12-1996

A Stronger Triangle Inequality

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MSTR 96-05

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December 1996

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The triangle inequality is basic for many results in real and complex analysis. The geometric form states that the sum of any two sides of a triangle is greater than the third. This was included as Proposition XX in the first book of Euclid's Elements. Many geometric triangle inequalities involving sides, angles, altitudes, inscribed circles and circumscribed circles have been found. Hundreds of these inequalities are summarized in [1] and [2]. A nice geometric proof of the triangle inequality is given in [3].

We will show that a stronger inequality,

$$a + b > c + h \tag{1}$$

is satisfied for most triangles with $\theta < \pi/2$, where (Figure 1) a, b, and c are the sides of a triangle, h is the altitude to side c, and θ is the angle opposite c. Our interest was drawn to this inequality by a classical problem of comparing the sizes of two squares inscribed in a right triangle, whose solution turns on proving that (1) is always false if $\theta = \pi/2$.

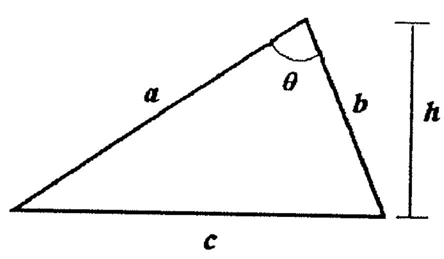

Figure 1

Let's begin with the inscribed square problem. Squares of sides s and t are inscribed in a right triangle with sides a, b, hypotenuse c and altitude h (Figures 2 and 3). The problem is to determine which of the two squares is larger.

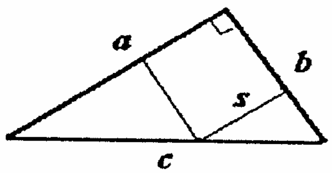

Figure 2

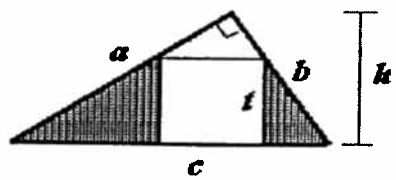

Figure 3

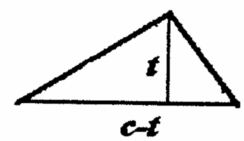

Figure 4

Form a new triangle (Figure 4) by combining the shaded areas in Figure 3. Then by similar triangles we have

$$\frac{s}{b-s} = \frac{a}{b} \qquad \Rightarrow \qquad s = \frac{ab}{a+b} \qquad \text{, and}$$

$$\frac{t}{c-t} = \frac{h}{c} \qquad \Rightarrow \qquad t = \frac{hc}{c+h}.$$

Note that the numerators, ab and hc, are equal since they are both equal to twice the area of the given triangle.

To compare the denominators, we set

$$D := (a+b)^2 - (c+h)^2 = a^2 + 2ab + b^2 - c^2 - 2hc - h^2.$$

Since $a^2 + b^2 = c^2$ and $hc = ab$, we have $D = -h^2 < 0$, thus $a + b < c + h$ and we conclude that $s > t$.

In brief, then, $s>t$ since inequality (1) is false for any right triangle. We now compare

$a{+}b$ with $c{+}h$ when the angle θ opposite side c is not required to be a right angle, as in Figure 1. Note that inequality (1) is equivalent to $D{:=}(a+b)^\wedge 2-(c+h)^\wedge 2>0$. We can express c and h in terms of a, b, and θ and by using the law of cosines,

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

and an equation relating two expressions for the area of a triangle,

$$\frac{1}{2}ch = \frac{1}{2}ab\sin\theta.$$

Substituting,

$$D = (a^2 + b^2 - c^2) + 2ab - 2ch - h^2$$

$$= 2ab\cos\theta + 2ab - 2ab\sin\theta - \frac{(ab\sin\theta)^2}{c^2}$$

$$= ab\left(2 + 2\cos\theta - 2\sin\theta - \frac{ab\sin^2\theta}{a^2 + b^2 - 2ab\cos\theta}\right).$$

Setting

$$g(\theta) := 2(1 + \cos\theta - \sin\theta).$$

and $R{:=}a/b$ yields

$$D = ab\left(g(\theta) - \frac{R\sin^2\theta}{R^2 + 1 - 2R\cos\theta}\right).$$

Now ab and $R^2 + 1 - 2R\cos\theta$ are positive, so inequality (1) holds if and only if $D(R^2 + 1 - 2R\cos\theta)/(ab) = g(\theta)R^2 - (2g(\theta)\cos\theta + \sin^2\theta)R + g(\theta)$ is positive. Figure 5 shows the graph of the surface $F(\theta, R) := g(\theta)R^2 - (2g(\theta)\cos\theta + \sin^2\theta)R + g(\theta)$, along with the plane z=0.

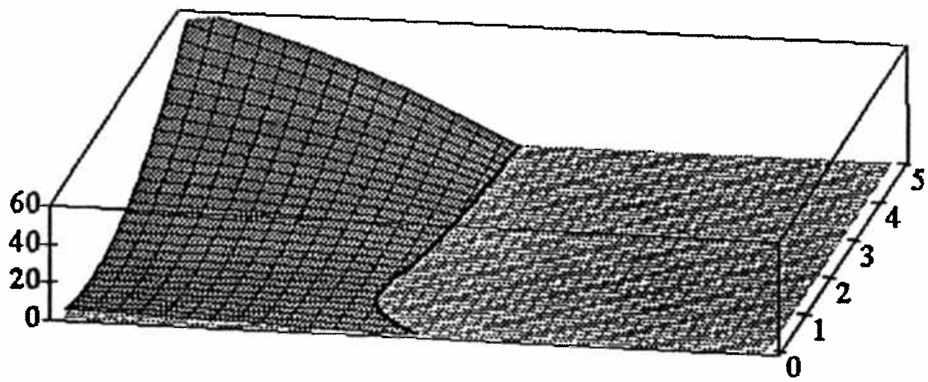

Figure 5

Inequality (1) then holds to the left of the level curve $F(\theta, R) = 0$ since the surface is above the z=0 plane in this region. Thus the strong inequality holds for most triangles with $\theta < \pi/2$, and fails for all triangles with $\theta \geq \pi/2$. This level curve is shown in the (θ,R)-plane in Figure 6.

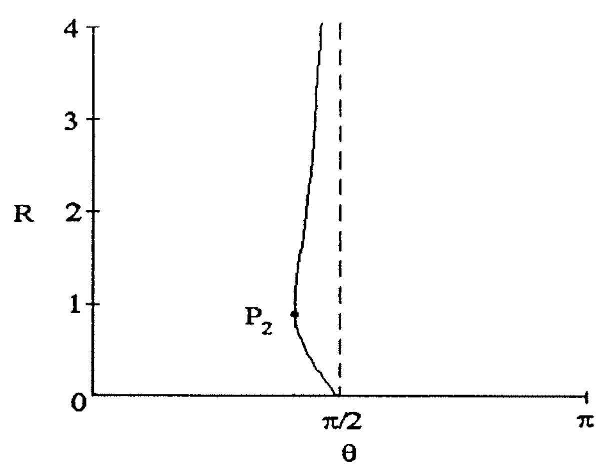

Figure 6

The graphs in Figures 5 and 6 are restricted to $R>0$ and $0<\theta<\pi$, since R and θ are associated with a triangle. The corresponding graphs without these restrictions are shown in Figures 7 and 8.

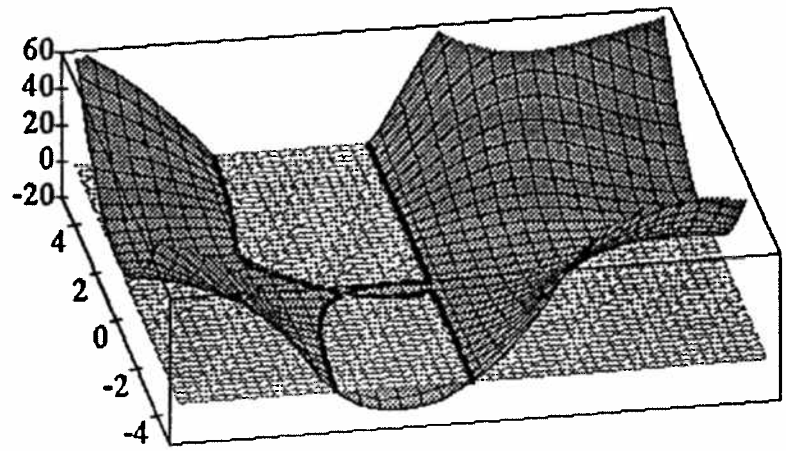

Figure 7

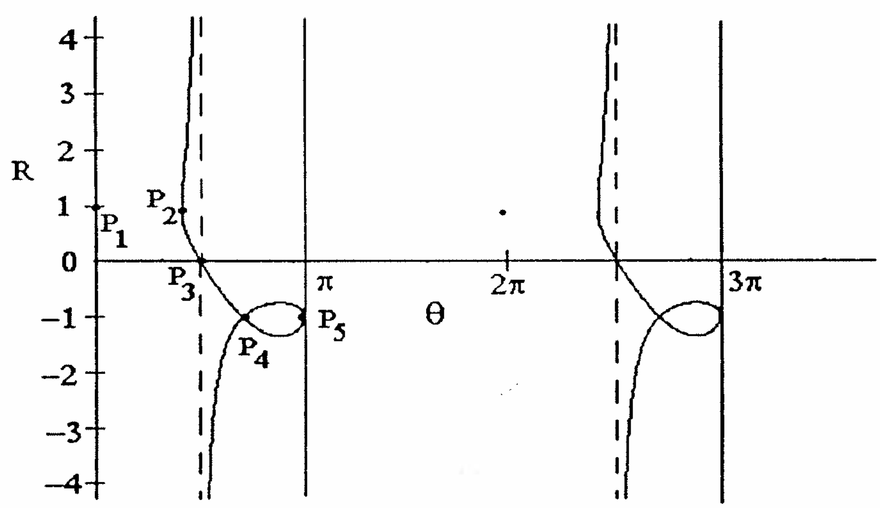

Figure 8

The solution set satisfying $F(\theta, R) = 0$ is periodic in θ with period 2π and the graph for two of its periods is shown in Figure 8. Concentrating on the first period ($0 \le \theta < 2\pi$), we consider two cases. If $g(\theta)=0$, then $\theta=\pi/2$ or π, and solving $F(\theta,R)=0$ for R gives two solutions: the point $P_3(\pi/2,0)$ and the line $\theta=\pi$. If $g(\theta) \neq 0$, then $F(\theta,R)=0$ is a quadratic in R. Its discriminant is $\left(2g(\theta)\cos\theta + \sin^2\theta\right)^2 - 4g(\theta)^2$, which can be factored, with the help of a computer algebra system, to

$$(1 - \cos\theta)(-3\cos\theta + 4\sin\theta - 3)(2 + 2\cos\theta - \sin\theta)^2 .$$

The zeros of the above discriminant occur at the four points: $P_1(0,1)$, $P_2(\arctan(24/7),1)$, $P_4(\pi - \arctan(4/3),-1)$ and $P_5((-1,\pi)$. These points along with P_3 and the asymptotes at $\pi/2$ and $5\pi/2$ are shown in Figure 8. The upper and lower branches of the quadratic meet at P_2, P_3, P_4, and P_5. The point $P_1(0,1)$ is an isolated solution of $F(\theta, R) = 0$.

References

1. O. Bottema et.al., Geometric Inequalities, Grouingen, 1969.

2. D.S. Mitrinovic, et al, Recent Advances in Geometric Inequalities, Kluwer Academic Publishers, 1989.

3. M.F. Smiley, The Proof of the Triangle Inequality, Am Math Monthly 70 (1963) 546.