

## Generalizations of Goursat's Theorem for Groups

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# GENERALIZATIONS OF GOURSAT'S THEOREM FOR GROUPS

KRISTINA KUBLIK

ABSTRACT. Petrillo's recent article in the College Mathematics Journal explained a theorem of Goursat on the subgroups of a direct product of two groups. In this note, we extend this theorem to commutative rings, and to modules over commutative rings and fields.

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## 1. INTRODUCTION

A recent article by J. Petrillo in the College Mathematics Journal, [4], explained a little-known result of Edouard Goursat that describes subgroups of a direct product of two groups. In this note we will extend the result to apply to the direct product of two commutative rings, and to the direct product of modules over a commutative ring. First we state Goursat's Theorem and outline the proof.

**Theorem 1.1. Goursat's Theorem** *If  $G_1$  and  $G_2$  are groups then there exists a bijection between the set  $S$  of all subgroups of  $G_1 \times G_2$  and the set  $T$  of all 5-tuples  $(A_1, B_1, A_2, B_2, \varphi)$  where  $A_i$  is a subgroup of  $G_i$ ,  $B_i$  is a normal subgroup of  $A_i$ , and  $\varphi$  is a group isomorphism mapping  $A_1/B_1$  to  $A_2/B_2$ .*

Let  $G_1$  and  $G_2$  be groups. Given a subgroup  $U$  of  $G_1 \times G_2$  there is a unique 5-tuple  $(A_{U_1}, B_{U_1}, A_{U_2}, B_{U_2}, \varphi_U)$  where  $B_{U_i} \trianglelefteq A_{U_i}$  are subgroups of  $G_i$  and  $\varphi_U : A_{U_1}/B_{U_1} \rightarrow A_{U_2}/B_{U_2}$  is a group isomorphism. This 5-tuple is formed as follows.

Define the projection maps  $\pi_i : G_1 \times G_2 \rightarrow G_i$  by

$$\pi_i : (g_1, g_2) \mapsto g_i.$$

Since the  $\pi_i$  are group homomorphisms, then the images of the  $\pi_i$  restricted to  $U$  are subgroups of  $G_i$ . Furthermore, the image of the kernel of  $\pi_i$  restricted to  $U$  is a normal subgroup of  $G_j$  with  $j \neq i$ .

Thus we have that with

$$\begin{aligned} A_{U_1} &= \text{Im}(\pi_1|_U) = \{g_1 \in G_1 \mid (g_1, g_2) \in U \text{ for some } g_2 \in G_2\}; \\ B_{U_1} &= \pi_1(\text{Ker}(\pi_2|_U)) = \{g_1 \in G_1 \mid (g_1, e_{G_2}) \in U\}; \\ A_{U_2} &= \text{Im}(\pi_2|_U) = \{g_2 \in G_2 \mid (g_1, g_2) \in U \text{ for some } g_1 \in G_1\}; \\ B_{U_2} &= \pi_2(\text{Ker}(\pi_1|_U)) = \{g_2 \in G_2 \mid (e_{G_1}, g_2) \in U\}. \end{aligned}$$

and with  $\varphi_U : A_{U_1}/B_{U_1} \rightarrow A_{U_2}/B_{U_2}$  defined by

$$\varphi_U(a_1 B_{U_1}) = a_2 B_{U_2} \text{ when } (a_1, a_2) \in U;$$

then  $\varphi_U$  is an isomorphism and so the 5-tuple  $(A_{U_1}, B_{U_1}, A_{U_2}, B_{U_2}, \varphi_U)$  lies in  $T$ .

Conversely, given  $(A_1, B_1, A_2, B_2, \varphi) \in T$  then the set  $U_\varphi$  defined as

$$U_\varphi = \{(a_1, a_2) \in A_1 \times A_2 \mid \varphi(a_1 B_1) = a_2 B_2\}$$

is a subgroup of  $G_1 \times G_2$ .

Define  $\alpha : S \rightarrow T$  by  $\alpha(U) = (A_{U_1}, B_{U_1}, A_{U_2}, B_{U_2}, \varphi_U)$  and then  $\alpha$  is a bijection with inverse  $\beta : T \rightarrow S; (A_1, B_1, A_2, B_2, \varphi) \mapsto U_\varphi$ .

The proof of this theorem is outlined as a series of exercises in [4]; full details can be found in [2].

We are now going to generalize Goursat's Theorem to direct products of commutative rings and to direct products of modules over commutative rings.

## 2. PRELIMINARIES

We assume that the reader is familiar with elementary group theory. A good reference for this material is [5].

**Definition 2.1.** A **commutative ring**  $R$  is a set with two operations, addition  $+$  and multiplication written  $\cdot$  or by concatenation such that  $(R, +)$  is an abelian group with identity written  $0$ , multiplication is associative, commutative, and has an identity element  $1 \neq 0$ , and the distributivity property

$$a(b + c) = ab + ac, \text{ holds for every } a, b, c \in R.$$

**Definition 2.2.** An **ideal** in a commutative ring  $R$  is a subgroup  $(I, +)$  of  $(R, +)$  such that if  $a \in I$  and  $r \in R$ , then  $ra \in I$ . An ideal  $I$  different from  $R$  or  $\{0\}$  is called a **proper ideal**.

**Definition 2.3.** If  $R_1$  and  $R_2$  are commutative rings, a **ring homomorphism** is a function  $f : R_1 \rightarrow R_2$  such that  $f$  is a group homomorphism from  $(R_1, +_1)$  to  $(R_2, +_2)$  and

- (i)  $f(1) = 1$ ;
- (ii)  $f(aa') = f(a)f(a')$  for all  $a, a' \in R_1$ .

If  $f$  is an bijection then it is called a **ring isomorphism** and we write  $R_1 \cong R_2$ .

**Definition 2.4.** If  $f : R_1 \rightarrow R_2$  is a ring homomorphism, then its **kernel** is

$$\text{Ker } f = \{a \in R_1 \mid f(a) = 0\}.$$

Note that  $\text{Ker } f$  is an ideal in  $R_1$ . The **image** of  $f$  is

$$\text{Im } f = \{r \in R_2 \mid r = f(a) \text{ for some } a \in R_1\},$$

where  $\text{Im } f$  is a subring of  $R_2$ . [5, p. 248]

**Definition 2.5.** Let  $R$  be a commutative ring. An **R-module** is a set  $M$  together with maps  $M \times M \rightarrow M$ ,  $(m, n) \mapsto m + n$ , and  $R \times M \rightarrow M$ ,  $(r, m) \mapsto rm$ , called addition and scalar multiplication such that  $(M, +)$  is an abelian group and for all  $r, s \in R$  and  $m, n \in M$ :

- (a)  $(r + s)m = rm + sm$ ,
- (b)  $(rs)m = r(sm)$ ,
- (c)  $r(m + n) = rm + rn$ .

An **R-submodule**  $N$  of  $M$  is a subset of  $M$  that is also an  $R$ -module with the same addition and scalar multiplication. [1, p. 318]

For example,  $R$  is itself an  $R$ -module under the addition in  $R$  and with multiplication in  $R$  as scalar multiplication. Ideals in  $R$  are submodules of the  $R$ -module  $R$ .

**Definition 2.6.** Let  $R$  be a ring and let  $M$  and  $N$  be  $R$ -modules. A map  $\xi : M \rightarrow N$  is an **R-module homomorphism** if  $\xi$  is a homomorphism of abelian groups from  $(M, +)$  to  $(N, +)$  and  $\xi(rx) = r\xi(x)$ , for all  $r \in R$ , and  $x \in M$ . We call  $\xi$  an **R-module isomorphism** if  $\xi$  is a bijection. [1, p. 326]

**Definition 2.7.** Let  $R$  be a ring,  $M$  be an  $R$ -module and  $N$  be a submodule of  $M$ . The (additive, abelian) quotient group  $M/N$  can be made into an  $R$ -module by defining an action of  $R$  on  $M/N$  by

$$r(x + N) = (rx) + N, \text{ for all } r \in R, x + N \in M/N.$$

The natural projection map  $\pi : M \rightarrow M/N$  defined by  $\pi(x) = x + N$  is an  $R$ -module homomorphism with kernel  $N$ .

*Remark 2.1.* The notation  $\mathbb{Z}_n$  will be used to denote  $\mathbb{Z}/n\mathbb{Z}$ , the integers modulo  $n$ .

**Definition 2.8.** A field  $F$  is a commutative ring with  $1 \neq 0$  in which every nonzero element  $a$  is a multiplicative unit; that is, there is  $a^{-1} \in F$  with  $a^{-1}a = 1$ . [5, p. 230]

For example, the complex numbers, the real numbers and the rational numbers are fields. The integers form a commutative ring but not a field.

*Remark 2.2.* If  $R$  is a field then an  $R$ -module is called a vector space.

### 3. GOURSAT'S THEOREM FOR THE DIRECT PRODUCT OF COMMUTATIVE RINGS

Let  $R_1$  and  $R_2$  be commutative rings, with multiplicative identities denoted  $1_1$  and  $1_2$ . It is easy to check that the direct product  $R_1 \times R_2$  is also a commutative ring with component-wise operations  $\cdot_i$  and  $+_i$ . We show that there exists a bijection between  $\mathcal{S}$  (set of subrings of  $R_1 \times R_2$ ) and  $\mathcal{T}$ , the set of 5-tuples  $(A_1, B_1, A_2, B_2, \varphi)$  where  $A_i$  is a subring of  $R_i$ ,  $B_i$  is an ideal in  $A_i$ , and  $\varphi : A_1/B_1 \rightarrow A_2/B_2$  is a ring isomorphism.

**Proposition 3.1.** Given  $V \in \mathcal{S}$ , we define  $A_{V_i}$  and  $B_{V_i}$  as in Section 1 by letting  $G_i = R_i$  and  $e_{G_i} = 0$ . Define  $\varphi_V : A_{V_1}/B_{V_1} \rightarrow A_{V_2}/B_{V_2}$  by

$$\varphi_V(a_1 + B_{V_1}) = a_2 + B_{V_2} \text{ when } (a_1, a_2) \in V.$$

Then the 5-tuple  $(A_{V_1}, B_{V_1}, A_{V_2}, B_{V_2}, \varphi_V)$  lies in  $\mathcal{T}$ .

*Proof.* Since the projection  $\pi_i$  is a ring epimorphism,  $A_{V_i}$  is the image of  $\pi_i|_V$  making  $A_{V_i}$  a subring of  $R_i$ . For  $j \neq i$ , the kernel of  $\pi_j|_V$  is  $(R_1 \times 0) \cap V$  or  $(0 \times R_2) \cap V$  so that  $B_{V_i}$  is an ideal of  $A_{V_i}$ .

By Goursat's Theorem for groups, Theorem 1.1,  $\varphi_V$  is a group isomorphism and therefore a bijection. It remains to show that the ring structure is respected.

Since  $V$  is a subring,  $(1, 1) \in V$  so then  $\varphi_V(1 + B_{V_1}) = 1 + B_{V_2}$  and  $\varphi$  maps the multiplicative unit in  $A_{V_1}/B_{V_1}$  to the multiplicative unit in  $A_{V_2}/B_{V_2}$ . For the multiplicative property, note that if  $(a_1, a_2)$ , and  $(a'_1, a'_2)$  are in  $V$ , then  $(a_1a'_1, a_2a'_2) \in V$ . Thus,

$$\varphi_V(a_1 + B_{V_1})\varphi_V(a'_1 + B_{V_1}) = (a_2 + B_{V_2})(a'_2 + B_{V_2}) = a_2a'_2 + B_{V_2} = \varphi_V(a_1a'_1 + B_{V_1}).$$

Therefore, we can construct the 5-tuple  $(A_{V_1}, B_{V_1}, A_{V_2}, B_{V_2}, \varphi_V)$  from a given subring  $V$  of  $R_1 \times R_2$ . □

**Proposition 3.2.** Given a 5-tuple  $(A_1, B_1, A_2, B_2, \varphi) \in \mathcal{T}$ , the set  $V_\varphi$  defined by  $V_\varphi = \{(a_1, a_2) \in R_1 \times R_2 \mid \varphi(a_1 + B_1) = a_2 + B_2\}$  is a subring of  $R_1 \times R_2$ .

*Proof.* From Goursat's Theorem, we already know that  $(V_\varphi, +)$  is a subgroup of  $(R_1 \times R_2, +)$  so we just need to prove it is also a ring. Since  $\varphi$  is a ring isomorphism, then  $\varphi(1 + B_1) = 1 + B_2$  and thus  $(1, 1) \in V_\varphi$ . Also,  $V_\varphi$  is closed under multiplication because if  $(a_1, a_2), (a'_1, a'_2) \in V_\varphi$ , then  $\varphi(a_1 + B_1)\varphi(a'_1 + B_1) = (a_2 + B_2)(a'_2 + B_2) = a_2a'_2 + B_2$  hence  $(a_1a'_1, a_2a'_2) \in V_\varphi$ .

Therefore  $V_\varphi$  is a subring of  $R_1 \times R_2$ . □

### Theorem 3.1. Goursat's Theorem for Commutative Rings

If  $R_1$  and  $R_2$  are commutative rings then there exists a bijection between the set  $\mathcal{S}$  (subrings of  $R_1 \times R_2$ ) and  $\mathcal{T}$ , the set of all 5-tuples  $(A_1, B_1, A_2, B_2, \varphi)$  where  $B_i$  is an ideal of  $A_i$ ,  $A_i$  is a subring of  $R_i$ , and  $\varphi : A_1/B_1 \rightarrow A_2/B_2$  is a ring isomorphism.

*Proof.* Define the mapping  $\alpha : \mathcal{S} \rightarrow \mathcal{T}$  by:  $\alpha(V) = (A_{V_1}, B_{V_1}, A_{V_2}, B_{V_2}, \varphi_V)$ ; and  $\beta : \mathcal{T} \rightarrow \mathcal{S}$  by:  $\beta(A_1, B_1, A_2, B_2, \varphi) = V_\varphi$ .

By Proposition 3,  $\alpha$  maps  $\mathcal{S}$  to  $\mathcal{T}$ , by Proposition 3.2,  $\beta$  maps  $\mathcal{T}$  to  $\mathcal{S}$  and by Goursat's Theorem for groups,  $\alpha$  and  $\beta$  are inverse bijections.  $\square$

**Example 3.1.** Subrings of  $\mathbb{Z}_2 \times \mathbb{Z}_6$ . Table 1 lists the subrings  $A_1$  and  $A_2$ , their ideals  $B_1$  and  $B_2$  and the quotients  $A_1/B_1$  and  $A_2/B_2$  that can be formed if  $R_1 = \mathbb{Z}_2$  and  $R_2 = \mathbb{Z}_6$ .

TABLE 1.  $R_1 = \mathbb{Z}_2$  and  $R_2 = \mathbb{Z}_6$

Subrings of $\mathbb{Z}_2$				Subrings of $\mathbb{Z}_6$			
$A_1$	$B_1$	$A_1/B_1$	Order	$A_2$	$B_2$	$A_2/B_2$	Order
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2/\mathbb{Z}_2$	1	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_6/\mathbb{Z}_6$	1
$\mathbb{Z}_2$	$\{0\}$	$\mathbb{Z}_2/\{0\}$	2	$\mathbb{Z}_6$	$\langle [2] \rangle$	$\mathbb{Z}_6/\langle [2] \rangle$	2
				$\mathbb{Z}_6$	$\langle [3] \rangle$	$\mathbb{Z}_6/\langle [3] \rangle$	3

The only subring of the form  $A_1 \times A_2$  is  $\mathbb{Z}_2 \times \mathbb{Z}_6$ . Applying Theorem 3.1 the isomorphism  $\varphi : \mathbb{Z}_2/\{0\} \rightarrow \mathbb{Z}_6/\langle [2] \rangle$  can be used to form the subring

$$U_\varphi = \{(0, 0), (0, 2), (0, 4), (1, 1), (1, 3), (1, 5)\}.$$

Note that this is the only nontrivial subring of  $\mathbb{Z}_2 \times \mathbb{Z}_6$ .

#### 4. GOURSAT'S THEOREM FOR THE DIRECT PRODUCT OF MODULES

Let  $R$  be a commutative ring and  $M_1, M_2$  be  $R$ -modules. The direct product  $M_1 \times M_2$  is also an  $R$ -module with component-wise addition and scalar multiplication given by  $r(m_1, m_2) = (rm_1, rm_2)$ . Define the projection  $\pi_i : M_1 \times M_2 \rightarrow M_i$  by  $(m_1, m_2) \mapsto m_i$ . It is easy to see that this is an  $R$ -module homomorphism onto  $M_i$  with the kernel of  $\pi_i$  isomorphic to  $M_j$ ,  $j \neq i$ .

We apply Goursat's Theorem for groups to find all  $R$ -submodules of  $M_1 \times M_2$ .

**Theorem 4.1.** *Let  $M_1$  and  $M_2$  be  $R$ -modules. There is a one to one correspondence between submodules  $N$  of the  $R$ -module  $M_1 \times M_2$  and 5-tuples  $(W_1, S_1, W_2, S_2, \varphi)$  where  $S_i \subseteq W_i$  are submodules of  $M_i$  and  $\varphi$  is an  $R$ -module isomorphism from  $W_1/S_1$  to  $W_2/S_2$ .*

*Proof.* Let  $N$  be an  $R$ -submodule of  $M_1 \times M_2$  and let  $(W_1, S_1, W_2, S_2, \varphi : W_1/S_1 \rightarrow W_2/S_2)$  be the tuple associated to the abelian group  $(N, +)$  by Goursat's Theorem. The image of  $\pi_i(N) = W_i$  so the subgroups  $(W_i, +)$  of  $(M_i, +)$  are  $R$ -submodules. The kernel  $J_i$  of the map  $\pi_i$  restricted to  $N$  is  $(0 \times M_j) \cap N$  and  $S_j = \pi_j(J_i)$  is a submodule of  $W_j$ . That is  $S_i \subseteq W_i$  are submodules of  $M_i$ . It remains to show that the isomorphism  $\varphi$  from Goursat's Theorem for groups preserves scalar multiplication. Recall that  $\varphi(w_1 + S_1) = w_2 + S_2$  if and only if  $(w_1, w_2) \in N$ . But then  $(rw_1, rw_2) \in N$  for any  $r \in R$  and thus

$$\varphi(rw_1 + S_1) = rw_2 + S_2 = r\varphi(w_1 + S_1)$$

as required.

Conversely we show that the subgroup  $N$  of  $(M_1 \times M_2, +)$  constructed from a 5-tuple  $(W_1, S_1, W_2, S_2, \varphi : W_1/S_1 \rightarrow W_2/S_2)$  is an  $R$ -submodule of  $M_1 \times M_2$ . Let

$(w_1, w_2) \in N$  if and only if  $\varphi(w_1 + S_1) = w_2 + S_2$ . Then since  $\varphi$  preserves scalar multiplication,  $\varphi(rw_1 + S_1) = rw_2 + S_2$  and  $(rw_1, rw_2) \in N$ . Thus  $N$  is closed under componentwise scalar multiplication and the proof of the theorem is complete.  $\square$

Recall that if the commutative ring is a field  $k$  then a  $k$ -module is called a vector space.

**Corollary 4.1.** For  $V_1 \times V_2$  a direct product of  $k$ -vector spaces, the subspaces of  $V_1 \times V_2$  are in one to one correspondence with the 5-tuples  $(W_1, S_1, W_2, S_2, \varphi)$  where  $S_i$  is a subspace of  $W_i$ ,  $W_i$  is a subspace of  $V_i$  and  $\varphi$  is an isomorphism as defined in Theorem 4.1.

**Example 4.1.** Using the notation from Theorem 4.1 let  $R = \mathbb{Z}_{18}$ , then  $M_1 \times M_2 = \mathbb{Z}_{18} \times \mathbb{Z}_{18}$  is an  $R$ -module. Let  $W_1 = W_2 = \langle [2] \rangle$ ,  $S_1 = S_2 = \langle [6] \rangle$ , and define  $\varphi : W_1/S_1 \rightarrow W_2/S_2$  to be the identity map. Then

$$N_\varphi = \langle [6] \rangle \times \langle [6] \rangle \cup ([2] + \langle [6] \rangle \times [2] + \langle [6] \rangle) \cup ([4] + \langle [6] \rangle \times [4] + \langle [6] \rangle),$$

or more explicitly,

$$\begin{aligned} N_\varphi = & \{(0, 0), (0, 6), (0, 12), (6, 0), (6, 6), (6, 12), (12, 0), (12, 6), (12, 12) \\ & (2, 2), (2, 8), (2, 14), (8, 2), (8, 8), (8, 14), (14, 2), (14, 8), (14, 14) \\ & (4, 4), (4, 10), (4, 16), (10, 4), (10, 10), (10, 16), (16, 4), (16, 10), (16, 16)\}. \end{aligned}$$

Note that  $S_1 \times S_2 \subseteq N_\varphi \subseteq W_1 \times W_2$ .

**Example 4.2.** Let  $R = \mathbb{Z}[x]$  the polynomials over  $\mathbb{Z}$ . If  $M_1 = \langle x \rangle$ , (polynomials with a zero constant term) and  $M_2 = \langle x^2 \rangle$  (polynomials where each term has degree 2 or higher) then  $M_1 \times M_2$  is an  $R$ -module.

Using the notation from Theorem 4.1, let  $W_i = M_i$ ,  $S_i = \{0\}$ , and define  $\varphi : W_1 \rightarrow W_2$  by  $p(x) \mapsto xp(x)$  where  $p(x) \in \mathbb{Z}[x]$  has a zero constant term. Clearly  $\varphi$  is a bijection and preserves scalar multiplication.

Then  $N_\varphi = \{(p(x), xp(x)) \mid \text{for all } p(x) \in \langle x \rangle\}$ .

**Example 4.3.** Let  $k = \mathbb{R}$  and  $V = \mathbb{R} \times \mathbb{R}$ . It is well-known that the only subspaces of  $\mathbb{R}^2$  are:  $\mathbb{R}^2$  itself, lines through the origin and the point  $(0, 0)$ . We use Goursat's Theorem for vector spaces together with the fact that the only subspaces of  $\mathbb{R}$  are  $\mathbb{R}$  and  $\{0\}$  to verify this.

- Case 1: Using the notation from Corollary 4.1 let  $W_1 = W_2 = \mathbb{R} = S_1 = S_2$ . The identity isomorphism  $\varphi : \mathbb{R}/\mathbb{R} \rightarrow \mathbb{R}/\mathbb{R}$  gives the entire space  $\mathbb{R}^2$ .
- Case 2: Let  $W_1 = W_2 = \mathbb{R}$ ,  $S_1 = S_2 = 0$ , and define  $\varphi : \mathbb{R}/0 \rightarrow \mathbb{R}/0$ ; by  $r \mapsto \alpha r$ , where  $\alpha$  is a nonzero scalar. Then the subspace  $U_\alpha = \{(r, \alpha r) \mid r \in \mathbb{R}\}$ , which we know is a line through the origin with slope  $\alpha$ .
- Case 2': If  $W_1 = S_1 = 0$  and  $W_2 = S_2 = \mathbb{R}$  then the resulting subspace is the line  $\{0\} \times \mathbb{R}$ . Similarly the subspace  $\mathbb{R} \times \{0\}$  is found when  $W_1 = S_1 = \mathbb{R}$  and  $W_2 = S_2 = 0$ . Note that  $\mathbb{R} \times \{0\}$  can also be found using the method in Case 2 by letting  $\alpha = 0$ .
- Case 3: When  $W_1 = W_2 = 0 = S_1 = S_2$  the resulting subspace is the origin,  $\{(0, 0)\}$ .

**Example 4.4.** When  $k = \mathbb{R}$  and  $V = \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$  the only subspaces of  $V$  are lines or planes passing through the origin.

Case 1: If  $W_1 = S_1$  and  $W_2 = S_2$ , then the subspace produced by applying Goursat's theorem for modules is  $W_1 \times W_2$  where  $W_1 \in \{\mathbb{R}^2, \mathbb{R} \times \{0\}, \{0\} \times \mathbb{R}, U_\alpha \text{ with } \alpha \neq 0, \{(0,0)\}\}$  and  $W_2 \in \{\mathbb{R}, \{0\}\}$ . (Here  $U_\alpha$  follows the notation used in Example 4.3.) The subspaces obtained using Goursat's Theorem are: all of  $\mathbb{R}^3$ , the  $x - y, y - z$  and  $x - z$  planes, the  $x, y$  and  $z$  axes, lines  $U_\alpha$  through the origin, and planes parallel to the  $z$ -axis and intersecting the  $x - y$  plane in the line  $U_\alpha$ .

Case 2: For some  $0 \neq \alpha \in \mathbb{R}$ , let  $W_1 = U_\alpha, S_1 = 0, W_2 = \mathbb{R}$ , and  $S_2 = 0$ . Define  $\phi : U_\alpha/0 \rightarrow \mathbb{R}/0$ ; by  $(r, \alpha r) + 0 \mapsto \delta r + 0$ , where  $\delta \in \mathbb{R}$  is a stretch factor.

This mapping gives the subspace  $V'$  where every element in the subspace has the form  $(r, \alpha r, \delta r)$  for all  $r \in \mathbb{R}$ . The line  $V' \cong \mathbb{R}$  passes through the origin.

Case 3: Let  $W_1 = \mathbb{R}^2$  and let  $S_1 = U_\alpha$  for some nonzero  $\alpha$ . Then the quotient space  $\mathbb{R}^2/U_\alpha$  is one dimensional and we may assume that any element in this quotient has the form  $(0, s) + U_\alpha$  since  $(x, y) + U_\alpha = (x, y) - (x, \alpha x) + U_\alpha$ . Define the isomorphism  $\psi : \mathbb{R}^2/U_\alpha \rightarrow \mathbb{R}/0$  by  $(0, s) + U_\alpha \mapsto \gamma s + \{0\}$  where  $0 \neq \gamma \in \mathbb{R}$  is a stretch factor. The resulting subspace is a plane  $V_\gamma$  that when cut along  $\mathbb{R}^2$  at the origin you get the line  $U_\alpha$ . Any point in  $V_\gamma$  has the form  $(s + r, s + \alpha r, \gamma s)$  as  $r$  and  $s$  go through  $\mathbb{R}$ .

Therefore, as expected, any non-trivial subspace of  $\mathbb{R}^3$  will be either a plane or a line passing through the origin.

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