21/32$, $\Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 0$, and $M(G) = 11/32 + 2(21/32) = 53/32$. □

We note that $M(D_4) = 53/32$. Clearly $M(G) < 6$: Any group contains triples (such as (e, e, e)) that yield only one distinct product. However, $M(G)$ can be arbitrarily close to 6.

Lemma 3 $\lim_{n\to\infty} M(S_n) = 6$.

PROOF: Since the number of conjugacy classes in S_n is equal to the number of partitions of n it follows that $k(S_n)/n! \to 0$ as $n \to \infty$ (see Andrews [1], p. 70). But from Theorem 1, we saw that $\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 \leq 12k/|G|$. So as n approaches infinity, $\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5$ approaches 0, and thus Δ_6 approaches 1; i.e., $M(S_n)$ approaches 6. □

The foregoing three lemmas suffice to establish Theorem 2. In closing, we note that Lemma 2 can easily be generalized to arbitrary n-tuples using the result from [6] cited in that proof. If we define $M_n(G)$ to be the extension of $M(G)$ to n-tuples, this yields the following result:

Fact 6 *For any non-Abelian group G, $M_n(G) \geq 2 - 3/2^n + 1/2^{2n-1}$.*

4 Characterization of Groups with $\Delta_i(G) = 0$

In this section, we prove a number of results governing when the various $\Delta_i(G)$ can be equal to zero. This allows us to characterize groups in terms of which of the $\Delta_i(G)$ are equal to zero. The results are summarized in the following theorem:

Theorem 3 *For any finite group G, the following hold:*

- $\Delta_1(G) \neq 0$,
- $\Delta_2(G) = 0 \Longrightarrow \Delta_3(G) = \Delta_4(G) = \Delta_5(G) = \Delta_6(G) = 0$,
- $\Delta_3(G) = 0 \Longrightarrow \Delta_4(G) = \Delta_5(G) = 0$.

For readability, the proof of Theorem 3 will again be broken up into a series of lemmas. We note in passing that it is clear that $\Delta_1(G) \neq 0$.

Lemma 4 $\Delta_2(G) = 0 \Rightarrow \Delta_3(G) = \Delta_4(G) = \Delta_5(G) = \Delta_6(G) = 0$.

PROOF: Suppose that $\Delta_2(G) = 0$, and suppose that G is not Abelian. Then there exist two elements x and y that do not commute. Consider the triple (x, y, e). Clearly this triple yields exactly two distinct products, xy and yx. So $\Delta_2(G) \neq 0$, a contradiction. This implies that G is Abelian, and thus $\Delta_3(G) = \Delta_4(G) = \Delta_5(G) = \Delta_6(G) = 0$. □

Lemma 5 $\Delta_3(G) = 0 \Rightarrow x^2 \in Z(G)$ *for each* $x \in G$.

PROOF: Suppose that $\Delta_3(G) = 0$ and that there exists $x \in G$ such that $x^2 \notin Z(G)$. Then there exists $y \in G$ such that x and y do not commute. Consider the triple (x, x, y). This triple yields the three products x^2y, xyx, and yx^2. Of these, x^2y and yx^2 cannot be equal, as that would imply that x^2 and y commute. Also, xyx cannot equal either of the other two products, as that would imply that x and y commute, and thus that x^2 and y commute. So this triple yields exactly three distinct products, and $\Delta_3(G) \neq 0$, a contradiction. □

Lemma 6 *If $x^2 \in Z(G)$ for each $x \in G$, then $\Delta_5(G) = 0$.*

PROOF: Suppose that $x^2 \in Z(G)$ for each $x \in G$ and that $\Delta_5(G) \neq 0$. Then there exists a triple (x, y, z) that yields precisely 5 distinct products. In this case, exactly one of the fifteen possible equalities between different formal products must hold.

Case 1: This equality is a commute. Then it is accompanied by a corresponding second equality (for example, if $xyz = yxz$, then $zxy = zyx$) and fewer than 5 distinct products are yielded.

Case 2: This equality is a conjugate equality. Say without loss of generality that $xyz = zyx$. Then

$$\begin{aligned} xyz &= zyx \Rightarrow \\ x^2yz &= xzyx \Rightarrow \\ yzx^2 &= xzyx \Rightarrow \\ yzx &= xzy \end{aligned}$$

So fewer than 5 distinct products are yielded.

Case 3: This equality is a flip equality. Say without loss of generality that $xyz = zxy$. Then

$$\begin{aligned} xyz &= zxy \Rightarrow \\ yx^2yz &= yxzxy \Rightarrow \\ x^2y^2z &= yxzxy \Rightarrow \\ zx^2y^2 &= yxzxy \Rightarrow \\ zx^2y &= yxzx \Rightarrow \\ zyx^2 &= yxzx \Rightarrow \\ zyx &= yxz \end{aligned}$$

So, again, fewer than 5 distinct products are yielded.

Thus, the triple (x, y, z) cannot exist, and $\Delta_5(G) = 0$. □

Lemma 7 $\Delta_3(G) = 0 \Rightarrow \Delta_4(G) = 0$.

PROOF: Suppose $\Delta_3(G) = 0$ and $\Delta_4(G) \neq 0$. It follows from Lemma 6 that the existence of a flip equality implies both other flip equalities and the existence of a conjugate equality implies a second conjugate equality from the other possible conjugacy class. Consulting the complete list of possible triple configurations, we see that there are only two ways that a triple yielding exactly four distinct products can be formed, given these conditions: either there are two corresponding conjugate equalities (for example, $xyz = yzx$ and $xzy = zyx$) and no others, or there is a commute and no other equalities. We will show that the existence of a triple satisfying either of these conditions implies the existence of a triple yielding exactly 3 distinct products.

Case 1: There are two corresponding conjugate equalities. Say without loss of generality that there exists a triple (x, y, z) such that $xyz = yzx$ and $xzy = zyx$ and no other equalities hold. Construct the triple (y, z, xz). The six products are $yzxz$, yxz^2, $zyxz$, $zxzy$, $xzyz$, and xz^2y. Note that $yzxz = xyz^2 = xz^2y$. Since this is a flip equality, it follows by Case 2 of Lemma 6 that $yxz^2 = zxzy$ and $zyxz = xzyz$. We now have at most three distinct products, xyz^2, yxz^2, and $zyxz$. If the first two are equal then x and y commute; if the first and third are equal then $yxz = zyx$; and if the last two are equal then $yxz = zyx$. None of these conditions hold, so (y, z, xz) yields exactly 3 distinct products.

Case 2: There is a commute. Say without loss of generality that there exists a triple (x, y, z) such that y and z commute and no other equalities hold. Construct the triple (xy, zx, x). The six products are $xyzx^2$, $xyxzx$, zx^2yx, zx^3y, x^2yzx, and xzx^2y. Note that $xyzx^2 = xzyx^2 = xzx^2y$. Since this is a flip equality, it follows by Case 2 of Lemma 6 that $xyxzx = zx^3y$ and $zx^2yx = x^2yzx$. So there are at most 3 distinct products, xzx^2y, zx^3y, and zx^2yx. If the first two are equal then x and z commute; if the first and third are equal then $yzx = xyz$; and if the last two are equal then x and y commute. None of these conditions hold, so (xy, zx, x) yields exactly 3 distinct products.

In either case $\Delta_3(G) \neq 0$, a contradiction. □

The foregoing four lemmas suffice to establish Theorem 3.

5 The $\Delta_4(G) = 0 \Rightarrow \Delta_5(G) = 0$ Conjecture

Conjecture 1 $\Delta_4(G) = 0 \Rightarrow \Delta_5(G) = 0$.

This conjecture is supported by experimental evidence (no counterexamples were found in checking all groups of order 100 or less, all groups of order 128, and all groups of order 256 using CAYLEY) but it seems difficult to prove. I present here the machinery developed working toward a proof.

The following lemma is analogous to Lemma 5:

Lemma 8 $\Delta_4(G) = 0 \Rightarrow x^2 \in Z(G)$ *or* $x^3 \in Z(G)$ *for each* $x \in G$.

PROOF: Suppose $\Delta_4(G) = 0$ and there exists $x \in G$ such that $x^2 \notin Z(G)$ and $x^3 \notin Z(G)$. Since the centralizers of x^2 and x^3 can each have order equal to at most half the group's and have nonempty intersection, there exists a y that does not commute with either x^2 or x^3. Now consider the triple (x, x^2, y). This yields the four products x^3y, x^2yx, xyx^2, yx^3, no two of which can be equal since this would imply that y commutes with x, x^2, or x^3. So $\Delta_4(G) \neq 0$, a contradiction, and the lemma follows. □

Unfortunately, the condition provided by this lemma does not necessarily imply that $\Delta_5(G) = 0$. Experiments with examples show that groups with $\Delta_4(G) = 0$ tend to fulfill one of two additional conditions:

- If $x^3 \in Z(G)$ and $y^3 \in Z(G)$, then $xy = yx$.
- $|G'| = 3$ and $x^2 \in Z(G)$ implies $x \in Z(G)$.

This problem is interesting because it would permit a complete determination of the possibile group configurations in terms of which of the Δ_i are zero. At present, the following configurations are known to be the only ones possible:

Configuration	Smallest Known Example
$\Delta_1 \neq 0;\ \Delta_2 = 0;\ \Delta_3 = 0;\ \Delta_4 = 0;\ \Delta_5 = 0;\ \Delta_6 = 0$	Trivial group (order 1)
$\Delta_1 \neq 0;\ \Delta_2 \neq 0;\ \Delta_3 = 0;\ \Delta_4 = 0;\ \Delta_5 = 0;\ \Delta_6 = 0$	D_4 (order 8)
$\Delta_1 \neq 0;\ \Delta_2 \neq 0;\ \Delta_3 \neq 0;\ \Delta_4 = 0;\ \Delta_5 = 0;\ \Delta_6 = 0$	D_3 (order 6)
$\Delta_1 \neq 0;\ \Delta_2 \neq 0;\ \Delta_3 \neq 0;\ \Delta_4 \neq 0;\ \Delta_5 = 0;\ \Delta_6 = 0$	D_5 (order 10)
$\Delta_1 \neq 0;\ \Delta_2 \neq 0;\ \Delta_3 \neq 0;\ \Delta_4 \neq 0;\ \Delta_5 \neq 0;\ \Delta_6 = 0$	group of order 20
$\Delta_1 \neq 0;\ \Delta_2 \neq 0;\ \Delta_3 \neq 0;\ \Delta_4 \neq 0;\ \Delta_5 \neq 0;\ \Delta_6 \neq 0$	group of order 21
$\Delta_1 \neq 0;\ \Delta_2 \neq 0;\ \Delta_3 = 0;\ \Delta_4 = 0;\ \Delta_5 = 0;\ \Delta_6 \neq 0$	group of order 128
$\Delta_1 \neq 0;\ \Delta_2 \neq 0;\ \Delta_3 \neq 0;\ \Delta_4 = 0;\ \Delta_5 = 0;\ \Delta_6 \neq 0$	group of order 54
$\Delta_1 \neq 0;\ \Delta_2 \neq 0;\ \Delta_3 \neq 0;\ \Delta_4 \neq 0;\ \Delta_5 = 0;\ \Delta_6 \neq 0$	group of order 128
$\Delta_1 \neq 0;\ \Delta_2 \neq 0;\ \Delta_3 \neq 0;\ \Delta_4 = 0;\ \Delta_5 \neq 0;\ \Delta_6 = 0$	none known
$\Delta_1 \neq 0;\ \Delta_2 \neq 0;\ \Delta_3 \neq 0;\ \Delta_4 = 0;\ \Delta_5 \neq 0;\ \Delta_6 \neq 0$	none known

For CAYLEY users, the five groups cited but not specified are stored as g20n2; g21n1; group number 7 in the gps128d3 library; g54n7; and group number 162 in the gps128d3 library.

6 Formulae for the Dihedral Groups

It is possible to find explicit formulae for the occurence of triples of each type in the specific case of the dihedral groups. The following theorem gives the details:

Theorem 4 *Let D_n denote the dihedral group on n symbols.*

$$\Delta_1(D_n) = \begin{cases} \frac{n^2+7}{8n^2} & \textit{if } n \textit{ is odd.} \\ \frac{n^2+28}{8n^2} & \textit{if } n \textit{ is even, but } 4 \not| n. \\ \frac{n^2+28}{8n^2} & \textit{if } n \textit{ is even, and } 4|n. \end{cases}$$

$$\Delta_2(D_n) = \begin{cases} \frac{15(n-1)}{8n^2} & \textit{if } n \textit{ is odd.} \\ \frac{15(n-2)}{4n^2} & \textit{if } n \textit{ is even, but } 4 \not| n. \\ \frac{3(5n-6)}{4n^2} & \textit{if } n \textit{ is even, and } 4|n. \end{cases}$$

$$\Delta_3(D_n) = \begin{cases} \frac{(n+10)(n-1)}{8n^2} & \textit{if } n \textit{ is odd.} \\ \frac{(n+20)(n-2)}{8n^2} & \textit{if } n \textit{ is even, but } 4 \not| n. \\ \frac{(n+22)(n-4)}{8n^2} & \textit{if } n \textit{ is even, and } 4|n. \end{cases}$$

$$\Delta_4(D_n) = \begin{cases} \frac{3(n-1)(n-3)}{4n^2} & \textit{if } n \textit{ is odd.} \\ \frac{3(n-2)(n-6)}{4n^2} & \textit{if } n \textit{ is even, but } 4 \not| n. \\ \frac{3(n-4)^2}{4n^2} & \textit{if } n \textit{ is even, and } 4|n. \end{cases}$$

$$\Delta_5(D_n) = \Delta_6(D_n) = 0.$$

PROOF: Suppose that n is odd. Exactly half of the elements of D_n are rotations, and exactly half are reflections. Further, note that two reflections always commute; a rotation and a reflection commute only if the rotation is the identity; and two reflections commute only if they are equal (in which case their product is equal to the identity). There are $8n^3$ possible triples of elements in D_n: n^3 triples of three rotations, $3n^3$ triples of two rotations and a reflection, $3n^3$ triples of a rotation and two reflections, and n^3 triples of three reflections.

First, consider a triple of three rotations. In this case, all three pairs of elements of the triple commute, and only one distinct product is yielded; so we have n^3 triples yielding 1 distinct product.

Next consider a triple of two rotations and one reflection; call the rotations R_1 and R_2 and the reflection F. Since R_1 and R_2 commute, $R_1R_2F = R_2R_1F$ and $FR_1R_2 = FR_2R_1$. There is a $1/n^2$ probability that $R_1 = R_2 = e$; in this case, only 1 distinct product is yielded. Also, there is a $2(n-1)/n^2$ probability that one, but not both, of $R_1 and R_2$ is the identity; then two distinct products are yielded, say R_1F and FR_1. There is a $(n-1)^2/n^2$ probability that neither R_1 nor R_2 is the identity; then there is only one commute. First, we note that $R_1R_2F = FR_1R_2$ if and only if $R_1R_2 = e$. Since we already know that R_1 and R_2 are not the identity, there is a $1/(n-1)$ probability that $R_1R_2 = e$. Thus, there is a $(n-1)/n^2$ probability that $R_1R_2F = FR_1R_2$, and a $(n-1)(n-2)/n^2$ probability that $R_1R_2F \neq FR_1R_2$.

It remains only to consider whether $R_1FR_2 = R_2FR_1$ (neither of these elements can equal any of R_1R_2F, R_2R_1F, FR_1R_2, or FR_2R_1 without implying a commute). This is true if and only if $R_1 = R_2$. If $R_1R_2 = e$, this is impossible (since no rotation in D_n, with n odd, is its own inverse). Otherwise, the chance that $R_1 = R_2$ is $1/(n-2)$ (since the cases $R_2 = e$ and $R_2 = R_1^{-1}$ have already been eliminated). So there is a triple yielding three distinct values with probability $2(n-1)/n^2$, and a triple yielding 4 distinct values with probability $(n-1)(n-3)/n^2$. The totals for the case of triples of two rotations and a reflection are thus (multiplying through by $3n^3$) $3n$ triples yielding 1 product, $6n(n-1)$ triples yielding 2 products, $6n(n-1)$ triples yielding 3 products, and $3n(n-1)(n-3)$ triples yielding 4 products.

Now consider a triple of two reflections and one rotation; call the reflections F_1 and F_2 and the rotation R. The rotation commutes with both of the reflections if it is the identity, and neither otherwise. Also, F_1 and F_2 commute if and only if $F_1 = F_2$. So if $F_1 = F_2$ and $R = e$, the triple yields only one value; the probability of this is $1/n^2$. If $R = e$ but $F_1 \neq F_2$, then the triple yields two products; the probability of this is $(n-1)/n^2$. Finally, if $F_1 = F_2$ but $R \neq e$, then $F_1F_2R = F_2F_1R = RF_1F_2 = RF_2F_1 = R$, and $F_1RF_2 = F_2RF_1$; the triple thus again yields 2 products.

There is a $(n-1)^2/n^2$ probability that $F_1 \neq F_2$ and $R \neq e$. But note that F_1F_2 and F_2F_1 are rotations, and thus commute with R. So $F_1F_2R = RF_1F_2$ and $F_2F_1R = RF_2F_1$. Also, these two pairs cannot be equal, as that would imply that F_1 and F_2 commute. The other two products, F_1RF_2 and F_2RF_1, cannot be equal, because they are equal only if $F_1 = F_2$, a case we have dispensed with. However, F_1RF_2 can be equal to $F_2F_1R = RF_2F_1$, or F_2RF_1 can be equal to $F_1F_2R = RF_1F_2$. (These equations cannot hold if F_1RF_2 and F_2RF_1 are switched, as that would imply a commute). If $F_1RF_2 = F_2F_1R$, then F_1R and F_2 commute, so $F_2 = F_1R$ and there is a $1/(n-1)$ probability of this. Likewise if $F_2RF_1 = RF_1F_2$, then $F_2 = RF_1$ and there is a $1/(n-1)$ probability of this. Obviously, both conditions cannot hold unless F_1 and R commute, a case already handled. So there is a $2/(n-1)$ probability that one of these conditions holds and the triple thus yields 3 distinct values. Otherwise, on the remaining $(n-1)(n-3)/n^2$ probability, the triple yields 4 products. So, the totals for this case (mulitplying through by $3n^3$) yield $3n$ triples giving 1 product, $6n(n-1)$ triples yielding 2 products, $6n(n-1)$ triples yielding 3 products, and $3n(n-1)(n-3)$ triples yielding 4 products.

Finally, the last possibility is that the triple consists of three reflections; call these F_1, F_2, and F_3. Here, we have three commuting pairs if, and only if, $F_1 = F_2 = F_3$; this case occurs with probability $1/n^2$, yielding only one product. Also, there is a $3(n-1)/n^2$ probability that one pair commutes, i.e. $F_1 = F_2$, $F_1 = F_3$, or $F_2 = F_3$. Examining the first case without loss of generality, we see that $F_1F_2F_3 = F_2F_1F_3 = F_3F_1F_2 = F_3F_2F_1 = F_3$ and $F_1F_3F_2{=}F_2F_3F_1$, so the triple yields two distinct products.

Suppose no pairs commute. Note that the product of three reflections is a reflection, and that a reflection is its own inverse. Now $(F_1F_2F_3)(F_3F_2F_1) = e$, so $F_1F_2F_3 = F_3F_2F_1$, $F_2F_1F_3 = F_3F_1F_2$, and $F_1F_3F_2 = F_2F_3F_1$. None of these pairs can be equal to each other without implying a commute so these triples yield 4 distinct products and the probability of such a triple is $(n-1)(n-2)/n^2$. So the totals for this case are n triples yielding one product, $3n(n-1)$ triples yielding 2 products, and $n(n-1)(n-2)$ triples giving 3 products.

Totalling the above values gives the quantities cited in the statement of the theorem above for odd n. Further, by Fact 1 and the fact that, for odd m, $D_{2m} \approx D_m \otimes Z_2$, the figures for even n that are not multiples of four follow.

It remains to establish the formulae for n divisible by 4. This proof is similar to the above (taking into consideration the added possibilities for commuting elements) and is omitted. $\square$

7 Classification of Groups

An open problem involves characterizing the groups that fall into each of the categories tabulated in Section 5. One approach centers around $I(G) = \max\{i : \Delta_i(G) \neq 0\}$; i.e., the maximum number of distinct products produced by any triple in the group G. Several results about $I(G)$ are immediate.

Fact 7 *$I(G) = 1$ if and only if G is Abelian.*

Fact 8 *$I(G) = 2$ if and only if $|G'| = 2$.*

PROOF: Suppose that $|G'| = 2$. Any permutation of the formal product xyz can be written as xyz multiplied by some element of the commutator subgroup. But the commutator subgroup has only two elements, so there are only two possible products yielded, and the number of distinct products is at most 2. If the number of distinct products were never 2, the group would be Abelian and $|G'|$ would be 1, a contradiction. So $I(G) = 2$.

Suppose, conversely, that $I(G) = 2$. Then, since it is impossible for a triple to be of the form $[5,1]$, the group is 3-rewritable (i.e. for any triple (x, y, z), the product xyz is equal to at least one of the five other formal products). It follows that $|G'| \leq 2$ [3]. Since $|G'| \neq 1$ (because then $I(G)$ would be 1), we have $|G'| = 2$. $\square$

Fact 9 *If $|G'| = 3$, then $I(G) = 3$.*

Proof: The proof is analogous to that of Fact 8. If $|G'| = 3$, then there can be no more than three distinct products yielded by any triple. Some triple must yield three products since otherwise $|G'| < 3$. So $I(G) = 3$. $\square$

The evidence from groups of order less than 100 suggests the following conjectures:

Conjecture 2 *If $I(G) = 3$, then $|G'| = 3$.*

Conjecture 3 *If $|G'| = 4$, then $I(G) = 4$.*

Conjecture 4 *$I(G) = 5$ if, and only if, $|G'| = 5$ and $k/|G| = 1/4$.*

Conjecture 5 *If $|G'| = 6$ and $k/|G| \leq 1/4$, then $I(G) = 6$.*

We finish this discussion of classification with the following corollary of Theorem 1:

Corollary 1 *If any of the following conditions hold, then $I(G) = 6$.*

- $12k/|G| < 1$
- $(12k - 11t)/|G| < 1$
- $(12k - 21s + 10t)/|G| < 1$

Proof: From Theorem 1, each of the above quantities forms an upper bound on the sum $\Delta_1(G) + \Delta_2(G) + \Delta_3(G) + \Delta_4(G) + \Delta_5(G)$. So, if any of them is less than 1, then $\Delta_6(G)$ must be nonzero and therefore $I(G) = 6$. $\square$

A Table of Triple Configurations

Conjugate equalities are abbreviated as "c.e.s".

Number of Distinct Products	Triple Type	Equalities
1	[6]	All fifteen
2	[3,3]	Two commutes, two corr. c.e.s
		All six conjugate equalities
	[4,2]	One commute, two corr. c.e.s, three flips
3	[4,1,1]	One commute, two corr. c.e.s, two flips
	[3,2,1]	Four conjugate equalities (3 from one class)
	[2,2,2]	One commute and one flip
		Two corr. c.e.s and one flip
		Three flips
4	[3,1,1,1]	Three c.e.s from one class
	[2,2,1,1]	One commute
		Two corr. c.e.s
		Two non-corr. c.e.s from diff. classes
		One c.e. and one flip
		Two flips
5	[2,1,1,1,1]	One conjugate equality
		One flip
6	[1,1,1,1,1,1]	No equalities

It should be noted that, although this information is not listed in the table above for brevity, an entry of the form "x and one flip" or "x and two flips" generally implies a specific flip equality.

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