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Tyson Lipscomb
Marshall University, lipscomb16@marshall.edu

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Gödel's Incompleteness Theorems: A Revolutionary View of the Nature of Mathematical Pursuits

Tyson Lipscomb

Marshall University

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Abstract

The work of the mathematician Kurt Gödel changed the face of mathematics forever. His famous incompleteness theorem proved that any formalized system of mathematics would always contain statements that were undecidable, showing that there are certain inherent limitations to the way many mathematicians studies mathematics. This paper provides a history of the mathematical developments that laid the foundation for Gödel's work, describes the unique method used by Gödel to prove his famous incompleteness theorem, and discusses the far-reaching mathematical implications thereof.

I. Introduction

For thousands of years mathematics has been the one field of human thought wherein truths have been known with absolute certainty. In other fields of thought, such as science and philosophy, there has always been a level of imprecision or speculation. In mathematics, however, there are truths that we can know beyond a shadow of a doubt due to the methods of proof first devised by the ancient Greeks [10]. The unparallelled certainty of mathematics has throughout the past millennia led many to accept the notion that it would be possible to answer any mathematical question through expansive enough development of mathematical techniques [16].

This seemingly intuitive belief was shown to be in error by the famous logician and mathematician Kurt Gödel in his 1931 paper "On Formally Undecidable Propositions of Principia Mathematica and Related Systems." In this groundbreaking work, he provided a pair of mathematical theorems, the incompleteness theorems, that proved that any consistent formal system capable of interpreting arithmetic that was sufficiently strong and computably axiomatizable would necessarily contain statements that the system can neither prove nor disprove and that no such system could prove itself consistent. The age-old belief that mathematics could someday be developed to a point that it would be able to answer all mathematical questions was disabused, raising many questions about what humanity could and could not know.

Gödel's theorems were surprising to many because the theorems differed from many other famous mathematical results in the way that they came completely unexpectedly and did not require particularly difficult mathematical thought [4]. Unlike many proofs whose impact were comparable to that of Gödel's, his did not require much knowledge beyond that of basic arithmetic and logic. Though complex and insightful, they do not overwhelm the reader with advanced mathematical concepts and present a surprisingly concise proof. For this reason Gödel's work stands out from many other groundbreaking mathematical proofs.

II. Historical Overview

Before we can fully grasp all of the intricacies of Gödel's famous theorems, an overview of the mathematical developments throughout history that laid the foundation for his work is necessary. As with all mathematicians and mathematical developments, Gödel and his incompleteness theorems built upon the work of countless other developments in mathematics that made up the mathematical culture of their day. Gödel's work harkens back to and incorporates many mathematical concepts that have developed over time, from the fundamental theory of axiomatic systems made famous by Euclid over two millennia ago to the modern mathematical concept of formalism established by David Hilbert. Gödel's work is the culmination of the thinking of his great predecessors and laid the foundation for many more mathematical developments that took place throughout the 20th century.

The most fundamental concept of modern mathematics is that of the axiomatic method. Though widely in use today, it has undergone many transformations throughout its history and has matured through the ages into what it is today [19]. Axioms, from a mathematical standpoint, are statements within a specific branch of mathematics that are taken to be true and accepted without proof. Initially, the thought of mathematicians accepting a collection of statements without demonstrating their veracity through a rigorous series of proofs may seem strange. After all, mathematics is often viewed as the one branch of scientific thought where we can be absolutely certain of anything, a field where we are not subject to the limitations of our observations. We are able to accept the axioms of most fields of mathematical study as true based on their intuitively true nature. Take for example one of Euclid's geometric axioms: All right angles are equal to each other. This seemingly trivial axiom is a good example of why the reliability of mathematics is not undermined when we accept statements without proof in axiomatic systems. They are simply too fundamental to reject.

In addition to the axioms themselves, axiomatic systems consist of two other major parts: rules of inference and the theorems within the system [10]. Obviously, a small set of unequivocally true statements is of little use to mathematicians unless something can be done with them. The rules of inference are a set of truth-preserving logical operations that can be performed on the axioms, usually mirroring the basic system of classical logic. In short, the rules of inference allow us to combine sequences of axioms to arrive at new, more complex, statements. If we use only our axioms and the rules of inference that govern their interaction, the true statements that we create using them will be certifiably true and are called theorems. The derivation of new theorems is the ultimate goal of any axiomatic system [19].

The process of deriving theorems can be expanded further by using collections of axioms and the theorems directly derived from them to derive new theorems. Since we know that the rules of inference preserve the truth of a collection of statements and the theorems derived directly from the axioms are true, it is obvious that the theorems derived from them must also be true. This process of deriving new theorems can be continued on indefinitely, revealing new sets of truths that would not necessarily be obvious from the axioms, but are the direct implications thereof.

Though the precise origin of the axiomatic method is unknown, Euclid is its most well-known founder [19]. In his famous book *Elements* he sets forth five geometric axioms and proceeds to build a system of geometry upon them that has been in use ever since. Euclid's method of establishing the field of geometry through the use of an axiomatic system provides valuable insight into the way that axioms can be used to create a highly complex system of mathematics from a few simple rules. His five famous axioms are as follows:

1. A straight line can be drawn between any two points.
2. A straight line segment can be extended indefinitely.
3. A circle can be described using any center point and distance.
4. All right angles are equal to each other.
5. Parallel lines do not intersect.

These statements may seem too basic or trivial to be of use in any developed branch of mathematics, but the axiomatic method demonstrates that the axioms upon which we base our mathematics need not be highly complex. Using this small set of axioms and the basic rules of inference, Euclid created a highly complex system of plane geometry that was widely used throughout the western world for millennia and continues to be used to this day.

Until recent times Euclid's geometry was the only branch of mathematics that had been thoroughly axiomatized. Other fields took a less formal approach and relied on basic human intuitions instead of a rigid set of axioms. Many mathematicians simply did not see the need to spend their time creating systems of axioms for concepts, such as number theory, that humans have been exploring since ancient times. Axiomatizing these systems seemed wasteful because our minds grasped them so intuitively.

This mathematical mindset begin to change during the 19^{th} century, however. David Hilbert's work during this time set the tone of the mathematical world and showed the importance of axiomatic systems to modern mathematics [19]. With new fields of mathematics emerging that did not necessarily parallel our natural intuitions, it was obvious that relying on our intuitions to develop these fields would be counterproductive. Mathematicians needed a system of rules to govern their thought processes and proofs of more abstract fields, such as elliptic or hyperbolic geometry. For this reason modern mathematics has adopted Euclid's famous axiomatic method, allowing rules to be set in place for nascent fields of study.

The modern view of axiomatic systems differs from the classical view in the sense that the modern view takes the axioms as merely statements about a concept, whereas the classic Euclidean view defined axioms as “obviously true” statements. The classical method did not allow any two mathematical systems to have contradictory or incompatible axioms, whereas in the modern view this is possible [19]. As an example, observe Euclidean and elliptic geometry. The axioms of Euclidean geometry state that parallel lines never intersect, whereas the axioms of elliptic geometry ensure that all parallel lines intersect. A concept that will undoubtedly catch the attention of some is that of allowing two systems of mathematics to have contradictory axioms. After all, would this not cause a division in mathematics, allowing two conflicting, yet equally valid, theories to emerge? Parallel fields of mathematics do not cause the problems that one might anticipate, however, and instead allow us to gain more insight into mathematics.

Take as an example Euclidean geometry and elliptic geometry, where the fifth of Euclid's axioms is discarded and no lines are considered parallel in the Euclidean sense. We can think of work with Euclidean geometry “occurring” on a flat plane, such as a table or piece of paper, and elliptical geometry “occurring” on a sphere, such as the surface of the earth. It is apparent that two parallel lines drawn on a plane and extended indefinitely will never intersect, but this is not the case on the surface of a sphere, where two parallel lines, such as lines of longitude, may intersect. These two seemingly conflicting branches of mathematics do not create any mathematical problems, if kept properly limited to their intended scopes, and instead offer new insight into branches of mathematics that would not be explored otherwise.

Elliptic geometry is an example of a mathematical field that begins to stretch the limits of our intuitive mathematical understanding. But what is the case when a new mathematical field goes beyond what we are able to experience or perceive? We are able to create analogues to some of the more abstract mathematical fields, such as in the previous example of elliptic geometry and spheres. However, problems arise when developing theories of mathematics become further divorced from our sensory perceptions, such as higher-dimensional geometry.

III. Mathematical Consistency

With many highly abstract fields of mathematics emerging throughout the 19th century and onward, the problem of consistency arose in many mathematical minds. Consistency as described in mathematics is the property a mathematical theory possesses when it is impossible to derive a theorem and its negation within that theory. In short, using a consistent system's axioms and rules of inference a proof for a theorem “A” and a proof for its negation “not-A” could never be created within the system. If mathematics is limited to describing the world around us, consistency does not cause much concern. The physical world dictates that 1+1 will never equal anything but 2, that two parallel lines on the same plane will never intersect and that we will never be able to combine certain principles to disprove anything that we have already proved.

In the more abstract fields of mathematics we simply do not have this same assurance. If a mathematician creates an axiomatic system describing a higher dimensional system of geometry, for example, how can anyone be sure that this system is consistent? Since there are no physical analogues to it, we could not hope to produce a proof for its consistency by comparing it to the world around us [16].

Though more basic branches of mathematics may be easier to prove consistent, mathematicians have always struggled with proofs of mathematical consistency, viewing them as much more difficult to achieve than proofs of usual mathematical statements [6]. This difficulty has not dissuaded mathematicians from conducting consistency proofs for various branches of mathematics, however. One technique widely utilized by mathematicians is that of a relative consistency proof. A relative consistency proof consists of “converting” the axioms of one branch of mathematics into another and showing that the concepts expressed in the first can be expressed in terms of the second. Using this method it is possible to prove the more abstract fields of mathematics, e.g., elliptic geometry, consistent. The drawback is that we must first assume the consistency of the first system, a fact that may or may not be established [16].

David Hilbert famously demonstrated that the axioms of Euclid's *Elements* can be converted into algebraic terms. Using the Cartesian coordinate system he was able to translate Euclid's system of geometry into a system composed of algebraic, instead of geometric, concepts. This relative consistency proof showed that Euclid's geometric system was consistent, but it was based on the assumption of the consistency of algebra [16]. The consistency of algebra, on the other hand, was still questionable in Hilbert's proof.

The problem with relative consistency proofs is obvious. They simply do not—and cannot—prove the consistency of one mathematical system without assuming the consistency of another. This then raises the question of whether or not the second system is consistent. Another relative consistency proof can then be performed, but would only assume the consistency of a third system of mathematics, which could then be called into question. This procedure could be carried on almost indefinitely and may be enough to allow us to gain a level of comfortable certainty about the consistency of one system, but many mathematicians would not be pleased with anything less than absolute certainty in mathematics. Without proving a mathematical assertion with absolute certainty we can never be assured that the implications that may follow will be consistent. Mathematics would therefore lose some of the certainty that allows us to rely so heavily upon it.

IV. Hilbert's Challenge

Not content to see mathematics go down a path that would potentially undermine its certitude, Hilbert began a program to establish beyond all doubt the consistency and verifiability of mathematics. Heavily influenced by the work he had completed in geometry during the late 19^{th} century, Hilbert believed that mathematicians should hold fast to axiomatic systems, espousing the belief that they were necessary for any science to be developed fully [20].

The beginning of Hilbert's program was marked by his speech "Mathematical Problems," delivered to the International Congress of Mathematicians in 1900 [1]. Hilbert's speech began with an expression of his thoughts on mathematics and the importance he believed rigorous proof of mathematical concepts possessed. He espoused the belief that rigorous proof provides more clarity than any alternative and that mathematicians should seek to find concrete, absolute proofs for any of the mathematical truths they encounter [12]. The strength of mathematics is found in its ability to prove beyond any doubt the truth of its statements. Other sciences, in contrast, must rely on observation and experiment, two methods that, compared to the rigor of mathematics, are too subjective to provide us with a level of assurance that parallels that of mathematics.

After having explained his views on mathematical exploration and the inexhaustibility thereof, he set forth a series of unresolved problems in mathematics and challenged mathematicians to find their answers, confident that the human mind was capable of living up to the challenge [12]. Before outlining the twenty-three problems that he felt mathematicians should seek to solve, he issued his famous call: "This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no *ignorabimus*" [12]. Hilbert's conviction of the importance of mathematical pursuits and faith in their continuation is summed up in this call and has served as an inspiration for countless mathematical minds, the most notable of these being Kurt Gödel.

The challenge that piqued the interest of Gödel was that to prove the axioms of mathematics "are not contradictory, that is, that a finite number of logical steps based upon them can never lead to contradictory results" [12]. In short, Hilbert called for a proof of consistency of the entirety of mathematics, seeking to do away with any uncertainty that may have arisen in years prior due to the development of both the increasingly complex fields of mathematical inquiry and the departure from intuitive mathematical models. Though many mathematicians may have traditionally accepted the consistency of mathematics, mathematical acceptance based on either faith or intuition did not coalesce with Hilbert's proposed system of rigorous proof [1]. Without a rigorous proof of mathematical consistency, Hilbert could argue, we would never have complete assurance.

In addition to the consistency of mathematics, Hilbert challenged mathematicians to provide a theory that was complete mathematically. Completeness in this regard meant that it would be possible to prove or disprove any meaningful statement created within the theory. Mathematicians using a complete theory would be assured that they would never encounter a proposition with a truth value that was undecidable. Every mathematical question they may ask within that theory could ultimately be answered. As with consistency, mathematical completeness is traditionally taken as true without the need of any proof. The thought of a mathematical proposition that could never be solved ran counter to Hilbert's beliefs about mathematics and would certainly make many other mathematicians uneasy.

V. Metamathematics

With the questions of consistency and completeness in mind, Hilbert and many of his contemporaries set out to prove the consistency and completeness of mathematics. To devise an undeniable proof of the consistency and completeness of mathematics was an undertaking unparalleled by any prior mathematical pursuit. For this reason it was necessary that a new approach be taken to mathematics. Hilbert's solution was metamathematics. If we think of mathematics as being a description of numbers and the way they interact, we can think of metamathematics as a description of mathematics and the various ways formulas and mathematical reasoning interact [16]. Metamathematics sought to approach mathematics with the rigor and certainty that mathematics approached numbers. If successful, Hilbert would be able to use his system of metamathematics to prove statements about mathematics, such as its consistency and completeness, with the same logical certitude and finesse as mathematicians prove statements about numbers, e.g., whether or not a number is prime, odd, even, etc. Once a strong enough system of metamathematics was established, proving the consistency of a theory of mathematics would only require proving that it was not possible to derive both a formula an its negation within the theory [19]. Similarly, proving a theory of mathematics complete would be comparably straightforward.

One tool that Hilbert felt would be invaluable in furthering his system of metamathematics was the system of formalism. The framework of formalism provides a technical description of the way that metamathematics should be carried out by providing a system of formal logic rules that can be used in the proof of various concepts, whether mathematical or metamathematical. Formalism differs from mathematics and strengthens metamathematics in the sense that it looks at the logical underpinnings of mathematics itself, an endeavor that had not been undertaken until relatively recent times [16].

Formalism may be thought of as being a form of a mathematical system in which all inherent meaning has been removed. Instead of dealing with numbers, angles, or geometric figures, formalism is concerned only with the interaction of inherently meaningless symbols that can be construed to signify or describe these mathematical concepts, but do not in reality symbolize them. The construction of a formal system consists of defining three main concepts that can be thought of as paralleling the basic components of an axiomatic system. The first thing that must be done in constructing a formal system is defining its inherently meaningless symbols that the system will use. Next, the way in which these symbols can be validly arranged in regard to each other to create “strings” of symbols must be properly defined. Finally, the rules of inference for arranging the various strings of symbols to logically deduce new strings of symbols must be laid out [10].

Since the basic symbols of formalism are not tied to the physical or observable world around us, it is completely free from the limitations placed on traditional mathematical systems by our intuitions. It is therefore clear that formalism can be viewed as a purely logical system. The divorce between pure mathematics and its application was evident to Hilbert and he felt that it was vitally important to his system of metamathematics. Mathematics, according to Hilbert, had evolved from empirical experience of the world around us and has grown, as the mathematical mind became "conscious of its independence" from the outside world, into a system transcendent and independent of the external world [12]. For further clarification of this concept, we may turn our thoughts to geometry. In Hermann Grassmann's book *Ausdehnungslehre* (*The Theory of Extension*), he argues that the mathematics of geometry itself is wholly independent of the physical world and is grounded only in pure mathematics. The "science of space," he would say, is "no branch of mathematics, but is an application of mathematics to nature" [19].

The question may arise of why we need to divest mathematics of its inherent meaning and why we cannot allow ourselves to rely on the intuitive knowledge that we gain from our experience, limiting ourselves to using foundations that are "clear and distinct," as Descartes prescribed. In mathematical pursuits it would be easy to simply assume the consistency of theories that rely only on axioms that seem clear to us, thereby solving Hilbert's problem of consistency, but oftentimes these theories prove themselves to be inconsistent [16]. One such example is the theory of infinite numbers explored by Georg Cantor, wherein contradictions began to arise based upon axioms that seemed to be intuitive. Our intuition about mathematics, therefore, cannot be relied upon to conduct absolute proofs of consistency.

The strength of formalism lies in the fact that it does not rely on any naturally occurring system of thought or our intuitions, but instead upon a strict, well-defined system of symbols and rules of interaction. It is thereby much easier to verify a formal system's consistency since it does not involve the fallibility of our intuitions.

Though the importance of formalism was not widely recognized at first, as mathematical logic developed, many began to see its usefulness [19]. Many mathematical developments in the 19th and 20th centuries, such as non-Euclidean geometry, ran counter to human intuitions about axiomatic systems, leading many to see the necessity and value of formalism [10]. With the tool of formalism in hand, many mathematicians felt better equipped to someday address many of mathematics' questions and problems, and Hilbert believed it would be the tool that would prove the most useful in finding assurance of the consistency and completeness of mathematics.

VI. Principia Mathematica

One of the first, and arguably most influential, formal mathematical systems developed was the one set forth in Bertrand Russell and Alfred North Whitehead's *Principia Mathematica*, a three-volume work published from 1910-1913. In their voluminous work, Russell and Whitehead provided a theory that supplied mathematicians with a tool to study the entirety of mathematics as a formalized system [16]. One of the more unique aspects of *Principia Mathematica*'s formal system of mathematics was that it provided a description of mathematics that was ultimately derived from nothing more than basic logic [1].

Since *Principia Mathematica* derived its mathematical system directly from logic, many mathematicians viewed it as very strong evidence that mathematics was consistent and complete, as the notion that logic itself was consistent was almost universally accepted and very few would imagine that logic would be in any way incomplete [16]. *Principia Mathematica* failed, however, to provide any concrete proof for its consistency or completeness, and without such a proof, Hilbert's original challenge was still unmet. Nonetheless, Hilbert was very impressed with

Russell and Whitehead's work, and, equipped with his formalist way of thinking, sought to prove their system consistent and complete [1]. The work of Kurt Gödel, however, would prove Hilbert's work futile.

VII. Kurt Gödel

Kurt Gödel's famous 1931 paper "On Formally Undecidable Propositions of Principia Mathematica and Related Systems" set forth a groundbreaking set of theorems that would serve to foil Hilbert's program of providing a system of mathematics that was both consistent and complete. The first of these theorems, often referred to as the first incompleteness theorem, formally demonstrates that any sufficiently strong, consistent formal system capable of interpreting arithmetic will contain propositions that are neither provable nor disprovable, that is, propositions that are undecidable. Quite obviously, this theorem undermined the notion that a fully complete system of mathematics could ever be constructed, effectively proving that Hilbert's program could not be completed. Next, Gödel proved what would become known as his second incompleteness theorem, which states that a sufficiently strong, consistent formal system capable of interpreting arithmetic could never prove itself consistent. The inability of a formal system to prove itself consistent showed that absolute proofs of consistency, in the traditional sense, could never be conducted within the system itself. The second incompleteness theorem does not, however, prove that it is impossible to prove that a given system is consistent. It only proved that a formal system cannot prove *itself* consistent.

As one might expect, Gödel's method of proving metamathematical concepts is very unique and uses concepts and strategies of proof that mathematicians had previously been unfamiliar with. Any complex subject, such as metamathematics, will undoubtedly require those dealing with it to think in ways that may not necessarily be immediately intuitive. Investigation of proofs about proofs, for example, will parallel our usual thought processes about mathematical proofs in certain aspects, but will also diverge in their methodology in others. When examining these proofs care must be taken to ensure we fully grasp the various intricacies and nuances of the proofs in order to avoid getting overwhelmed or lost in the more technical aspects of the discussion. For this reason it would be prudent to observe an outline of Gödel's proof before we involve ourselves in considering the more technical aspects of his proof.

VIII. Gödel's Incompleteness Theorems: An Overview

In the opening pages of "On Formally Undecidable Propositions of Principia Mathematica and Related Systems", Gödel provides a brief, nontechnical overview of the proofs that would follow and his method for proving them. He begins by explaining how the formalization of mathematics that had taken place in the decades before his paper had allowed mathematical proofs to be "carried out according to a few mechanical rules" [8], allowing proofs to be carried out in a mechanical, logical fashion. One of the strongest such formal systems, Gödel believed, was the one set forth in Russell and Whitehead's *Principia Mathematica* (hereinafter referred to as PM), whose system he would use throughout his paper. These formal systems contained formalizations of "all methods of proof used in mathematics," reducing them to a set of several "axioms and rules of inference" [8]. It is thereby obvious that these systems were capable of carrying out a vast array of mathematical proofs.

Gödel then proceeded to explain that, though some mathematicians may believe that any properly formed mathematical statement constructed within one of these systems could be determined to be true or false by using the formal rules of the system, this was simply not the case, as he would demonstrate in his paper [8]. Though no names are mentioned, it is obvious that the aim of Gödel's paper was to disprove the notions held by Hilbert and his supporters that any mathematical question could be decided within some thorough enough formal system of mathematics. If Gödel proved his proposition about the undecidability of some mathematical statements within a formal system, he would effectively end the Hilbert program.

The formal system that Gödel would use throughout his paper would be that of PM , wherein formulas consist of fundamentally no more than a series of the system's basic signs [8]. Proofs for formulas, viewed from within the formal framework, can thereby be thought of as simply a finite series of other formulas arranged in a valid fashion. Since Gödel would be working within the rules of formal systems, he was free to use any symbols he desired to construct the various formulas he would need. For his work Gödel chose the natural numbers as his symbols.

Though he could have arbitrarily chosen any set of symbols to use in his formal system, Gödel's choice was anything but random. With formulas reduced to simple series of natural numbers, Gödel revealed, "[m]etamathematical concepts and propositions thereby become concepts and propositions concerning natural numbers or series of them" [8]. This was the most pivotal point of Gödel's paper. Gödel continued, explaining that he would further demonstrate "that the concepts 'formula,' 'proof-schema', [and] 'provable formula' are definable in the system of PM" [8]. With the ability to express metamathematical statements as mathematical statements, Gödel would be able to use mathematics to express certain facts about itself. No metamathematical development before Gödel allowed such a unique perspective of mathematics and without this new view Gödel would not have been able to accomplish what he did within the context of his paper.

The next important concept we must define in providing a sketch of Gödel's proof is that of the "class-sign." A class sign can be thought of as simply a function of one variable, such as "x is divisible by 2" or "y is a prime number." This is obviously a very basic concept, but Gödel would be able to expand this idea to accomplish his very intriguing results.

With the previous information in hand, we are equipped to begin our overview of Gödel's proof, as he provided. First, let us assume that all of the class-signs definable in a theory can be arranged in some order. Though the exact order is of no consequence, it is clear that we can arrange the class-signs in numerical order, as they are simply series of numbers. Now, let us define R(n) as the n^{th} formula in this list. Further, allow us to to define [α; n] for any class-sign α as the formula obtained by replacing the free variable of α with a natural number n.

To clarify this concept, let us observe an example of how this notation can be used. Assume that the class-sign that denotes "*x* is a prime number" is the i^{th} class-sign in our set of values for R(n). Then we can see that R(i) corresponds to the previously mentioned class-sign "*x* is a prime number" where *x* is the free variable of R(i). Further, we can see that the statement [R(i); 7] corresponds to the mathematical statement wherein 7 is substituted for R(i)'s free variable, specifically “7 is a prime number.”

Next, let us define a class of natural numbers K such that

$$n \in K \equiv \neg provable[R(n); n] \tag{1}$$

(with the symbol ¬ representing negation). With the assumption that the formula “provable(x)” can be expressed within Gödel's system, which will be demonstrated later, K is simply the set of class-signs where the formula obtained when the free variable of R(n) is substituted with the natural number n is not provable within the system. We can then define a class-sign S such that [S; n] represents the formula $n \in K$. Since S is a class-sign, and we have an enumerated list of all class-signs, S must therefore correspond to some q such that $S = R(q)$.

At this point we have not delved very deeply into the mathematical interpretation of the aforementioned concepts, but will demonstrate the ability to express them within the system of mathematics when we perform a more technical analysis of Gödel's paper. All that is necessary for us to continue our overview of his method of proof is that we take the previous definitions as valid.

With our class-sign R(q) we can proceed in demonstrating the undecidability of certain mathematical statements. Let us observe the statement [R(q); q]. As with any properly formed mathematical statement, it is either provable within a theory of mathematics, and thereby true, or it is disprovable, i.e., its negation is provable, and thereby false. Assume that [R(q); q] is provable within our system. Then the statement [R(q); q] must therefore be true. The proof of [R(q); q] then implies that $q \in K$, which means that [R(q); q] is unprovable, which is an obvious contradiction to our assumption that [R(q); q] is provable. For further clarification, observe:

$$[R(q); q] \equiv [S; q] \equiv q \in K \equiv \neg provable[R(q); q] .$$

The truth and provability of [R(q); q], oddly enough, implies that it is unprovable. This contradiction shows us that we cannot, therefore assume [R(q); q] to be provable.

Let us then assume that ¬[R(q); q] is provable. This would imply that ¬[R(q); q] was not part of the set K, meaning that it was provable. This can be clearly seen in the following formula:

$$\neg[R(q); q] \equiv \neg[S; q] \equiv q \notin K \equiv \neg(\neg provable[R(q); q]) \equiv provable[R(q); q] .$$

Again this contradicts our initial assumption, showing that ¬[R(q); q] is not provable within our system of mathematics.

Since neither [R(q); q] nor its negation ¬[R(q); q] are formally provable within our system, it is obvious that we have produced an undecidable proposition. Moreover, if we take into account the fact that [R(q); q] implies that it is unprovable, not only is it undecidable, it is also *true*. Obviously, since the statement is undecidable, it is unprovable. Since the statement itself states that it is unprovable, it is therefore true. This demonstrates that not only are there undecidable statements within mathematics, there are undecidable statements that are also true.

We should bear in mind that the previous outline does not serve as a formal mathematical proof, and would do little in convincing many serious mathematicians of the fact that there are undecidable statements within any formal theory of mathematics. It was merely meant to serve as the framework for the following discussion wherein we will observe the more technical aspects of Gödel's proof. One of the most important concepts to define within a system of mathematics is that of "provable(x)." Without being able to properly formalize this concept, the above proof would prove meaningless.

IX. Gödel's System P

Within the second section of Gödel's paper, he proceeds to define a formal system P of mathematics "obtained by superimposing on the Peano axioms the logic of PM" [8]. The system of PM would be sufficient enough by itself to conduct the proof, but the addition of the Peano axioms served to thoroughly simplify Gödel's method of proof [8].

Gödel began defining his system P the way that the definition of any formal system is started, by defining the inherently meaningless symbols that will form the system. The symbols comprising P consist of two types: constants and variables. The constants of Gödel's system enable meaningful sentences to be constructed and consist of the following symbols: "¬" (not), "∨" (or), "∀" (for all), "0" (zero), "*f*" (the successor of), "(", ")" (parentheses) [8]. The set of symbols for variables is notably more complex and is comprised of multiple types. Variables of the first type, representing natural numbers, are denoted by symbols of the form "x_1," "y_1" and "z_1". Variables of the second type, representing sets of natural numbers, are denoted by symbols of the form "x_2", "y_2" and "z_2". Likewise, variables of the third type, which represent sets of sets of numbers are denoted by symbols of the form "x_3", "y_3" and "z_3". This system of definition can thereby be carried out for variables of higher types, consisting of sets of sets of sets of numbers, and so on. In addition, a sign of n^{th} type is understood to be the same as a variable of n^{th} type.

Gödel continues the definition of his formal system by establishing formal definitions for the formulas in his system. These definitions do not differ from those commonly used in any other system of mathematics. They are essentially statements about different qualities of numbers or collections of numbers and can occur in two different forms. The first is a proposition formula, in which there occur no free variables and the second is a formula with any number of free variables, which Gödel defined as an "n-place relation-sign." In the context of our previous explanation of his proof, a 1-place relation-sign is simply a class sign. An example of a proposition formula would be "7 is a prime number," whereas a 2-place relation-sign would be "x is divisible by y."

With these definitions in place, Gödel had laid the foundation of his formal system. As it stood, however, there were not many mathematical applications. With a system of basic signs and definitions of formulas it is not possible to construct meaningful mathematical statements. To give any formal system the ability to express mathematical truths, a system of axioms that these truths will be derived from and a set of inference rules must first be established. This is precisely what Gödel proceeded to do.

At the core of Gödel's system P are three axioms taken directly from the Peano axioms. The first of these three is defined as $\neg(fx_1=0)$, meaning that the zero has no predecessor and we will thereby begin counting in the system P at 0 and progress through its successor, its successor's successor, and so on. The second axiom is defined as $fx_1=fx_2 \Rightarrow x_1=x_2$, interpreted simply as, if two numbers have the same successor, they are equal. Finally, an axiom for induction is established: $x_2(0) \wedge (\forall x_1, x_2(x_1) \Rightarrow x_2(f\ x_1)) \Rightarrow \forall x_1 . x_2(x_1)$, outlining the concept that if any property x_2 holds for 0 and if the property holding for any number x_1 implies it holds for the successor of x_1, then the property holds for all numbers. The aforementioned axioms established enough arithmetical foundation to perform the mathematics necessary for Gödel to complete his proofs.

In addition to this subset of the Peano axioms, Gödel included a set of induction rules that do not differ in any significant way from those of classical logic. It may seem redundant for Gödel to define these self-evident truths within his system P as they have been more or less universally accepted by both mathematicians and logicians for thousands of years. However, for any formal axiomatic system to be fully defined it must be very explicit in outlining specifically what rules of inference are valid within its framework. With these rules set there is no room for questioning whether or not a given statement is derivable with the system. Using the axioms and the strict rules of inference it will always be possible to determine whether or not a statement within the system is derived correctly. In short, a statement can be derived within the system if, using only the axioms and the rules of inference, a finite series of steps can be taken to show that the axioms imply the statement.

Until this point, the way Gödel developed his formal system did not diverge in the least bit from the techniques used by other mathematicians in the development of other formal systems. The next step in preparing his system to prove the incompleteness theorems, however, was likely the most unique approach that Gödel took in his method of proof. To each sign within the system P a unique natural number would be assigned. Series of these strings would then be constructed in such a way that every variable and statement, whether an axiom, formula, or any other combination of symbols, would have its own unique number associated with it. This unique process of assigning numbers to every statement within a formal system has become known as Gödel numbering. As we will soon see, Gödel's innovative technique allowed him to allow the formulas within his system, formulas that would speak of various numerical properties, to talk about metamathematical statements, represented as numbers with discernible numerical properties.

The first step in defining the system of Gödel numbering was to assign unique numbers to the basic symbols of the system. All of the formula of the system would be created from these signs, so it is obvious that once these basic signs had been assigned unique numbers, it would be possible to create any formula from them. To these basic symbols the following numbers were assigned [8]:

"0" … 1	"$\vee$" … 7	"(" … 11
"f" … 3	"$\forall$" … 9	")" … 13
"~" … 5		

Additionally, to each variable of type *n* would be assigned a number p^n for any prime number greater than 13. For example, to variables of type 1, such as x_1 or y_1, where x_1 and y_1 correspond to number signs, we assign the numbers 17 or 19. For variables of type 2, say x_2 and y_2, we assign the numbers $17^2=289$ and $19^2=361$, continuing this process with additional variables of these types and of higher number types in a similar fashion. This ensures that "to every finite series of basic signs (and so also to every formula) there corresponds, one-to-one, a finite series of natural numbers" [8].

Next, the way in which the Gödel number of individual formulas would be generated would be defined. It would not be possible to simply append the numbers corresponding to the signs of each symbol in a formula to each other in the order they appear in the formula. We would have no assurance that no two formulas would have adjacent terms that would have the same series of Gödel numbers. For example, if the signs ($0 \vee x_1$) appeared in that order in a formula the number associated with that part of the formula would be 11171713. This is the same series of numbers that would be assigned to the sequence of signs (x_1 x_1). Similar problems could potentially arise with taking the sum of the terms or the product of all of the terms.

Gödel's method of constructing the Gödel numbers of his formulas removed this potential for duplication. With the series of natural numbers corresponding to the signs of a formula, n_1, n_2, …, n_k, where n_i is the Gödel number of the i^{th} term in the sequence of signs, the number $2^{(n_1)} \cdot 3^{(n_2)} \cdot \ldots \cdot p_k^{(n_k)}$ would be assigned, where p_k is the p^{th} prime number.

For clarification of the concept of Gödel numbering, let us examine the Gödel number of the formula ($0 \vee f(0)$), which is equivalent to the English statement "zero or one." Though this statement holds no important mathematical information, we shall consider it only as an example of the construction of a Gödel number without taking into account the meaning of the statement itself.

The basic symbols of this formula are, in order, "(", "0", "∨", "f", "(", "0", ")", ")", which correspond, respectively, to the natural numbers 11, 1, 7, 3, 11, 1, 13, 13. Using the formula $2^{(n_1)} \cdot 3^{(n_2)} \cdot \ldots \cdot p_k^{(n_k)}$ for creating the Gödel number of a formula, we see that the Gödel number of the formula ($0 \vee f(0)$) equates to $2^{11} \cdot 3^1 \cdot 5^7 \cdot 7^3 \cdot 11^{11} \cdot 13^1 \cdot 17^{13} \cdot 19^{13}$ which equates to a definite, albeit extremely large, natural number.

We can be assured of the uniqueness of the natural numbers associated with the formulas in P due to the qualities of prime factorization of natural numbers. The prime factorization of any number is unique to that number and that number alone. Since the formula for generating Gödel numbers, $2^{(n_1)} \cdot 3^{(n_2)} \cdot \ldots \cdot p_k^{(n_k)}$, essentially generates the prime factorization of a number using the Gödel numbers of the symbols in the formula for the powers of each prime number, and because each of these numbers is unique to a single sign, we see that the Gödel number for any formula will be unique to that formula.

With the ability to convert formulas about numbers into numbers themselves, Gödel was able to use mathematical statements within his system P to answer questions about the attributes of other formulas within the system, and could even create formulas that addressed questions about themselves. With mathematics able to produce formulas referencing its own formulas, all Gödel had to do was to translate statements about metamathematical properties into statements about numerical properties of natural numbers to put the final piece of his proofs groundwork into place. Choosing the right mathematical properties would allow these statements to accurately mimic metamathematical properties, thereby permitting the system P to address questions about itself and, since P accurately modeled mathematics, mathematics itself.

One of the final steps Gödel took in defining his formal system was defining a certain type of recursively-defined functions, now called primitive recursive functions. Gödel defined a function $\phi(x_1, x_2, \ldots, x_n)$ to be recursive if it could be defined by the functions $\psi(x_1, x_2, \ldots, x_{n-1})$ and $\mu(x_1, x_2, \ldots, x_{n+1})$ such that the following hold:

$$\begin{gathered}\phi(0, x_2, \ldots, x_n) = \psi(x_2, \ldots, x_n) \\ \phi(k+1, x_2, \ldots, x_n) = \mu(k, \phi(k, x_2, \ldots, x_n), \ldots, x_n)\end{gathered} \tag{2}$$

In simpler terms, a function is defined to be primitive recursive if it can either be defined in terms of a base case, in this example, $\psi(x_1, x_2, \ldots, x_{n-1})$ or by another function that will, after a finite number of computations, reduce the function to another function of its base case, in this example represented by $\mu(x_1, x_2, \ldots, x_{n+1})$.

For a clarification of this concept, let us observe a function $fact(n) = (n)!$. We can then define $fact(n)$ as

$$\begin{gathered}fact(1) = 1 \\ fact(n+1) = (n+1)[fact(n)]\end{gathered}$$

In this definition, the function corresponding to Gödel's function Φ is $fact(n)$, with Ψ , the base case, corresponding to 1, and μ , the reduction function, corresponding to $(n+1)[fact(n)]$.

Equipped with the system of Gödel numbering and the ability to define primitive recursive functions, Gödel set out to define the mathematical equations he would use to signify provability within P. Through a series of forty-six formulas Gödel would create a system for analyzing natural numbers that would allow making meaningful metamathematical statements about the Gödel numbers of properly constructed formulas, and therefore the formulas themselves. Within these formulas, mathematics was given the ability to make statements about itself. Once complete, it was possible to express metamathematical concepts using nothing but numbers.

Beginning with basic formulas to determine various properties of numbers, such as whether or not one number is divisible by another, or finding the n^{th} prime factor in the prime factorization of a number (formulas 1 and 3, respectively), Gödel began formalizing the ability to access basic information about mathematical formulas represented by Gödel numbers. Slowly building upon these basic formulas, Gödel was able to define formulas for more meaningful metamathematical concepts, such as whether or not a statement was a formula (formula 23) or if a specific variable was bound or free within a particular formula (formulas 24 and 26, respectively). With each new formula added to his list, the range of statements that could be made about mathematical formulas and their Gödel numbers increased, bringing the goal of proving Gödel's theorems ever closer.

With formal definitions of the process needed to determine whether or not a statement was a formula or not, Gödel could then start expanding the system to make reflections about the system itself. Formulas 34-42 give the ability to determine whether or not a formula is one of the axioms that make up P itself. We can see that P was given a basic ability to make assessments, however primitive they may be at this point, about itself. Gödel's formulas had at this point provided the ability for P to define itself using its own rules. Arithmetic was essentially describing itself using only its own powers of numerical analysis, a concept that had previously never been heard of.

With the rules of inference of P defined for Gödel numbers in formula 43, the final three formulas, for determining if a specific Gödel number represents a proof figure, if a the Gödel number of a series of statements proves the statement represented by another Gödel number, and, finally, for determining whether a given formula was provable or not (formulas 44, 45, and 46, respectively), were easily defined.

Gödel had thus created a system wherein a large subset of metamathematical concepts had been defined in terms of a formal axiomatic system. Numbers were given the ability to express statements about themselves and mathematics the ability to express various properties about itself and its formulas within the confines of its own logical and mathematical rules. The metamathematical statements that it produced could then be analyzed with the same amount of credibility and certainty that mathematical formulas were. One could be assured that a mathematical statement was a theorem of a mathematical theory, and thereby true, given the reliability of the theory, by examining whether or not it could be derived from the axioms of the theory using its rules of induction. Clearly, since metamathematical statements could be expressed using mathematical methods, metamathematical truths could be derived with the same absolute certainty. The metamathematical proof that would follow using Gödel's mathematically-defined metamathematical statements would be indisputable, given the reliability of mathematics.

The final step in laying the groundwork for his proof is defined in Proposition V of his proof. In this proposition he states that any recursive relation $R(x_1, \ldots, x_n)$, defined as a relation that can be expressed as a primitive recursive function, can be expressed within P using a relation sign r, whose free variables are $u_1, \ldots, u_n$, such that the following hold for all natural numbers $x_1, \ldots, x_n$:

$$R(x_1, \ldots, x_n) \Rightarrow provable(subst(r \frac{u_1 \ldots u_n}{number(x_1) \ldots number(x_n)})) \tag{3}$$

$$\neg R(x_1, \ldots, x_n) \Rightarrow provable(not(subst(r \frac{u_1 \ldots u_n}{number(x_1) \ldots number(x_n)}))) \tag{4}$$

where $subst(r \frac{u_1 \ldots u_n}{number(x_1) \ldots number(x_n)})$ is the relation sign generated by substituting the free variables of r, namely $u_1, \ldots, u_n$, with the Gödel numbers for the free variables of R, $x_1, \ldots, x_n$. In other words, it is possible to prove every true, recursive mathematical statement within P.

With Proposition V proved, an exercise beyond the scope of this work, Gödel was then fully equipped to prove the incompleteness theorem. The following is an examination of this process.

X. Gödel's Incompleteness Theorem

First, let us observe any formalized axiomatic system, *c*. Let Conseq(*c*) represent the set of all of the axioms of *c* and the theorems that can be derived from them, closed under the relation "immediate consequence," i.e., all of the statements that are derivable within *c*. We will then define the class c as ω-consistent if for every existential formula that is an element of Conseq(c), that is, a formula that makes a claim that a number with a certain property exists, there is actually a natural number that satisfies the property of the formula. This implies that the system is also consistent by the usual mathematical definition of consistency.

Gödel's incompleteness theorem can then be stated: To every ω-consistent recursive class c of formulas there correspond recursive class signs r, such that neither $\forall v, r$ nor $\neg\forall v, r$ are contained within Conseq(c), where v is the free variable of r. This implies that any consistent mathematical system will have statements within itself that are undecidable, implying that the system is incomplete.

To prove this theorem, Gödel defined a collection of new formulas using the basic group of 46 formulas defined previously along with several relation signs expressing various relationships and arranged them in such a way to demonstrate the validity of his theorem.

The first new formula to be defined was

$$isProofFigure_c(x) \tag{5}$$

which, using several of of the 46 formulas, defined a number x as a proof figure if, for each of the statements in the series of statements whose Gödel number corresponds to x, each of the statements is either an axiom, an element of c, or is implied by two previous statements within x.

Next,

$$proofFor_c(x, y) \tag{6}$$

was defined, providing the ability to evaluate whether or not the statements represented by x proved the statement y. The formula $proofFor_c(x, y)$ is defined simply by checking whether or not x is a proof figure, functionality provided by the formula $isProofFigure_c(x)$, and then checking to see if y is the last statement in the series of statements corresponding to x. The formula

$$provable_c(x) \tag{6.1}$$

is simply defined as $provable_c(x) \Leftrightarrow \exists y, proofFor_c(y, x)$.

With these formulas defined, we can see that the following hold:

$$\forall x (provable_c(x) \Leftrightarrow x \in Conseq(c)) \tag{7}$$
$$\forall x (provable(x) \Rightarrow provable_c(x)) \tag{8}$$

since

$$\begin{gathered} provable_c(x) \Leftrightarrow \exists y, proofFor_c(y, x) \\ \Leftrightarrow there\, is\, some\, natural\, number\, z\, such\, that\, proveFor(z, x)\, is\, true\ , \\ \Leftrightarrow x\, is\, a\, theorem\, of\ c \Leftrightarrow x \in Conseq(c) \end{gathered}$$

because the fact that c is ω-consistent, the statement $\exists y, proofFor_c(y, x)$ implies that there is a natural number z such that $proofFor(z, x)$ holds, and since $provable_c(x)$ implies that x is either an axiom, an element of c, or is implied by two previous statements in x and $provable(x)$, defined in formula 46, implies that x is either an axiom or is implied by two previous statements in x.

We are free to then define the relation Q(x, y):

$$Q(x, y) \Leftrightarrow \neg(proofFor_c(x, subst(y \frac{19}{number(y)}))) \qquad (8.1)$$

Before continuing, let us examine this relation. We can see that the variable y is assumed to be a class-sign with the free variable 19, noting that numbers of the form p_i represent free numbers within Gödel's system P. We can think of y as being the function "$y(19)$" in the same sense that we think of a function f as "$f(x)$", where x is the free variable of f. The expression $subst(y \frac{19}{number(y)})$ therefore corresponds to the number obtained by substituting the Gödel number for y, represented by *number*(y), into the free variable of y, in a sense evaluating as "$y(y)$." This process of substituting a function itself as its argument may seem odd, but we must bear in mind that the class sign y, or any other formula within P, is simply a formula whose free variable is a natural number. Since we can convert the class sign y into a number, we can then use the Gödel number of y for its free variable. Taking into account (6) we see that Q(x, y) is equivalent to the statement "x does not prove $y(y)$." All the terms in this statement are expressible within P, so it is then possible to create a relation sign q whose free variables are 17 and 19 that represents our relation Q(x, y).

We then have, from Proposition V:

$$\neg(proofFor_c(x, subst(y \frac{19}{number(y)}))) \Rightarrow provable_c(subst(q \frac{17}{number(x)} \frac{19}{number(y)})) \qquad (9)$$

$$(proofFor_c(x, subst(y \frac{19}{number(y)}))) \Rightarrow provable_c(not(subst(q \frac{17}{number(x)} \frac{19}{number(y)}))) \qquad (10)$$

Next, Gödel defined

$$p \Leftrightarrow forall(17, q) \qquad (11)$$

where the formula $forall(x, y)$ is defined in P via Gödel's formula 15 as equivalent to the statement $\forall x, y$ and where p is a class sign with the free variable 19. In essence, if we view the variable 19 as an analogue to y, we have:

$$P(y) \Leftrightarrow \forall x, Q(x, y) \Leftrightarrow \forall x, x\ \overline{proves}\ Y(y) \Leftrightarrow Y(y)\ is\ unprovable$$

(where, for clarity, we write $x\ \overline{proves}\ y$ for $\neg(x\ proves\ y)$ for any x, y).

Similarly, he defined:

$$r \Leftrightarrow subst(q \frac{19}{number(p)}) \qquad (12)$$

with r being a class sign with free variable 19. In a similar fashion to our explanation of (11), we have $R(x) \Leftrightarrow Q(x, y=p) \Leftrightarrow x\ \overline{proves}\ P(p)$. Since $P(y) \Leftrightarrow Y(y)\ is\ unprovable$, we can see that $P(p) \Leftrightarrow P(p)\ is\ unprovable$ and $R(x) \Leftrightarrow x\ \overline{proves}(P(p)\ is\ unprovable)$.

The reader should take note at this point that although the previously defined formula and variables may seem cumbersome or nonsensical, since they are all based on the concepts that Gödel has demonstrated to be mathematically expressible, they are all valid mathematical statements, however cumbersome they may seem.

With the previous definitions, Gödel could then make the following observation:

$$subst(p, \frac{19}{number(p)}) \Leftrightarrow subst(forall(17,q), \frac{19}{number(p)}) \Leftrightarrow \\ forall(17, subst(q \frac{19}{number(p)})) \Leftrightarrow forall(17,r) \tag{13}$$

Looking at this from a higher level, we can see that $P(p) \Leftrightarrow \forall x, Q(x, y=p)$ from (11), and from (12), $\forall x, Q(x, y=p) \Leftrightarrow \forall x, R(x) \Leftrightarrow \forall x, x\ \overline{proves}\ P(p) \Leftrightarrow (P(p)\ is\ unprovable)$. We can therefore see that the formula P(p), interpreted within Gödel's system P, states its own unprovability.

Also,

$$subst(q, \frac{17 \quad 19}{number(x) \quad number(p)}) \Leftrightarrow subst(r, \frac{17}{number(x)}) \tag{14}$$

Similarly to (13) we see that $Q(x, y=p) \Leftrightarrow x\ \overline{proves}\ P(p) \Leftrightarrow R(x)$.

Then observe the results when p is substituted for y in (9) and (10):

$$\neg(proofFor_c(x, subst(p \frac{19}{number(p)}))) \Rightarrow provable_c(subst(q \frac{17 \quad 19}{number(x)\, number(p)}))$$

$$(proofFor_c(x, subst(p \frac{19}{number(p)}))) \Rightarrow provable_c(not(subst(q \frac{17 \quad 19}{number(x)\, number(p)})))$$

Taking (13) and (14) into account, this yields:

$$\neg(proofFor_c(x, forall(17,r))) \Rightarrow provable_c(subst(r \frac{17}{number(x)})) \tag{15}$$

$$(proofFor_c(x, forall(17,r))) \Rightarrow provable_c(not(subst(r \frac{17}{number(x)}))) \tag{16}$$

With these results and definitions we can then see that the formula $forall(17,r)$ is undecidable within *c*. Let us examine why.

Assume that $forall(17,r)$ is *c*-provable. Then, as implied by (6.1) and (7), $\exists n, proofFor_c(n, forall(17,r))$. Examining (16) shows us that this implies $provable_c(not(subst(r \frac{17}{number(n)})))$. The *c*-provability of $forall(17,r)$, however, would imply $provable_c(subst(r \frac{17}{number(n)}))$. This, however, contradicts our assumption of ω-consistency of *c*, so $forall(17,r)$ is not *c*-provable.

Alternatively, let us assume that $not(forall(17,r))$ is *c*-provable. Since, as demonstrated above, $forall(17,r)$ is not *c*-provable, we have $\forall n, \neg proofFor_c(n, forall(17,r))$, which implies, by (15), $\forall n, provable_c(subst(r, \frac{17}{number(n)}))$. We assumed $provable_c(not(forall(17,r)))$, however, which leads to a contradiction in the ω-consistency of

c.

The incompleteness theorem is thereby proved by demonstrating a class sign r can be constructed within an ω-consistent class of formulas that is undecidable.

XI. Conclusions

Gödel's incompleteness theorem has many far reaching implications and has found applications in fields as diverse as mathematics and philosophy. Ever since its first publication in 1930, thinkers have drawn a myriad of conclusions from it. Many have been justified conclusions. Some, on the other hand, have been falsely drawn without fully understanding all of the intricacies of the theorem. A thorough analysis of what the theorem truly says and implies and what topics are not addressed by it may clear up any uncertainties of what conclusions can be drawn from the incompleteness theorem.

The first and most obvious implication of Gödel's incompleteness theorem is that any effective, consistent formal system capable of expressing basic arithmetic is inherently incomplete. This implication is among the most fundamentally counterintuitive of all the results of Gödel's work. Most, if not all, mathematicians before Gödel's time would likely have assumed that any statement consisting of basic arithmetic symbols arranged in a sensible way within a formal system could be demonstrated to be either true or false using either PM or a stronger system. Gödel demonstrated that this notion simply did not reflect reality. The proof of the incompleteness theorem involved constructing just such a statement. The series of statements that were created to provide Gödel with the undecidable statement that would demonstrate the validity of his incompleteness theorem were nothing more than arithmetical statements about natural numbers. It is therefore evident that the undecidable statement was an undecidable statement not simply about a metamathematical concept, but about natural numbers themselves.

With Gödel's proof that no effective, consistent formal system of mathematics that interprets arithmetic can be consistent and complete, Hilbert's program was irrefutably undermined [1]. The mathematical community was thoroughly shaken by this revelation, as the formalist view of mathematics had convinced many that formalized mathematical systems would be adequate to fully express both all that we knew about mathematics and all that *could* be known about mathematics. Hilbert and his followers would therefore be forced to adopt a new way of thinking as far as creating the strongest possible formal system of mathematics, knowing full well that any such system would be necessarily limited, assuming it was consistent.

One of the core tenets of Hilbert's program was that human knowledge about mathematics could be fully formalized. With the demise of the program after Gödel's work, it became obvious that human reason could never be fully formalized using the methods of Hilbert and other formalists [16]. Further, if we examine Gödel's statement $forall(17, r)$ in the context of his proof, we can see that the statement, when viewed from the metamathematical perspective of Gödel numbering, states that it is unprovable. We can see that this statement is therefore true, but that its truth cannot be deduced from the axioms and rules of inference of the formal system in which it is created. We can see that Gödel's theorem shows that "an axiomatic approach to number theory cannot fully characterize the nature of number-theoretical truth" and that what we know and understand about mathematics transcends what can be expressed through our mathematical systems [16], a fact was not lost on Gödel [10].

Though there have been many groundbreaking conclusions drawn from Gödel's theorem, there are many limitations that are often ignored. Many people have taken Gödel's incompleteness theorem and applied it to areas wherein it was never meant to be applied [5]. At the heart of these incorrect conclusions is a lack of understanding of the technical details of the theorem itself and of the definition of terms such as “system,” “complete,” “consistent,” etc. within Gödel's theorems. If a conclusion is to be drawn from Gödel's incompleteness theorem, care must be taken to ensure that the more subtle points of the theorem are not ignored or misinterpreted in any way.

One point that is often missed is the fact that Gödel's theorem applies only to the arithmetic portion of a formal system. An effective, consistent formal system that interprets arithmetic will always be incomplete regarding arithmetic, but will not necessarily be incomplete in any other respects [5]. Take, for example, a formalized system of physics that includes a formalization of all of basic arithmetic. Gödel's theorem does not imply that this system is incomplete with regard to physics or that it would be unable to answer a meaningful question about physics. It simply states that the system would be unable to answer some questions regarding arithmetic.

Many have also taken Gödel's theorem to imply that every consistent formal system is incomplete, but this is not the case. In fact, many formal systems have been proven to be both complete and consistent [5]. One such example of a formal system that has been proven both complete and consistent is the system of mathematics known as Presburger arithmetic. In 1928 Moses Presburger demonstrated that the formal mathematical system defined by the Peano axioms, taken without multiplication, is a consistent and complete formal system [4]. It should be noted that this result does not conflict with Gödel's proof, which only applies to a system capable of expressing a sufficient amount of arithmetic. Presburger arithmetic does not provide any description of multiplication, omitting a large portion of the basic arithmetic structure necessary for the application of Gödel's results. Other subsets of mathematics have also been proven consistent and complete [19], but none of these involve a formalization of a sufficient amount of arithmetic.

Even though mathematics as a whole has been shown to be incomplete through Gödel's work, its fundamental reliability has not in any way been undermined. One important fact to keep in mind is that Gödel's incompleteness theorem provides only a contrived undecidable statement and does not prove that any meaningful arithmetic statement is undecidable [4]. This demonstrates how Gödel's work has in no way detracted from the strength of mathematics. His result does not, and was never intended to, undermine the reliability of mathematics.

A further testament to the unaltered reliability of mathematics is the fact that no meaningful arithmetic proposition has to date been proven to be undecidable in the standard system of set theory employed in mathematics [5]. There still remain many conjectures and questions within arithmetic, some outlined by Hilbert in his “Mathematical Problems.” Mathematicians, however, have no reason to believe that the answers to these questions will never be found and continue seeking the resolution of these problems [6].

Some thinkers outside of mathematics, on the other hand, have not taken such a positive viewpoint in light of Gödel's theorem. Many postmodern thinkers have wrongly drawn the conclusion that Gödel's theorem suggests or even proves that there are universally unknowable truths, whether inside or outside of mathematics. This conclusion is erroneous because Gödel's theorem amounts to nothing more philosophical than a mathematical theorem. Its direct reach goes no further than the truths of mathematics. The only thing that Gödel proved is that there are always formally undecidable statements within a formalized system of mathematics that is capable of describing a sufficient portion of arithmetic and that does not refer to any other system of thought [5]. Interestingly enough, the formally undecidable statement that Gödel created in his proof could be shown to be true using methods of reasoning that could not be formalized within his system.

Furthermore, the technical nature of the theorem precludes it from being applicable to any system outside of a formalization of mathematics. There have been many arguments since its first publication concerning the consistency and completeness of various other fields, such as philosophy or legal systems, that often stem from a misunderstanding of the terms used in the incompleteness theorem [5]. At the core of these misunderstandings is the way in which many of the terms used in Gödel's work can be equivocally used in other systems. Quite obviously, systems of philosophy or laws can be viewed as being consistent and inconsistent and many legal documents can be described as formal. It is easy to see how one might misconstrue a theorem about a formal system's completeness to refer to the consistency of a "formal" system of laws, for example. When applying Gödel's theorem it is vitally important to bear in mind that the type of "formal" system it refers to is defined to be an axiomatic system with a set of formal rules of induction. Any system that does not fit this description and does not describe basic arithmetic is entirely unaffected by the results of the incompleteness theorem.

With Hilbert's program shown to be inadequate, mathematicians since Gödel's time have been working in a new era of mathematics. Though the inexaustibility of mathematics espoused by Hilbert has not in any way been refuted, the fact that mathematics as it has been practiced for millennia is inherently limited demonstrates that the methods many mathematicians have grown accustomed to using do not fully encompass the entirety of the mathematical knowledge of the human mind.

Though the effects of Gödel's work have been rippling throughout mathematics for over half a century, the full scope of the implications of the incompleteness theorem has not been fully realized [16]. Though fundamentally unchanged, mathematics must be expanded, in at least some capacity, in order to better parallel human understanding. After publishing his revolutionary work on completeness, Gödel felt that mathematics should be allowed to extend in order to include new axioms and methods of approaching mathematical questions that may not necessarily be self-evident, but may be justified to explain certain observations. This may reduce some of the certitude of mathematics, but Gödel felt that this had at least partially been eroded away by "the modern criticism of the foundations" of mathematics [15].

His views were not meant to encourage any sort of postmodern or uncertain view of mathematics, however. Instead, they were meant to encourage mathematicians to continue to push the boundaries of their thought and continue to pursue innovative new ways of solving some of mathematics' most elusive questions. Gödel's mindset coalesces with Hilbert's desire to continually push the boundaries of fields of mathematical study and continually search for new questions and new answers that he expressed in his famous “Mathematical Problems” speech: “As long as a branch of science offers an abundance of problems, so long is it alive; a lack of problems foreshadows extinction or the cessation of the independent development” [12]. Gödel's work should not be viewed as an encouragement to cease the search for new problems, but as a call to search for them in ever more innovative ways.

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