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Counting Order Classes of Triple Products in Finite Groups

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Abstract

Let G be a finite group and let W_G denote the proportion of triples, (x, y, z), in G^3 for which xyz, xzy, yxz, zxy, yzx, and zyx have the same order. The following results are established.

i) G is abelian if, and only if, $W_G = 1$.

ii) W_G can be arbitrarily close to 1. In particular, if p is an odd prime, then $W_{D_p} = 1 - 3(p-1)/4p^2$ where D_p is the dihedral group on p symbols.

iii) $W_G \geq 1 - \dfrac{(|G'|-1)(2|G'|-3)}{2|G'|^2} \cdot \dfrac{|G|-|Z(G)|}{|G|}$.

1 Introduction

The products, xy and yx, of two elements of a group are conjugate

$$xy = y^{-1}(yx)y$$

and, therefore, have the same group theoretic order

$$|xy| = |yx|$$

because conjugation is a group automorphism. However the products,

$$xyz, xzy, yxz, zxy, yzx \text{ and } zyx,$$

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are not necessarily of the same order. For example

$$|(1,2,3)(1,2)(2,3)| = 1$$

and

$$|(1,2,3)(2,3)(1,2)| = 3$$

in the symmetric group on three symbols. In this paper we are concerned with the proportion of triples in a finite group for which all six products are of the same order.

Definition. Let G be a finite group and let $(x, y, z) \in G^3$.

$$\omega_{(x,y,z)} = |\{|xyz|, |xzy|, |yxz|, |zxy|, |yzx|, |zyx|\}|.$$

$$W_G = \frac{|\{(x,y,z) \in G^3 : \omega_{(x,y,z)} = 1\}|}{|G|^3}.$$

Note that xyz, yzx, and zxy are conjugate and that xzy, zyx, and yxz are conjugate. Thus, $\omega_{(x,y,z)}$ is one or two because $\omega_{(x,y,z)} = |\{|xyz|, |xzy|\}|$. Moreover, $\omega_{(x,y,z)}$ is one if any one of x, y, or z is in the centralizer of at least one of the other two. It follows that $W_G \geq \mathrm{Pr}_2(G)$ where $\mathrm{Pr}_2(G)$ denotes the proportion of commuting pairs, (x, y), in G^2.

2 Fundamental Theorems

Theorem 1 *G is abelian if and only if $W_G = 1$.*

Proof. If G is abelian, then $xyz = xzy$ for all $x, y, z \in G$, so $|xyz| = |xzy|$ for all $x, y, z \in G$. Hence, $W_G = 1$. Conversely, if G is non-abelian, then there exist $x, y \in G$ such that $xy \neq yx$. Note that $|xy(x^{-1}y^{-1})| = |[x^{-1}, y^{-1}]| \neq 1$. However, $|x(x^{-1}y^{-1})y| = 1$, so $\omega_{(x,y,x^{-1}y^{-1})} = 2$, and $W_G \neq 1$. •

Theorem 2 *If* $|G'| = n$*, then*

$$W_G \geq 1 - \left(\frac{(n-1)(2n-3)}{2n^2}\right)\left(1 - \frac{|Z(G)|}{|G|}\right).$$

Proof. We find an upper bound for the number of triples $(x, y, z) \in G^3$ such that $|xyz| \neq |xzy|$. If $y \in C(x), z \in C(x)$, or $z \in C(y)$, then $|xyz| = |xzy|$.

Thus, by counting the maximum number of triples $(x, y, z) \in G^3$ such that $y, z \notin C(x)$, and $z \notin C(y)$, we find the upper bound we desire. The maximum number of choices for y such that $y \notin C(x)$ is obtained when $|C(x)|$ is as small as possible. Note that because $|G'| = n$ (and hence the largest possible conjugacy class cardinality is n), the smallest possible centralizer cardinality is $\frac{|G|}{n}$. Assume there exists $x \in G$ such that $|C(x)| = \frac{|G|}{n}$. Choose $y \notin C(x)$. Since z must be chosen so that $z \notin C(x)$ and $z \notin C(y)$, the maximum number of choices for z such that $|xyz| \neq |xzy|$ is realized when $|C(x) \cup C(y)|$ is as small as possible.

By the principal of inclusion and exclusion, $|C(x) \cup C(y)| = |C(x)| + |C(y)| - |C(x) \cap C(y)|$. Since $|C(x) \cap C(y)|$ divides $|C(x)| = \frac{|G|}{n}$ and $C(x) \cap C(y) \neq C(x)$ (otherwise, $x \in C(y)$ so $y \in C(x)$) the maximum for $|C(x) \cap C(y)|$ is $\frac{|G|}{2n}$. Thus,

$$|C(x) \cup C(y)| \geq \frac{3|G|}{2n}.$$

We now perform the count of the triples. There are $|G - Z(G)|$ choices for x such that unequal order class triples may be produced. We assume if $x \notin Z(G)$, then $|C(x)| = \frac{|G|}{n}$. Therefore, the maximum number of ways in which $y \notin C(x)$ is $|G| - \frac{|G|}{n}$, and the maximum number of ways in which $z \notin C(x) \cup C(y)$ is $|G| - \frac{3|G|}{2n}$.

Hence, we obtain that the maximum number of triples such that $|xyz| \neq |xzy|$ is

$$(|G - Z(G)|)\left(|G| - \frac{|G|}{n}\right)\left(|G| - \frac{3|G|}{2n}\right).$$

Dividing this expression by $|G|^3$, it is straightforward to check that

$$1 - W_G \leq \left(\frac{(n-1)(2n-3)}{2n^2}\right)\left(1 - \frac{|Z(G)|}{|G|}\right),$$

from which the desired result immediately follows. •

Notice in Theorem 2 that if $|G'| = 1$ (that is, G is abelian), $W_G = 1$ as expected. In Section 4, we will see that this theorem is also important in determining W_G for 3-rewriteable groups.

Theorem 3 *If A is an abelian group, then $W_G \leq W_{G\times A}$.*

Proof. Let $x, y, z \in G$ and $a, b, c \in A$. It suffices to show that if $\omega_{(x,y,z)} = 1$, then $\omega_{((x,a),(y,b),(z,c))} = 1$. Suppose that $\omega_{(x,y,z)} = 1$; that is, $|xyz| = |xzy|$. Assume that $\omega_{((x,a),(y,b),(z,c))} \neq 1$; that is, there exist distinct integers n and m such that $|(x,a)(y,b)(z,c)| = n$ and $|(x,a)(z,c)(y,b)| = m$. Without loss of generality, let $n < m$. Since $(xyz, abc)^n = ((xyz)^n, (abc)^n) = 1$, $|xyz| \mid n$ and $|abc| \mid n$. Consequently, $|xzy| \mid n$ and $|acb| \mid n$, so $(xzy)^n = (acb)^n = 1$. We conclude that $((xzy)^n, (acb)^n) = (xzy, acb)^n = 1$, which implies that $m = |(x,a)(z,c)(y,b)| \mid n$. So $m \leq n$, a contradiction. Hence, $\omega_{((x,a),(y,b),(z,c))} = 1$. •

3 Dihedral groups

We will use the following notational conventions for the dihedral group on n symbols:

$$D_n = \langle r, s : r^n = s^2 = 1, rs = sr^{-1} \rangle;\ n > 2.$$

The following lemma is crucial to the study of W_G in dihedral groups and will be applied several times throughout this section.

Lemma 4 *Let $x, y, z \in D_n$. If $|xyz| \neq |xzy|$, exactly one of x, y, z has the form r^i, where i is any integer.*

Proof. The proof is easily completed by eliminating the cases where zero, two, or three of x, y, z have the form r^i. If zero or two of x, y, z have the form r^i, we use the relations for D_n to write xyz in the form sr^{j_1}, where j_1 is an integer. Similarly, we can also write xzy in the form sr^{j_2}, where j_2 is an integer. Since all elements in D_n of the form sr^j are of order two, $|xyz| = |xzy|$. On the other hand, suppose that $x = r^{i_1}$, $y = r^{i_2}$, and $z = r^{i_3}$. Since all elements of the form r^i commute with one another, $|xyz| = |r^{i_1}r^{i_2}r^{i_3}| = |r^{i_1}r^{i_3}r^{i_2}| = |xzy|$. $\square$

We use this lemma to prove the following theorem which demonstrates that W_G is not bounded away from one in non-abelian groups.

Theorem 5 *If p is an odd prime, then $W_{D_p} = 1 - \frac{3(p-1)}{4p^2}$.*

Proof. We count the number of triples $(x, y, z) \in {D_p}^3$ such that $\omega_{(x,y,z)} \neq 1$. From Lemma 4, if $(x, y, z) \in {D_p}^3$ is a triple such that $\omega_{(x,y,z)} \neq 1$, $\{x, y, z\}$ consists of exactly one rotation and two reflections. Now let $i, j, k \in \mathbf{Z}$ such that $0 \leq i, j, k \leq p-1$ and consider the following three cases:

Case 1: $x = r^i$, $y = sr^j$, $z = sr^k$.

Clearly, $xyz = r^i sr^j sr^k = r^{i-j+k}$. Hence, since $xyz \in \langle r \rangle$, $|xyz| = 1$ or p. Similarly, since $xzy \in \langle r \rangle$, $|xzy| = 1$ or p. We count triples (x, y, z) such that $|xyz| \neq |xzy|$, so let us first count triples where $|xyz| = 1$ and $|xzy| = p$.

In this case, $|r^{i-j+k}| = 1$. We consider the number of ways of selecting i. If $r^i \in Z(D_p)$, then clearly $|xyz| = |xzy|$. Thus, $i \neq 0$, leaving $p-1$ choices for i. Now because $|xyz| = 1$, $|xyz| = |xzy|$ if, and only if, $xyz = xzy$. $C(x) = C(r^i) = \langle r \rangle$; hence, sr^j does not commute with r^i for any j. Thus, we may choose j freely from $\{0, 1, ..., p-1\}$. Since $|r^{i-j+k}| = 1$, $i - j + k \equiv_p 0$. It follows that once i and j are chosen, k is fixed. Consequently, we have $p(p-1)$ ways of choosing $x = r^i$, $y = sr^j$, and $z = sr^k$ such that $|xyz| = 1$ and $|xzy| = p$.

Now consider triples (x, y, z) where $|xyz| = p$ and $|xzy| = 1$. By applying the same argument again, we observe that an additional $p(p-1)$ triples have $\omega_{(x,y,z)} \neq 1$, all of which are distinct from the previously counted $p(p-1)$ triples because now $|xyz| = p \neq 1$. Hence a total of $2p(p-1)$ triples are produced in Case 1.

Case 2: $x = sr^j$, $y = r^i$, $z = sr^k$.

Observe that $xyz = r^{-j-i+k} = r^{j+i-k} \in \langle r \rangle$. Similarly $xzy \in \langle r \rangle$. Thus, Case 2 is essentially equivalent to Case 1, except that the order of the components in the triples is different, yielding different triples. Hence, an additional $2p(p-1)$ triples are produced.

Case 3: $x = sr^j$, $y = sr^k$, $z = r^i$.

This case is also equivalent to Case 1, and again yields different triples from Cases 1 and 2. Again $2p(p-1)$ triples are produced.

Consequently, $6p(p-1)$ triples in ${D_p}^3$ have the property that $\omega_{(x,y,z)} \neq 1$.

But the total number of triples in ${D_p}^3$ is $(2p)^3$. Hence, $1 - W_{D_p} = \frac{6p(p-1)}{8p^3} = \frac{3(p-1)}{4p^2}$, and $W_{D_p} = 1 - \frac{3(p-1)}{4p^2}$. •

Corollary 6 *As $p \to \infty$, $W_{Dp} \to 1$.*

We can also produce sequences of groups for which $W_G \to \frac{7}{8}$ as the cardinalities of the groups increase; namely, sequences of nilpotent dihedral groups (a dihedral group D_m is nilpotent if and only if $m = 2^n$ for some positive integer n).

Theorem 7 *If n is a positive integer, then* $W_{D_{2^n}} = \dfrac{7}{8} + \dfrac{1}{2^{2n+1}}$.

Proof. Our strategy is to construct all triples (x, y, z) in $(D_{2^n})^3$ such that $\omega_{(x,y,z)} \neq 1$.

Consider such a triple $(x, y, z) \in (D_{2^n})^3$. Recall from Lemma 4 that exactly one of x, y, z is of the form r^i. We now examine three cases corresponding to the three possible positions in the triple (first, middle, last) for the rotation.

Case 1 : $x \in \langle r \rangle$.

Let $x = r^i$, $y = sr^j$, and $z = sr^k$, where i,j,$k \in \mathbf{Z}$ and $0 \leq i, j, k \leq 2^n - 1$.

Definition. Suppose $i \neq 0$ and let h be the largest nonnegative integer such that $2^h \mid i$. We say class(i) $= h$. Let class(0) $= n$.

Lemma 8 *There are 2^{n-h-1} ways to select i such that class(i) $= h$.*

Proof. We wish to find the number of integers i such that $i = 2^h t$ for some odd integer t. Suppose class$(i) = h$. Let $m \in \mathbf{Z}$ and observe that class$(i + m2^{h+1}) = class(2^h t + m2^{h+1}) = h$.

We show now that only integers which can be expressed the form $i + m2^{h+1}$ have class h. Consider any integer of the form $i + m2^h$, where m is odd. Claim: $i + m2^h$ will not have class h. Notice $i + m2^h = 2^h t + m2^h = 2^h(t + m)$, and since t and m are odd, $t + m$ is even, so class $(i + m2^h) > h$. Finally, any integer of the form $i + m2^{h-d}$ (again assume m is odd and now $0 < d \leq h$) will have class $h - d$. Reason: $i + m2^{h-d} = 2^h t + m2^{h-d} = 2^{h-d}(2^d t + m)$.

Since $2^d t + m$ must be odd, class$(i + m2^{h-d}) = h - d$. So, all integers of class h can be expressed in the form $i + m2^{h+1}$. Between 0 and $2^n - 1$, then, there are clearly $\frac{2^n}{2^{h+1}} = 2^{n-h-1}$ integers of class h corresponding to $\frac{2^n}{2^{n-1}}$ unique choices for m. □

Let $h =$ class(i), and consider the following three subcases:

Subcase A : $h = 0$.

This implies that i is odd.

Lemma 9 *Let class$(i) = h = 0$. $j \equiv_2 k$ if and only if $|r^i s r^j s r^k| = |r^i s r^k s r^j|$.*

Proof. Suppose that $j \equiv_2 k$ (i.e., $j - k$ is even). Now $|r^i s r^j s r^k| = |r^{i-j+k}| = \frac{2^n}{(2^n, i-j+k)}$ since $|r| = 2^n$.

Similarly, we obtain that $|r^i s r^k s r^j| = \frac{2^n}{(2^n, i+j-k)}$. Because i is odd and $j - k$ is even, $i + j - k$ and $i - j + k$ are odd. We conclude that $|r^i s r^j s r^k| = 2^n = |r^i s r^k s r^j|$.

Conversely, suppose that $j \not\equiv_2 k$ (i.e., $j - k$ is odd) and assume that $|r^i s r^j s r^k| = |r^i s r^k s r^j|$. Since $\frac{2^n}{(2^n, i-j+k)} = \frac{2^n}{(2^n, i+j-k)}$, class$(i - j + k) =$ class$(i + j - k) = a$, for some non-negative integer a. We may assume that $0 \leq i - j + k$, $i + j - k \leq 2^n - 1$. If $i + j - k = 0$, then $i - j + k = 0$ since, by definition, both have class n. Since $i - j + k = i + j - k$, we have $j = k$, a contradiction. Thus $i + j - k \neq 0$, so we know there exist odd integers b and c such that $i - j + k = 2^a b$ and $i + j - k = 2^a c$. Observe that since i is odd and $j - k$ is odd, $a \neq 0$. Now $(i - j + k) - (i + j - k) = 2(k - j) = 2^a(b - c)$.

Hence, we simplify the last equality as follows: $k - j = 2^{a-1}(b-c)$. Now since $k-j$ is odd, $k \neq j$, so $b \neq c$. However, since both b and c are odd, $b-c$ is even. We conclude that $k - j = 2^{a}(\frac{b-c}{2})$, a contradiction. $\square$

When $x \in \langle r \rangle$ and $h = 0$, then, we can count the number of triples in $(D_{2^n})^3$ for which $\omega_{(x,y,z)} \neq 1$ as follows: From Lemma 8, we see that the number of rotations r^i such that class(i) $= 0$ is 2^{n-1}. Lemma 9 proves that $|r^i s r^j s r^k| \neq |r^i s r^k s r^j|$ if and only if one of j and k is even and the other is odd. So there are 2^n choices for j and 2^{n-1} choices for k. So in Subcase A, the number of triples $(x,y,z) \in (D_{2^n})^3$ with the property that $\omega_{(x,y,z)} \neq 1$ is $(2^{n-1})(2^n)(2^{n-1}) = 2^{3n-2}$.

Subcase B : $0 < h < n-1$.

First notice that i is even. Arbitrarily select $x = r^i$ such that class(i) $= h$. Since $h < n-1$, $r^i \notin Z(D_{2^n})$, so $C(r^i) = \langle r \rangle$. Thus, $y = sr^j$ and $z = sr^k$ do not commute with r^i for any j or k. Consequently, sr^j may be arbitrarily chosen from any of the 2^n reflections.

Lemma 10 *Suppose class(i) $= h > 0$. If $j \not\equiv_2 k$, then $|r^i s r^j s r^k| = |r^i s r^k s r^j|$.*

Proof. Recall that $|r^i s r^j s r^k| = \frac{2^n}{(2^n, i-j+k)}$ and that $|r^i s r^k s r^j| = \frac{2^n}{(2^n, i+j-k)}$. Since i is even and $j \not\equiv_2 k$, $i-j+k$ and $i+j-k$ are both odd. Hence, $|r^i s r^j s r^k| = 2^n = |r^i s r^k s r^j|$. $\square$

Lemma 10 proves that if j is even, k must also be even in order to produce a triple (x,y,z) such that $\omega_{(x,y,z)} \neq 1$. Similarly, if j is odd, k must also be odd. More simply, j and k must have the same parity. These remarks motivate the following definition:

Definition. $E_{j,v} = \Big\{k \in \{0,1,...,2^n-1\} : k = j + m2^{h+1} + 2v;\ m, v \in \mathbf{Z}\Big\}$.

Since $\displaystyle\bigcup_{0 \leq v \leq 2^h - 1} E_{j,v}$ contains all possible k values with the same parity as j, we see that this union contains all possible k values that could produce a triple such that $\omega_{(x,y,z)} \neq 1$.

Lemma 11 *The only set $E_{j,v}$ which produces triples (x,y,z) such that $\omega_{(x,y,z)} \neq 1$ is $E_{j,2^{h-1}}$, and in fact, all elements of $E_{j,2^{h-1}}$ produce such triples.*

Proof. Suppose $k \in E_{j,v}$. Then $k = j + m2^{h+1} + 2v$ for some $m \in \mathbf{Z}$. Note that $|r^i sr^j sr^k| = \frac{2^n}{(2^n, i-j+k)} = \frac{2^n}{(2^n, i+m2^{h+1}+2v)}$. Similarly, $|r^i sr^k sr^j| = \frac{2^n}{(2^n, i-m2^{h+1}-2v)}$. Now let class$(v) = d$ and write $v = 2^d u$, where u is an odd integer.

Suppose first that $v = 0$. Then

$$|r^i sr^j sr^k| = \frac{2^n}{(2^n, i + m2^{h+1})} = \frac{2^n}{(2^n, 2^h t + m2^{h+1})} = \frac{2^n}{(2^n, 2^h(t + 2m))} = 2^{n-h}.$$

Also,

$$|r^i sr^k sr^j| = \frac{2^n}{(2^n, i - m2^{h+1})} = \frac{2^n}{(2^n, 2^h t - m2^{h+1})} = \frac{2^n}{(2^n, 2^h(t - 2m))} = 2^{n-h}.$$

In this case, then, $\omega_{(x,y,z)} = 1$.

Suppose next that $v \neq 0$ and $v \neq 2^{h-1}$. Since we may assume $0 \leq v \leq 2^h - 1$, $d < h - 1$. Recall that because class$(i) = h$, $i = 2^h t$ for some odd t. Thus, using the fact that u is odd, we obtain $\frac{2^n}{(2^n, i+m2^{h+1}+2v)} = \frac{2^n}{(2^n, 2^h t+m2^{h+1}+2^d u)} = \frac{2^n}{(2^n, 2^d(2^{h-d}t+m2^{h+1-d}+u))} = \frac{2^n}{2^d} = 2^{n-d}$. Similarly, $\frac{2^n}{(2^n, i-m2^{h+1}-2v)} = 2^{n-d}$. Thus, if $v \neq 2^{h-1}$, then $|r^i sr^j sr^k| = |r^i sr^k sr^j|$.

Now consider the case where $v = 2^{h-1}$. Then, using the fact that $t+1$ is even, we obtain

$$\frac{2^n}{(2^n, i + m2^{h+1} + 2v)} = \frac{2^n}{(2^n, 2^h t + m2^{h+1} + 2^h)} = \frac{2^n}{(2^n, 2^h(t + 2m + 1))} = \frac{2^n}{(2^n, 2^{h+1}(\frac{t+1}{2} + m))}.$$

Similarly, using the fact that $t - 1$ is even, we see that

$$\frac{2^n}{(2^n, i - m2^{h+1} - 2v)} = \frac{2^n}{(2^n, 2^h t - m2^{h+1} + 2^h)} = \frac{2^n}{(2^n, 2^h(t - 2m - 1))} = \frac{2^n}{(2^n, 2^{h+1}(\frac{t-1}{2} - m))}.$$

Now one of $\frac{t+1}{2}$ and $\frac{t-1}{2}$ is odd and the other is even. Hence, one of $\frac{t+1}{2} + m$ and $\frac{t-1}{2} - m$ is odd and the other is even. Consequently, a factor of 2 can be taken out of the denominator of one of the expressions on the far right above, but not the other. Hence, the denominator of one of the expressions is 2^{h+1}, but the denominator of the other expression is greater that 2^{h+1}. Thus, $|r^i sr^j sr^k| \neq |r^i sr^k sr^j|$. □

The number of ways of choosing k, then, is just $|E_{j,2^{h-1}}|$. Using the same argument as in Lemma 8, we note that $|E_{j,2^{h-1}}| = 2^{n-h-1}$. Thus, for any $h < n - 1$, the number of triples (x, y, z)

such that $\omega_{(x,y,z)} \neq 1$ is just $(2^{n-h-1})(2^n)(2^{n-h-1}) = 2^{3n-2h-2}$. This is simply a generalization of Subcase A.

Subcase C : $h \geq n-1$.

The only two rotations in D_{2^n} such that class$(i) \geq n-1$ are r^0, $r^{2^{n-1}} \in Z(D_{2^n})$. Hence, no triples can be produced with the property that $\omega_{(x,y,z)} \neq 1$. The total number of triples produced in Case 1 is the sum of the numbers of triples produced in Subcases A, B, and C, which is $\sum_{h=0}^{n-2} 2^{3n-2h-2} = (2^{3n}) \sum_{h=0}^{n-2} \frac{1}{4^{h-1}}$.

Case 2 : $y \in \langle r \rangle$.

Next suppose that $x = sr^j$, $y = r^i$, and $z = sr^k$. We can show, for any h, that precisely the same number of triples is produced in this case as in Case 1. To see this, observe that $|sr^j r^i sr^k| = |s(r^{j+i-k})s| = |r^{-j-i+k}| = |r^{j+i-k}|$. The last equality holds because elements in any group have the same order as their inverses. Similarly, we find that $|sr^j sr^k r^i| = |r^{-j+k+i}|$. In both simplified products, the exponents of the rotations are the same as those considered in Case 1. Thus, Case 2 is equivalent to Case 1, and the same number of triples will be produced.

Case 3 : $z \in \langle r \rangle$.

Lastly, assume $x = sr^j$, $y = sr^k$, and $z = r^i$. Using a similar argument to that of Case 2, we can conclude that Case 3 is equivalent to Case 1. The same number of triples will again be produced.

As a result, the number of triples $(x, y, z) \in (D_{2^n})^3$ such that $\omega_{(x,y,z)} \neq 1$ is $3(2^{3n}) \sum_{h=0}^{n-2} \frac{1}{4^{h+1}}$. Since the total number of triples in $(D_{2^n})^3$ is $(2^{n+1})^3 = 2^{3n+3}$, the fraction of triples counted is $\frac{3}{8} \sum_{h=0}^{n-2} \frac{1}{4^{h+1}}$. Hence,

$$\begin{aligned}
W_{D_{2^n}} &= 1 - \tfrac{3}{8} \sum_{h=0}^{n-2} \frac{1}{4^{h+1}} = 1 - \frac{3}{32} \sum_{h=0}^{n-2} \frac{1}{4^h} \\
&= 1 - \tfrac{3}{32} \left(\tfrac{1-(\frac{1}{4})^{n-1}}{1-\frac{1}{4}} \right) = 1 - \tfrac{1}{8} \left(1 - \left(\tfrac{1}{4}\right)^{n-1} \right) \\
&= \frac{7}{8} + \left(\frac{1}{8}\right) \left(\frac{1}{4}\right)^{n-1} = \frac{7}{8} + \frac{1}{2^{2n+1}}. \bullet
\end{aligned}$$

Corollary 12 *As* $n \to \infty$, $W_{D_{2n}} \to \frac{7}{8}$.

We now consider direct products of dihedral groups with abelian groups (cf. Theorem 3). In particular, we examine direct products of D_4 with copies of C_2, the cyclic group of order 2.

Lemma 13 *Let* $(x,a),(y,b),(z,c) \in (D_4 \times C_2^n) \times C_2$*:*

$$|(x,a)(y,b)(z,c)| \neq |(x,a)(z,c)(y,b)| \text{ if, and only, if, } |xyz| \neq |xzy| \text{ and } abc = 1$$

Proof. First, suppose $|xyz| \neq |xzy|$ and $abc = 1$. Note that $|(x,a)(y,b)(z,c)| = \text{l.c.m. } (|xyz|, |abc|) = \text{l.c.m. } (|xyz|, 1) = |xyz|$. Since $abc = acb$, $|acb| = 1$. Thus, $|(x,a)(z,c)(y,b)| = \text{l.c.m. } (|xzy|, |acb|) = \text{l.c.m. } (|xzy|, 1) = |xzy|$. But $|xyz| \neq |xzy|$, so $|(x,a)(y,b)(z,c)| \neq |(x,a)(z,c)(y,b)|$.

Conversely, suppose that $|(x,a)(y,b)(z,c)| \neq |(x,a)(z,c)(y,b)|$. By Theorem 3, $|xyz| \neq |xzy|$. To show that $abc = 1$, we need the following lemma:

Lemma 14 *Let* $x,y,z \in D_4 \times (C_2^n)$*. If* $|xyz| \neq |xzy|$*, then* $|xyz|, |xzy| \neq 4$.

Proof. Suppose by way of contradiction that $|xzy| \neq |xyz| = 4$. We make this assumption without loss of generality. Since $x,y,z \in D_4 \times (C_2^n)$, we may write $x = (d_x, a_{x_1}, a_{x_2}, ..., a_{x_n})$, $y = (d_y, a_{y_1}, a_{y_2}, ... a_{y_n})$, and $z = (d_z, a_{z_1}, a_{z_2}, a_{z_n})$, where $d_x, d_y, d_z \in D_4$ and $a_{x_i}, a_{y_i}, a_{z_i} \in C_2$. Let $d = d_x d_y d_z$ and $a_i = a_{x_i} a_{y_i} a_{z_i}$ for all i. Since $xyz = (d, a_1, a_2, \ldots a_n)$, $|xyz| = \text{l.c.m. } (|d|, |a_1|, |a_2|, \ldots |a_n|)$. But since $|a_i| \in \{1,2\}$ and $|xyz| = 4$, $|d| = 4$.

We proceed to obtain a contradiction by showing that $|d| \neq 4$. From Theorem 3, we see that $|d_x d_y d_z| \neq |d_x d_z d_y|$. As a result, none of d_x, d_y, or d_z can be in the centralizer of either of the other two.

By examining the centralizers of the elements of D_4, we see that for this condition to be satisfied, d_x, d_y, and d_z must be chosen from distinct elements of $\{\{r, r^3\}, \{s, sr^2\}, \{sr, sr^3\}\}$. Using the group relations, we may write $d_x d_y d_z = r^i$ and $d_x d_z d_y = r^j$, for some integers i and j. It is easy to check that i and j must be even, because each is a sum of two odd numbers and an even number. Therefore, since $|r^0| = 1$ and $|r^2| = 2$, $|r^i| = |d_x d_y d_z| = |d| \neq 4$, a contradiction. $\circ$

Since $|xyz| \neq |xzy|$ and the order of every element in $D_4 \times (C_2^n)$ is 1,2, or 4, Lemma 14 allows us to assume without loss of generality that $|xyz| = 1$ and $|xzy| = 2$. We want to show $abc = 1$, so assume instead that $abc \neq 1$. Because $abc \in C_2$, $|abc| = 2$. Now, $|(x,a)(y,b)(z,c)| =$ l.c.m. $(|xyz|, |abc|) =$ l.c.m. $(1,2) = 2$. Also, $|(x,a)(z,c)(y,b)| =$ l.c.m. $(|xzy|, |acb|) =$ l.c.m. $(2,2) = 2$. But we assumed that $|(x,a)(y,b)(z,c)| \neq |(x,a)(z,c)(y,b)|$,yielding a contradiction. $\square$

Theorem 15 *Let $G = D_4 \times (C_2^n)$. $W_{G \times C_2} = \dfrac{W_G + 1}{2}$. In particular, $W_{D_4 \times (C_2^n)} \to 1$ as $n \to \infty$.*

Proof. Let $x, y, z \in G$ and $a, b, c \in C_2$. From Lemma 13, we know that $|(x,a)(y,b)(z,c)| \neq |(x,a)(z,c)(y,b)|$ if and only if $|xyz| \neq |xzy|$ and $abc = 1$. There are eight possible choices for (a,b,c), four of which have $abc = 1$. Hence, we know that $1 - W_{G \times C_2} = \frac{1}{2}(1 - W_G)$, so $W_{G \times C_2} = \dfrac{W_G + 1}{2}$. The second statement follows easily from the first. $\bullet$

Theorem 15 allows us to write a closed-form formula for $W_{D_4 \times (C_2^n)}$.

Corollary 16 *Let $G = D_4 \times (C_2^n)$. $W_G = \dfrac{2^{n+5} - 3}{2^{n+5}}$.*

Proof: We proceed by induction on n. If $n = 0$, then by Theorem 7,

$$W_G = \frac{7}{8} + \frac{1}{32} = \frac{29}{32} = \frac{2^5 - 3}{2^5}.$$

Now suppose $W_G = \dfrac{2^{n+4} - 3}{2^{n+4}}$. By Theorem 15, we observe that

$$W_G = \frac{\frac{2^{n+4} - 3}{2^{n+4}} + 1}{2} = \frac{2^{n+5} - 3}{2^{n+5}}. \quad \square$$

Theorem 17 *Let A be an abelian group and let $(x,a),(y,b),(z,c) \in D_4 \times A$:*

$|(x,a)(y,b)(z,c)| \neq |(x,a)(z,c)(y,b)|$ if, and only, if, $|xyz| \neq |xzy|$ *and* $(|abc|, 4) = 1$

Proof: Suppose first that $|xyz| \neq |xzy|$ and $(|abc|, 4) = 1$. Let $k = |(x,a)(y,b)(z,c)|$, $l = |(x,a)(z,c)(y,b)|$, and suppose by way of contradiction that $k = l$. Since $|xyz| \neq |xzy|$, Lemma 14 guarantees that $|xyz|, |xzy| \in \{1, 2\}$. Without loss of generality, let $|xyz| = 1$ and $|xzy| = 2$. Then $k = |(x,a)(y,b)(z,c)| = \text{l.c.m. } (|xyz|, |abc|) = \text{l.c.m. } (1, |abc|) = |abc|$. Since $(|abc|, 4) = 1$, k is odd. Because $k = l$, l is also odd. But $l = \text{l.c.m. } (|xzy|, |acb|)$, so $2 = |xzy| \mid l$, so l is even. This is a contradiction.

Now suppose $k \neq l$. From Theorem 3, $|xyz| \neq |xzy|$. Thus, it suffices to show that $(|abc|, 4) = 1$. Lemma 14 proves that $|xyz|, |xzy| \in \{1, 2\}$. Without loss of generality, let $|xyz| = 1$ and $|xzy| = 2$. It is enough to show $|abc|$ is odd, so assume that $|abc|$ is even. $k = \text{l.c.m. } (|xyz|, |abc|) = |abc| = |acb|$ and $l = \text{l.c.m. } (|xzy|, |acb|) = \text{l.c.m. } (2, |acb|) = |acb|$, since $|abc|$ is even. Hence, $k = l$, a contradiction. We conclude that $|abc|$ is odd and $(|abc|, 4) = 1$. •

Corollary 18 *If class($|A|$) = 1, then* $W_{D_4 \times A} = \dfrac{W_{D_4} + 1}{2}$.

Proof. If class($|A|$) = 1, then exactly half of the elements of A have even order. Clearly, $|abc|$ is even if and only if $(|abc|, 4) \neq 1$. Thus, from Theorem 17 we observe that in half the cases where $|xyz| \neq |xzy|$, we still obtain $|(x,a)(y,b)(z,c)| = |(x,a)(z,c)(y,b)|$. The result now follows immediately. □

We conclude this section with a result involving the direct product of D_p (where p is prime) with an abelian group.

Theorem 19 *Let A be an abelian group, p an odd prime, and $(x,a), (y,b), (z,c) \in D_p \times A$:*

$$|(x,a)(y,b)(z,c)| \neq |(x,a)(z,c)(y,b)| \iff |xyz| \neq |xzy| \text{ and } (|abc|, p) = 1$$

Proof: Suppose first that $|xyz| \neq |xzy|$ and $(|abc|, p) = 1$. Let $k = |(x,a)(y,b)(z,c)|$, $l = |(x,a)(z,c)(y,b)|$. Without loss of generality, let $|xyz| = 1$ and $|xzy| = p$. Then $k = |(x,a)(y,b)(z,c)| = \text{l.c.m. } (|xyz|, |abc|) = \text{l.c.m. } (1, |abc|) = |abc|$. Also, $l = |(x,a)(z,c)(y,b)| =$

l.c.m. $(|xzy|, |acb|)$ = l.c.m. $(p, |abc|)$. But $(|abc|, p) = 1$, so l.c.m. $(p, |abc|) = p|abc| \neq |abc|$, so $k \neq l$.

Now suppose $k \neq l$. From Theorem 3, $|xyz| \neq |xzy|$. Without loss of generality, let $|xyz| = 1$ and $|xzy| = p$. It suffices to show that $(|abc|, p) = 1$; suppose on the other hand that $(|abc|, p) \neq 1$. Then $p \mid |abc|$. Note that $k = |(x,a)(y,b)(z,c)|$ = l.c.m. $(|xyz|, |abc|) = |abc|$, and $l = |(x,a)(z,c)(y,b)|$ = l.c.m. $(|xzy|, |acb|)$ = l.c.m. $(p, |abc|) = |abc|$. This is a contradiction. •

We have a corollary to this theorem similar to the one for Theorem 17.

Corollary 20 *If class($|A|$) = 1, then* $W_{D_p \times A} \leq \dfrac{W_{D_p} + 1}{2}$.

Proof. As in the proof of Corollary 18, we note that exactly half of the elements of A have even order. Clearly, if $|abc|$ is even, then $(|abc|, p) = 1$. Hence, in at most half of the cases where $|xyz| \neq |xzy|$, $(|abc|, p) \neq 1$. From this and Theorem 19, the conclusion is clear. □

4 3-Rewriteable Groups

We begin by stating an immediate corollary to Theorem 2:

Corollary 21 *If G is a 3-rewriteable group, then* $W_G \geq \frac{7}{8}$.

Proof. Since G is 3-rewriteable, $|G'| \leq 2$. If $|G'| = 1$, G is abelian so by Theorem 1, $W_G = 1$. If $|G'| = 2$, we apply Theorem 2 with $n = 2$ and obtain $W_G \geq 1 - \frac{1}{8}\left(1 - \frac{|Z(G)|}{|G|}\right)$. Since $\left(1 - \frac{|Z(G)|}{|G|}\right) < 1$, we immediately obtain the desired result. □

There are many examples to prove that the converse of this corollary is false. To illustrate, $W_{D_7} = \frac{89}{98}$, $W_{A_4} = \frac{8}{9}$ and $W_{S_3 \times C_3} = \frac{17}{18}$, but none of these groups are 3-rewriteable.

In the case where G is a 3-rewriteable 2-group, we can improve considerably the lower bound for W_G given in Corollary 21. Before we state the theorem, however, we give the following lemma, crucial to the proof of the theorem:

Lemma 22 *Let G be a 3-rewriteable group. Let $x,y,z \in G$, and let $n = |xyz|$ and $m = |xzy|$. Then $m \neq n$ if and only if $\min\{m,n\}$ is odd and $\max\{m,n\} = 2\min\{m,n\}$.*

Proof. Since $m, n \neq 0$, if $2\min\{m,n\} = \max\{m,n\}$, then $m \neq n$.

Conversely, assume without loss of generality that $m < n$. Since G is 3-rewriteable, $|G'| \leq 2$, and G is nilpotent of class 1 or 2. Thus, since $m \neq n$, $|[z,y]| = |[y,z]| = 2$. Also note that $G' \leq Z(G)$. Therefore, $(xyz)^m[z,y]^m = (xyz[z,y])^m = (xzy)^m = 1$. Observe that if $[z,y]^m = 1$, then $(xyz)^m = 1$, which contradicts the assumption that $m < n$. So $[z,y]^m \neq 1$. Therefore, since $|[z,y]| = 2$, m must be odd, from which we see that m and $|[y,z]|$ are relatively prime. Using this observation and the fact that xzy and $[y,z]$ commute, $n = |xyz| = |xzy[y,z]| = (|xzy|)(|[y,z]|) = 2m$. $\square$

Theorem 23 *If G is a 3-rewriteable 2-group, then $W_G = 1 - \dfrac{2(1 - Pr_2(G))}{|G|}$.*

Proof: Since G is a 2-group, the identity is the only element of odd order. Thus, according to Lemma 22, we see that if $|xyz| \neq |xzy|$, then either $xyz = 1$ or $xzy = 1$, but not both. The number of triples (x,y,z) satisfying this condition is

$$|\{(x,y,z) : xyz = 1\}| + |\{(x,y,z) : xzy = 1\}| - 2|\{(x,y,z) : xyz = xzy = 1\}|.$$

Observe that $|\{(x,y,z) : xyz = 1\}| = |G|^2$ (since x and y may be arbitrary and then $z = y^{-1}x^{-1}$). Similarly, $|\{(x,y,z) : xzy = 1\}| = |G|^2$. Furthermore $|\{(x,y,z) : xyz = xzy = 1\}|$ is the number of triples such that $yz{=}zy$, namely $|G|^2 Pr_2(G)$. Hence,

$$|\{(x,y,z) \in G^3 : |xyz| \neq |xzy|\}| = |G|^2 + |G|^2 - 2|G|^2 Pr_2(G) = 2|G|^2(1 - Pr_2(G)).$$

Thus, $1 - W_G = \dfrac{2(1 - Pr_2(G))}{|G|}$. •

Corollary 24 *If G is a 3-rewriteable 2-group, then $W_G \geq 1 - \dfrac{1}{|G|}$.*

Proof. Since G is 3-rewriteable if and only if $\mathrm{Pr}_2(G) > 1/2$, we substitute $\mathrm{Pr}_2(G) = 1/2$ into Theorem 23 to obtain the lower bound. $\square$

5 A Connection between $\mathrm{Pr}_2(G)$ and W_G

Lemma 25 *If* $3\sum_{x\in G}|C(x)|^2 \le 2k^2|G| + k|G|^2$, *then* $W_G \ge 1-(1-\mathrm{Pr}_2(G))^2$.

Proof. Suppose that $3\sum_{x\in G}|C(x)|^2 \le 2k^2|G| + k|G|^2$. First, note that if $|xyz| \neq |xzy|$, then $y,z \notin C(x)$ and $z \notin C(y)$. As usual, we place an upper bound on $1-W_G$ by finding the maximum number of triples (x,y,z) such that $|xyz| \neq |xzy|$. To do this, we find the maximum number of triples such that $y,z \notin C(x)$ and $z \notin C(y)$. We fix $x \in G$, choose $y \notin C(x)$ and choose $z \notin C(x)\cup C(y)$. Because $y \notin C(x)$, and $C(x)\cap C(y) \le C(x)$, the minimum for $|C(x)\cup C(y)|$ is $\frac{3}{2}|C(x)|$. Thus, for each $x \in G$, the maximum number of triples such that $|xyz| \neq |xzy|$ is $(|G|-|C(x)|)\left(|G|-\frac{3}{2}|C(x)|\right)$. Therefore,

$$\begin{aligned}
1-W_G &\le \frac{1}{|G|^3}\left(\sum_{x\in G}(|G|-|C(x)|)\left(|G|-\tfrac{3}{2}|C(x)|\right)\right)\\
&\le \frac{1}{|G|^3}\left(\sum_{x\in G}|G|^2-\sum_{x\in G}\tfrac{5}{2}|G||C(x)|+\sum_{x\in G}\tfrac{3}{2}|C(x)|^2\right)\\
&\le \frac{1}{|G|^3}\left(|G|^3-\tfrac{5}{2}|G|(k|G|)+\tfrac{3}{2}\textstyle\sum_{x\in G}|C(x)|^2\right)\\
&\le 1-\frac{5k}{2|G|}+\frac{3}{2|G|^3}\sum_{x\in G}|C(x)|^2\\
&\le 1-\frac{5k}{2|G|}+\frac{1}{2|G|^3}\left(2k^2|G|+k|G|^2\right)\\
&\le 1-\frac{5k}{2|G|}+\frac{k^2}{|G|^2}+\frac{k}{2|G|}\\
&\le \left(1-\frac{k}{|G|}\right)^2\\
&\le (1-\mathrm{Pr}_2(G))^2
\end{aligned}$$

Hence, $W_G \ge 1-(1-\mathrm{Pr}_2(G))^2$. □

Theorem 26 *If* $|G'|$ *is prime, then* $W_G \ge 1-(1-\mathrm{Pr}_2(G))^2$.

Proof: By Lemma 25, it suffices to show that $3\sum_{x\in G}|C(x)|^2 \leq 2k^2|G| + k|G|^2$. Note that $\sum_{x\in G}|C(x)|^2 = |G|^2\sum_{\bar{x}}\frac{1}{|\bar{x}|}$, where the $\bar{x}$ denotes the conjugacy class of x. Since $|G'| = p$, a prime, we have that $|\bar{x}|$ is one or p. Therefore:

$$\begin{aligned} k &= |Z(G)| + \frac{1}{p}\left(|G| - |Z(G)|\right) \\ &= \frac{1}{p}|G| + \frac{p-1}{p}|Z(G)|. \end{aligned}$$

$$\begin{aligned} \sum_{\bar{x}\in G}\frac{1}{s_{\bar{x}}} &= |Z(G)| + \frac{1}{p^2}\left(|G| - |Z(G)|\right) \\ &= \frac{1}{p^2}|G| + \frac{p^2-1}{p^2}|Z(G)|. \end{aligned}$$

Let $|G/Z(G)| = n$. (Note that $n \geq 2$.) Using the facts above, we find:

$$\begin{aligned} 3\sum_{x\in G}|C(x)|^2 &\leq 2k^2|G| + k|G|^2 \\ &\Longleftrightarrow \\ 3\sum_{\bar{x}\in G}\frac{1}{s_{\bar{x}}} &\leq \frac{2k^2}{|G|} + k \\ &\Longleftrightarrow \\ 3\left(\frac{1}{p^2}|G| + \frac{p^2-1}{p^2}|Z(G)|\right) &\leq \frac{2}{|G|}\left(\frac{1}{p}|G| + \frac{p-1}{p}|Z(G)|\right)^2 + \left(\frac{1}{p}|G| + \frac{p-1}{p}|Z(G)|\right) \\ &\Longleftrightarrow \\ \frac{3}{p^2}|G| + \frac{3p^2-3}{p^2}|Z(G)| &\leq \frac{2}{|G|}\left(\frac{1}{p^2}|G|^2 + \frac{2p-2}{p^2}|G||Z(G)| + \frac{p^2-2p+1}{p^2}|Z(G)|^2\right) \\ &\quad + \frac{1}{p}|G| + \frac{p-1}{p}|Z(G)| \\ &\Longleftrightarrow \\ \frac{3}{p^2}|G| + \frac{3p^2-3}{p^2}|Z(G)| &\leq \frac{2+p}{p^2}|G| + \frac{p^2+3p-4}{p^2}|Z(G)| + \frac{2p^2-4p+2}{p^2}\frac{|Z(G)|^2}{|G|} \\ &\Longleftrightarrow \\ 0 &\leq \frac{p-1}{p^2}|G| + \frac{-2p^2+3p-1}{p^2}|Z(G)| + \frac{2p^2-4p+2}{p^2}\frac{|Z(G)|^2}{|G|} \\ &\Longleftrightarrow \end{aligned}$$

$$\begin{aligned}
0 &\leq (p-1)|G|^2 + (-2p^2+3p-1)|Z(G)||G| + (2p^2-4p+2)|Z(G)|^2 \\
&\Longleftrightarrow \\
0 &\leq (p-1)|G|^2 + \frac{-2p^2+3p-1}{n}|G|^2 + \frac{2p^2-4p+2}{n^2}|G|^2 \\
&\Longleftrightarrow \\
0 &\leq (p-1)n^2 + (-2p^2+3p-1)n + (2p^2-4p+2)
\end{aligned}$$

Showing that the last inequality holds is sufficient to prove the theorem. The equation $(p-1)n^2 + (-2p^2+3p-1)n + (2p^2-4p+2) = 0$ has solutions exactly when $n = 1$ and when $n = 2p-2$. When $n = 4$ and $p = 2$, we can easily verify that the inequality holds. Thus the inequality must hold for $n \geq 2p-2$. Hence, it is enough to show that $n > 2p-2$. For clarity, we now abbreviate $Z = Z(G)$.

Note that $G' \cong (G/Z)'$, so $|G'| = \left|(G/Z)'\right|$. Since $(G/Z)' \leq G/Z$, $\left|(G/Z)'\right|$ divides $|G/Z|$. $\left|(G/Z)'\right| \neq |G/Z|$, because in that case $|G/Z| = p$, so G/Z would be cyclic, making G abelian. But if G is abelian, then $|G'| = 1 \neq p$, a contradiction. Therefore, $|G/Z| \geq 2\left|(G/Z)'\right|$, so $n = |G/Z| \geq 2|G'| = 2p > 2p-2$. •

6 The Orderlizer of Group Elements

Definition. For $x \in G$, the orderlizer of x is

$$\mathcal{O}(x) = \{(y,z) \in G^2 : |xyz| = |xzy|\}.$$

Intuitively, the orderlizer of an element of a group is to W_G what the centralizer of an element is to $\Pr_2(G)$. Unlike $C(x)$, however, $\mathcal{O}(x) \not\leq G$: there are many examples of groups G for which $\frac{|G|}{2} < \mathcal{O}(x) < |G|$, where $x \in G$. The relevance of the orderlizer to our topic is clear:

$$|G|^3 W_G = \sum_x |\mathcal{O}(x)|.$$

This is analogous to

$$|G|^2 Pr_2(G) = \sum_x |C(x)|.$$

Note also that if $y \in C(x)$, then $(y, z) \in \mathcal{O}(x)$ for all $z \in G$. Thus, the sizes of centralizers of elements in G give us information about the sizes of their orderlizers. We formalize this with the following theorem:

Theorem 27 *Let $x \in G$. $|\mathcal{O}(x)| \geq 2|G||C(x)| - |C(x)|^2$.*

Proof: Let $y, z \in G$. Recall that if $y \in C(x)$ or $z \in C(x)$, then $|xyz| = |xzy|$. If $y \in C(x)$, then all pairs (y, z) where $z \in G$ are in $\mathcal{O}(x)$. The number of ways to choose y such that $y \in C(x)$ is $|C(x)|$, thus producing a total of $|G||C(x)|$ pairs that are elements of $\mathcal{O}(x)$. Similarly, there are $|C(x)|$ choices for z such that $z \in C(x)$. Thus, an additional $|G||C(x)|$ pairs (not necessarily distinct from the first set of pairs) are also elements of the orderlizer.

All pairs (y, z) such that $y \in C(x)$ and $z \in C(x)$ are counted twice. We therefore subtract $|C(x)|^2$ (the number of pairs (y, z) such that $y \in C(x)$ and $z \in C(x)$), yielding the desired minimum number of distinct pairs. •

There is another connection between centralizers and orderlizers of elements in G. Just as $|C(x)|$ is constant for all $x \in \overline{x}$, where $\overline{x}$ denotes the conjugacy class containing x, Theorem 28 implies a similar result for orderlizers.

Theorem 28 *Let $x \in G$. $|\mathcal{O}(x)| = |\mathcal{O}(x^G)|$.*

Proof: Let $y, z \in G$. It suffices to show that $(y, z) \in \mathcal{O}(x)$ if and only if $(g^{-1}yg, g^{-1}zg) \in \mathcal{O}(g^{-1}xg)$ for all $g \in G$. Let g be an arbitrary element of G.

$$\begin{aligned} (y, z) \in \mathcal{O}(x) &\iff |xyz| = |xzy| \\ &\iff |g^{-1}(xyz)g| = |g^{-1}(xzy)g| \end{aligned}$$

$$\iff |(g^{-1}xg)(g^{-1}yg)(g^{-1}zg)| = |(g^{-1}xg)(g^{-1}zg)(g^{-1}yg)|$$

$$\iff (g^{-1}yg, g^{-1}zg) \in \mathcal{O}(g^{-1}xg)$$

Thus, there is a one-to-one correspondence between the elements of $\mathcal{O}(x)$ and the elements of $\mathcal{O}(g^{-1}xg)$, so $|\mathcal{O}(x)| = |\mathcal{O}(g^{-1}xg)|$. •

7 Questions and Generalizations

We present a few questions and conjectures that arose as a result of our research.

(1) We have exhibited several sequences of groups $G_0, G_1, ...G_n$ such that as $n \to \infty$, $W_{G_n} \to 1$. We are interested in finding a sequence of groups $G_0, G_1, ...G_n$ such that as $n \to \infty$, $W_{G_n} \to 0$. From our work, it is clear that the "more abelian" a group is, the closer W_G is to 1. Hence, we suggest that any sequence of groups such that $W_{G_n} \to 0$ consists of groups which are particularly non-abelian in nature. A sequence of symmetric groups $S_3, S_4, ...S_n$, for example, appears to be a likely candidate. Our data supports this conjecture (see appendix).

(2) Our research entailed much work with direct products. Let C_n denote the cyclic group on n symbols. Our data supports the following conjecture concerning direct products of groups with cyclic groups:

$$W_{G\times C_n \times C_m} = W_{G\times C_{nm}}.$$

While this conjecture is clearly true in the case where n and m are relatively prime, we believe the equation is true in general.

(3) In Theorem 26, we showed that if $|G'| = p$ for any prime p, then $1 - W_G \leq (1 - Pr_2(G))^2$. We have not discovered an example of a group for which this is not true in general. It is easy to show that $W_G \geq Pr_2(G)$, but it appears that the stronger bound is also valid. If so, then it allows us to place sharper lower bounds on W_G for various groups in which information about $\mathrm{Pr}_2(G)$ is

known.

(4) A natural extension of the notion of studying orders of products of three elements in G is to study orders of products of more than three elements in G. We take a brief look at this extension to orders of products of four elements of G. Consider a four-tuple $(x, y, z, w) \in G^4$. Instead of two sets of conjugate elements, we have six. This gives rise to the following statements, based on the fact that conjugate elements have equal orders:

$$[1] \quad |xyzw| = |yzwx| = |zwxy| = |wxyz|$$

$$[2] \quad |xywz| = |ywzx| = |wzxy| = |zxyw|$$

$$[3] \quad |xwzy| = |wzyx| = |zyxw| = |yxwz|$$

$$[4] \quad |wyzx| = |yzwx| = |zxwy| = |xwyz|$$

$$[5] \quad |xzyw| = |zywx| = |ywxz| = |wxzy|$$

$$[6] \quad |yxzw| = |xzwy| = |zwyx| = |wyxz|.$$

We may also extend the definition of $\omega_{(x,y,z)}$ to $\omega_{(x,y,z,w)}$ as follows:

$$\omega_{(x,y,z,w)} = |\{|xyzw|, |xywz|, |xwzy|, |wyzx|, |xzyw|, |yxzw|\}|.$$

Intuitively, the more four-tuples (x, y, z, w) for which $\omega_{(x,y,z,w)} = 1$, the more commutativity the group enjoys. However, the problem is more complex in the case of four-tuples because there are 6 possible values for $\omega_{(x,y,z,w)}$. Rather than simply computing the proportion of four-tuples for which $\omega_{(x,y,z,w)} = 1$ to gain information about the group's commutativity, it becomes necessary to take weighted averages of the number of four-tuples for which $\omega_{(x,y,z,w)} = 1, 2, 3, 4, 5$, and 6. Certainly, G is abelian if and only if this average is 1. This is the analogous result to Theorem 1. We also expect that the larger this average is, the "less" abelian the group is. We expect the average for dihedral groups, for example, to be lower, due to their relatively abelian nature.

Theorem 29 *Let* $x,y,z,w \in D_n$.

$$|\{|xyzw|, |xywz|, |xzyw|, |xzwy|, |xwyz|, |xwzy|\}| \leq 4.$$

Proof: Using the same argument as in Lemma 4, we see that if $\{x, y, z, w\}$ contains exactly zero, one, or three elements of the form sr^j, then $|\{|xyzw|, |xywz|, |xzyw|, |xzwy|, |xwyz|, |xwzy|\}| = 1$.

Suppose $\{x, y, z, w\}$ contains exactly two elements of the form sr^j. From the list of conjugate four-tuples above, we clearly may assume without loss of generality that $x = sr^{j_1}$, $y = sr^{j_2}$, $z = r^{i_1}$, and $w = r^{i_2}$, where j_k, $i_k \in \mathbf{Z}$. Then since z and w commute with each other, $|xyzw| = |xywz|$ and $|xwzy| = |xzwy|$. Hence, $|\{|xyzw|, |xywz|, |xzyw|, |xzwy|, |xwyz|, |xwzy|\}| \leq 4$.

Finally, suppose $x = sr^{j_1}$, $y = sr^{j_2}$, $z = sr^{j_3}$, and $w = sr^{j_4}$. Then $|sr^{j_1}sr^{j_2}sr^{j_3}sr^{j_4}| = |r^{j_2+j_4-j_1-j_3}| = |sr^{j_1}sr^{j_4}sr^{j_3}sr^{j_2}|$. Hence, $|xyzw| = |xwzy|$. Similarly, $|xywz| = |xzwy|$, and $|xzyw| = |xwyz|$. Thus, $|\{|xyzw|, |xywz|, |xzyw|, |xzwy|, |xwyz|, |xwzy|\}| \leq 3$. •

(5) Conjecture: If G is a nilpotent group, then $W_G \geq \frac{7}{8}$. We have already seen in Theorem 7 that the result is true for nilpotent dihedral groups (which also proves that if a $\frac{7}{8}$ bound exists, it is sharp). Using CAYLEY, we performed random samples of 1000 triples from each non-abelian, nilpotent group of order ≤ 100 and discovered, in each case, that at least $\frac{7}{8}$ of the triples (x, y, z) satisfied $\omega_{(x,y,z)} = 1$.

A Data

We present data supporting Conjecture 1 in Section 6.

n	W_{S_n} for random sample of 1000 triples
1	1
2	1
3	.832
4	.566
5	.407
6	.435
7	.266
8	.232
9	.149
10	.163
11	.111
12	.113
13	.106
14	.072
15	.076
16	.072
17	.070
18	.056
19	.059
20	.050
30	.037
40	.034