Cut-sets and Cut-vertices in the Zero-Divisor Graph of $\prod \mathbb{Z}_n$

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CUT-SETS AND CUT-VERTICES IN THE ZERO-DIVISOR GRAPH OF $\prod_{i=1}^{m} \mathbb{Z}_{n_{i}}$

B. COTÉ, C. EWING, M. HUHN, C. M. PLAUT, D. WEBER

Abstract. We examine minimal sets of vertices which, when removed from a zero-divisor graph, separate the graph into disconnected subgraphs. We classify these sets for all direct products of $\Gamma \left( \prod_{i=1}^{m} \mathbb{Z}_{n_{i}} \right)$.

1. Introduction and Definitions

All rings in the paper are commutative with unity. An element $a \in R$ is a zero-divisor if there exists a nonzero $r \in R$ such that $ar = 0$; we denote the set of all zero-divisors in $R$ as $Z(R)$. For a graph $G$, define $V(G)$ as the set of vertices in $G$, and $E(G)$ as the set of edges in $G$. We define a path between two elements $a_1, a_m \in V(G)$ to be an ordered sequence of distinct vertices $\{a_1, a_2, \ldots, a_n\}$ of $G$ such that there is an edge incident to $a_{i-1}$ and $a_i$, denoted $a_{i-1} - a_i$ for each $i$. For $x, y \in V(G)$, the number of edges crossed to get from $x$ to $y$ in a path is called the length of the path; the length of the shortest path between $x$ and $y$, if it exists, is called the distance between $x$ and $y$ and is denoted $d(x, y)$. If such a path does not exist then $d(x, y) = \infty$. The diameter of a graph is $\text{diam}(G) = \max\{d(x, y) \mid, y \in V(G)\}$. A graph is connected if a path exists between any two distinct vertices.

A zero-divisor graph, denoted $\Gamma(R)$, is a graph whose vertices are all the nonzero zero-divisors of $R$. Two vertices $a$ and $b$ are connected by an edge in $\Gamma(R)$ if and only if $ab = 0$. In $R$, we define the annihilator of $a$, $\text{ann}(a)$, by $\text{ann}(a) = \{b \in R \mid ba = 0\}$, so that the neighbors of $a$ in $\Gamma(R)$ are the nonzero elements of $\text{ann}(a)$. A vertex $a$ is looped if and only if $a^2 = 0$. By [1], we know that $\Gamma(R)$ is always connected and $\text{diam}(\Gamma(R)) \leq 3$ for any ring $R$.

Definition 1.1. A vertex, $a$, in a connected graph $G$ is a cut-vertex if $G$ can be expressed as a union of two subgraphs $X$ and $Y$ such that $E(X) \neq \emptyset$, $E(Y) \neq \emptyset$, $E(X) \cup E(Y) = E(G)$, $V(X) \cup V(Y) = V(G)$, $V(X) \cap V(Y) = \{a\}$, $X \setminus \{a\} \neq \emptyset$, and $Y \setminus \{a\} \neq \emptyset$.

Definition 1.2. A set $A \subseteq Z(R)^*$, where $Z(R)^* = Z(R) \setminus \{0\}$, is said to be a cut-set if there exist $c, d \in Z(R)^* \setminus A$ where $c \neq d$ such that every path in $\Gamma(R)$ from $c$ to $d$ involves at least one element of $A$, and no proper subset of $A$ satisfies the same condition.

Another way to define a cut-set is as a set of vertices $\{a_1, a_2, a_3, \ldots\}$ in a connected graph $G$ where $G$ can be expressed as a union of two subgraphs $X$ and $Y$.

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such that $E(X) \neq \emptyset$, $E(Y) \neq \emptyset$, $E(X) \cup E(Y) = E(G)$, $V(X) \cup V(Y) = V(G)$, $V(X) \cap V(Y) = \{a_1, a_2, a_3, \ldots\}$, $X \setminus \{a_1, a_2, a_3, \ldots\} \neq \emptyset$, $Y \setminus \{a_1, a_2, a_3, \ldots\} \neq \emptyset$, and no proper subset of $\{a_1, a_2, a_3, \ldots\}$ also acts as a cut-set for any choice of $X$ and $Y$. A cut-vertex can be thought of as a cut-set with only one element. For a cut-set $A$ in $\Gamma(R)$, a vertex $a \notin A$ is said to be isolated, or an isolated point, if $\text{ann}(a) \setminus \{0\} \subseteq A$.

Example 1.3. Consider $\Gamma(Z_{12})$ shown in Figure 1. In this graph, 6 is a cut-vertex isolating 2 and 10. In addition, $\{4, 8\}$ is a cut-set isolating 3 and 9.

![Figure 1. $\Gamma(Z_{12})$ generated using [6].](image1)

Example 1.4. Consider $\Gamma(Z_{30})$ shown in Figure 2. In this graph, 15 is a cut-vertex isolating 2 among other vertices. Observe that the set $\{6, 12, 18, 24\}$ is a cut-set isolating 5 and 25. In addition, $\{10, 20\}$ is a cut-set isolating 3, 9, 21 and 27.

![Figure 2. $\Gamma(Z_{30})$ generated using [6].](image2)

The study of cut-vertices in a zero-divisor graph began in [3], where it was proven that if a vertex, $a$, is a cut-vertex of $\Gamma(R)$ for any commutative ring, $R$, then $\{0, a\}$
forms an ideal in R. We will generalize this notion and expand on many of the results from [3]. This paper will classify cut-vertices and cut-sets of zero-divisor graphs of finite commutative rings of the form \( \Pi(\mathbb{Z}_n) \). In section 2 we classify \( \Gamma(\mathbb{Z}_n) \), and apply our findings to cut-vertices of \( \Gamma(\Pi(\mathbb{Z}_n)) \). Section 3 classifies cut-sets of \( \Gamma(\Pi(\mathbb{Z}_n)) \) by examining cut-sets of \( \Gamma(\mathbb{Z}_n) \).

2. Cut-vertices in \( \Gamma \left( \prod_{i=1}^{m} \mathbb{Z}_{n_i} \right) \)

This section begins with an examination of cut-vertices in the ring \( \mathbb{Z}_n \) in preparation for generalizing to direct products. Recall that for a commutative ring \( R \), if \( a, b \in R^* \), where \( R^* \) is \( R \setminus \{0\} \), such that \( ab = 0 \), then \( a(-b) = 0 \). Also, in \( \mathbb{Z}_n \), let \( p \in \mathbb{Z} \) be a prime that divides \( n \). Then \( \text{ann}(p) \subseteq \text{ann}(ap) \) for any \( a \in \mathbb{Z} \).

**Theorem 2.1.** An element \( a \) is a cut-vertex of \( \Gamma(\mathbb{Z}_n) \) if and only if \( 2a = n \) with \( n \geq 6 \).

**Proof.** (\( \Rightarrow \)) Observe that \( \Gamma(\mathbb{Z}_n) \) has no cut-vertex for \( n < 6 \) [5]. So let \( n \geq 6 \), and assume that \( a \) is a cut vertex of \( \Gamma(\mathbb{Z}_n) \). Then \( \Gamma(\mathbb{Z}_n) \) is split into two subgraphs \( X \) and \( Y \), which are distinct except for their common vertex \( a \). Let \( V(X) = \{a, x_1, x_2, \ldots, x_m\} \) and \( V(Y) = \{a, y_1, y_2, \ldots, y_l\} \). Since \( a \) is a cut vertex, there exists some \( x_i \in V(X) \) and some \( y_j \in V(Y) \) such that \( x_i = a - y_j \). Since \( x_i \) is a cut-vertex, \( a = -a \) or \( 2a = 0 \) in \( \mathbb{Z}_n \). Thus, \( 2a = n \).

(\( \Leftarrow \)) It suffices to show \( \text{ann}(2) = \{0, a\} \) in \( \mathbb{Z}_{2a} \) with \( a \geq 3 \). Assume \( 2m = 0 \). This implies \( m = 0 \) or \( m = a \). Since \( a \neq 2, 2 \) is a vertex isolated by the cut-vertex \( a \).

**Theorem 2.2.** Let \( \mathbb{Z}_n \times \mathbb{Z}_m \neq \mathbb{Z}_2 \times \mathbb{Z}_2 \). Then \( (a, 0) \) is a cut-vertex of \( \Gamma(\mathbb{Z}_n \times \mathbb{Z}_m) \) if and only if \( 2a = n \).

**Proof.** (\( \Rightarrow \)) Assume \( (a, 0) \) is a cut-vertex of \( \Gamma(\mathbb{Z}_n \times \mathbb{Z}_m) \) separating subgraphs \( X \) and \( Y \). There exists some \( (x_{i_1}, x_{i_2}) \in V(X) \) and some \( (y_{j_1}, y_{j_2}) \in V(Y) \) such that \( (x_{i_1}, x_{i_2}) = (a, 0) = (y_{j_1}, y_{j_2}) \). But then, \( (x_{i_1}, x_{i_2}) = (-a, 0) = (y_{j_1}, y_{j_2}) \). Since \( (a, 0) \) is a cut-vertex, \( (-a, 0) \), which means \( a = -a \). This implies \( 2a = 0 \), in \( \mathbb{Z}_m \), so \( 2a = n \).

(\( \Leftarrow \)) Assume \( 2a = n \). In the case that \( n = 2 \), there is a cut-vertex at \( (1, 0) \) since it is the only element adjacent to \( (0, 1) \) and it is also adjacent to \( (0, 2) \) since \( \mathbb{Z}_n \times \mathbb{Z}_m \neq \mathbb{Z}_2 \times \mathbb{Z}_2 \). For the last case assume that \( n > 2 \) and consider \( \text{ann}((2, 1)) \). Clearly, \( \text{ann}((2, 1)) = \{0, (a, 0)\} \). Therefore, \( (a, 0) \) isolates \( (2, 1) \) and is a cut-vertex by definition.

**Theorem 2.3.** Consider \( R = \prod_{i=1}^{m} \mathbb{Z}_{n_i} \) for \( m \geq 3 \). Then \( (0, 0, \ldots, a_i, \ldots, 0) \) is a cut-vertex of \( \Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_i} \times \cdots \times \mathbb{Z}_{n_m}) \) if and only if \( 2a_i = n_i \).

**Proof.** (\( \Rightarrow \)) Assume \( (0, 0, \ldots, a_i, \ldots, 0) \) is a cut-vertex of \( \Gamma(R) \). Then \( \Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_i} \times \cdots \times \mathbb{Z}_{n_m}) \) is split into two subgraphs \( X \) and \( Y \), which are distinct except for their common vertex \( (0, 0, \ldots, a_i, \ldots, 0) \). There exists some
Theorem 3.1. Let \( \Gamma(\{x_1, x_2, \ldots, x_n\}) \) be a graph such that \( (x_1, x_2, \ldots, x_n) \in V(X) \) and \( (y_1, y_2, \ldots, y_m) \in V(Y) \) such that \( (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_m) \in (0, 0, \ldots, a_i, 0, \ldots, 0) \). Hence the following path must also exist: \( (x_1, x_2, \ldots, x_n) \in (0, 0, \ldots, -a_i, 0, \ldots, 0) \). Since \( (0, 0, \ldots, a_i, 0, \ldots, 0) \) is a cut-vertex, \( (0, 0, \ldots, a_i, 0, \ldots, 0) = (0, 0, \ldots, -a_i, 0, \ldots, 0) \), which implies \( a_i = -a_i \), or \( 2a_i = 0 \) in \( \mathbb{Z} \). Therefore \( 2a_i = n_i \).

Corollary 2.5. Let \( R = \prod_{i=1}^{m} \mathbb{Z}_{n_i} \) for \( m \geq 3 \). Then \( a = (a_1, a_2, \ldots, a_m) \) is a cut-vertex if and only if \( 2a_i = n_i \) for some \( i \), \( 1 \leq i \leq m \).

Proof. Let \( a \) be a cut-vertex of \( R \) with \( a = (a_1, a_2, \ldots, a_m) \). Assume \( a_i, a_j \neq 0 \) with \( i \neq j \). Since \( a \) is a cut-vertex, there exists \( \alpha, \beta \in \mathbb{Z}(R) \) such that the only path between them is \( \alpha = a = \beta \). Consider the ring element \( b = (0, 0, \ldots, a_i, 0, \ldots, 0) \). Then \( \alpha - b = \beta \), a contradiction.

The next corollary follows from Theorems 2.3 and 2.4.

Theorem 3.2. Let \( R = \prod_{i=1}^{m} \mathbb{Z}_{n_i} \) for \( m \geq 3 \). Then \( a = (a_1, a_2, \ldots, a_m) \) is a cut-vertex if and only if \( 2a_i = n_i \) for some \( i \), \( 1 \leq i \leq m \).

Proof. Let \( R = \prod_{i=1}^{m} \mathbb{Z}_{n_i} \) for \( m \geq 3 \).

(\( \Rightarrow \)) Let \( a = (a_1, a_2, \ldots, a_m) \) be a cut-vertex. Then by Theorem 2.4, \( a = (0, 0, \ldots, a_i, 0, \ldots, 0) \) for some \( 1 \leq i \leq m \). Since \( a = (0, 0, \ldots, a_i, 0, \ldots, 0) \) is a cut-vertex, then by Theorem 2.3, \( 2a_i = n_i \).

(\( \Leftarrow \)) Let \( a = (0, 0, \ldots, a_i, 0, \ldots, 0) \) where \( 2a_i = n_i \). Then by Theorem 2.3 \( a \) is a cut-vertex.

3. Cut-Sets in \( \Gamma\left(\prod_{i=1}^{m} \mathbb{Z}_{n_i}\right) \)

In this section we generalize the idea of a cut-vertex to that of a cut-set. Many results on cut-vertices generalize to cut-sets, and we may consider all theorems in the previous section as corollaries to the following theorems on cut-sets.

Note that when \( n = p \), \( p \) a prime, the ring \( \mathbb{Z}_n \) is a field so \( \Gamma(\mathbb{Z}_n) \) is empty. When \( n = 2p \), \( p > 2 \), \( \Gamma(\mathbb{Z}_n) \) is a star-graph, where the only cut-set is \( A = \text{ann}(2) \\{0\} = \{p\} \). For example, Figure 3 shows \( \Gamma(\mathbb{Z}_4) \). Notice that \( \text{ann}(p) \\{0\} = V(\Gamma(\mathbb{Z}_n)) \\{p\} \). When \( n = p^2 \), \( \Gamma(\mathbb{Z}_n) \) is a complete graph; whence there are no cut-sets.

Theorem 3.1. Let \( n \in \mathbb{Z}^+ \) such that \( n \neq p, 2p, p^2 \) for any prime \( p \). A set \( A \) is a cut-set of \( \Gamma(\mathbb{Z}_n) \) if and only if \( A = \text{ann}(p) \\{0\} \) for some prime \( p \) which divides \( n \).

Proof. (\( \Rightarrow \)) Let \( A = \text{ann}(p) \\{0\} \) for some prime \( p \in \mathbb{Z} \) that divides \( n \). Observe that \( p \notin \text{ann}(p) \) since \( n \neq p^2 \). Then \( p \) is only connected to \( A \) in \( \Gamma(\mathbb{Z}_n) \), so when \( A \) is removed, \( p \) is isolated.
Notice that \( n-p \neq p \), and \( n-p \) is connected to all elements in \( A \), but \((n-p)p \neq 0 \) since \( n \neq p^2 \). This implies that \( A \) splits \( \Gamma(Z_n) \) into two subgraphs.

Suppose some subset \( B \) of \( A \) splits the graph similarly, and let \( a \in A \setminus B \). Then \( p-a-(n-p) \), a contradiction.

(⇒) Assume \( A \) is a cut-set of \( \Gamma(Z_n) \) separating subgraphs \( X \) and \( Y \). Take any \( x \in X \setminus A \) and \( y \in Y \setminus A \) where \( x \neq a \), and \( y = a \), with \( a, y \in A \). Rewrite \( x \) and \( y \) as \( x = rp \) and \( y = sp \) where \( p \) and \( q \) are primes dividing \( n \).

First assume that \( p \neq q \), where \( p \) does not divide \( y \) and \( q \) does not divide \( x \), and take any nonzero element of \( \text{ann}(p) = \{k(n/p) \mid k \in Z_{p^2}\} \), say, \( c(n/p) \). This is a multiple of \( p \), so we have that \( \text{ann}(p) \subseteq \text{ann}(c(n/p)) \) and that \( x \in \text{ann}(c(n/p)) \), since \( x \) is a multiple of \( p \). Thus for any nonzero \( b \in \text{ann}(p) \) we have \( y = b - c(n/p) - x \). Since \( A \) is a cut-set separating \( X \) and \( Y \), \( b \in A \) or \( c(n/p) \in A \). If \( b \in A \) this implies that \( \text{ann}(p) \setminus \{0\} \subseteq A \), and if \( c(n/p) \in A \) this implies that \( \text{ann}(p) \setminus \{0\} \subseteq A \). Since \( p \) does not divide \( y \) and \( q \) does not divide \( x \), \( x \) is not connected to any element of \( \text{ann}(p) \setminus \{0\} \) and \( y \) is not connected to any element of \( \text{ann}(p) \setminus \{0\} \). If \( \text{ann}(p) \setminus \{0\} \subseteq A \), then since \( x \) is connected to an element of \( A \) and not to any element of \( \text{ann}(p) \setminus \{0\} \), there is at least one additional element in \( A \). Similarly, if \( \text{ann}(p) \setminus \{0\} \subseteq A \), then since \( y \) is connected to an element of \( A \) and not to any element of \( \text{ann}(p) \setminus \{0\} \), there is at least one additional element in \( A \). Since \( \text{ann}(p) \setminus \{0\} \) and \( \text{ann}(p) \setminus \{0\} \) are cut-sets, in either case \( A \) would contain a cut-set as a proper subset, a contradiction.

We may therefore assume that \( p = q \). Then since \( \text{ann}(p) \subseteq \text{ann}(x) \) and \( \text{ann}(p) \subseteq \text{ann}(y) \), for any nonzero \( a \in \text{ann}(p) \) we have \( x = a = y \), which implies that \( \text{ann}(p) \setminus \{0\} \subseteq A \). Since \( \text{ann}(p) \setminus \{0\} \subseteq A \) and \( \text{ann}(p) \setminus \{0\} \) is a cut-set by the first direction of this proof, then \( \text{ann}(p) \setminus \{0\} = A \) by definition of a cut-set.

\[\square\]

**Lemma 3.2.** In \( Z_n \), let \( p \in \mathbb{Z} \) be a prime that divides \( n \). Then \( |\text{ann}(p)| = p \).

**Proof.** In the ring \( Z_n \), \( \text{ann}(p) = \{an/p \mid a \in \mathbb{Z}_p\} \) since \( p(n/p) = 0 \), \( (p+1)(n/p) = n/p \), and so on. There are \( p \) distinct elements in this set. \[\square\]
Because every cut-set of $\Gamma(Z_n)$ is an annihilator of some prime that divides $n$, by Lemma 3.2, the size of any cut-set in $\Gamma(Z_n)$ is known. The following lemmas and corollaries will be useful in the proofs of some of the next theorems in this section.

**Lemma 3.3.** Let $R \cong R_1 \times R_2 \times \cdots \times R_n$. If $a \in R$ with $a = (a_1, \ldots, a_i, \ldots, a_n)$ and $a' = (a_1, \ldots, 0, \ldots, a_n)$ then $\text{ann}(a) \subseteq \text{ann}(a')$.

**Proof.** Let $b \in \text{ann}(a)$. Then $b \cdot a_l = 0$ for all $1 \leq l \leq n$, and $b_i \cdot 0 = 0$. Thus $b \in \text{ann}(a)$. □

**Corollary 3.4.** Let $R \cong R_1 \times R_2 \times \cdots \times R_n$. Let $A$ be a cut-set of $R$. If $a \in A$ with $a = (a_1, \ldots, a_i, \ldots, a_n)$ and $a' = (a_1, \ldots, 0, \ldots, a_n)$, then $a' \in A$.

**Proof.** Assume $a' \notin A$. Observe that for $b \in Z(R)^*$, if $b \sim a$, then $b \sim a'$ by Lemma 3.3. Thus $A \setminus \{a\}$ is (or contains) a cut-set - a contradiction. □

**Lemma 3.5.** Let $R \cong R_1 \times R_2 \times \cdots \times R_n$. If $a \in R$ with $a = (a_1, \ldots, a_i, \ldots, a_n)$ and $a' = (a_1, \ldots, 1, \ldots, a_n)$ then $\text{ann}(a') \subseteq \text{ann}(a)$.

**Proof.** Let $b \in \text{ann}(a')$. Then $b \cdot a_l = 0$ for all $1 \leq l \leq n$ and in particular $b_i = 0$. This implies $b \cdot a_l = 0$, and thus $b \in \text{ann}(a)$. □

**Theorem 3.6.** Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ with $n \geq 2$. If $A$ is a cut-set of $\Gamma(R)$ then there exists some $i$, $1 \leq i \leq n$ such that $a = (0, \ldots, 0, a_i, 0, \ldots, 0)$ for every $a \in A$.

**Proof.** Let $A$ be a cut-set of $R$ which splits $\Gamma(R)$ into $X$ and $Y$. Without loss of generality, assume there exists some $b = (b_1, b_2, \ldots, b_n) \in A$ with $b_1 \neq 0$, $c = (c_1, c_2, \ldots, c_n) \in A$ with $c_2 \neq 0$.

Consider the set of all elements in $Z(R)$ with a 0-entry in the $1^{st}$ position; let this set be denoted by $\{1\}$. Let $1_1$ denote the element $(1, 0, \ldots, 0)$ and similarly denote $1_2$, and so on. Notice $\text{ann}(1_1) = \{0\}$. Denote by $(1_1)^*$ all elements with 0 everywhere except the $1^{st}$ position; i.e., $(1_1)^*$ is the ideal generated by $1_1$, omitting 0. If $1_1 \in A$ then $(1_1)^* \subseteq A$ since any element which annihilates $1_1$ also annihilates any element in $(1_1)^*$.

Consider the element $1_1 = (0, 1, \ldots, 1)$. Notice $\text{ann}(1_1) = (1_1)^* \cup \{0\}$. Therefore, $(1_1)^*$ forms a cut-set by isolating $1_1$. Notice $(1_1)^* \subseteq A$ since $c \in A$, a contradiction. Therefore, $1_1 \notin A$ and we can similarly show $1_i \notin A$ for every $1 \leq i \leq n$. Without loss of generality, $1_1, 1_2 \in X \setminus A$ since $1_1 \sim 1 - 1_2$. Similarly every $1_i \in X \setminus A$ for $1 \leq i \leq n$. This implies that if $y \in Y \setminus A$ then $y_i \neq 0$ for any $1 \leq i \leq n$, since otherwise it would be connected to an element in $X \setminus A$, namely $1_i$.

Consider $\alpha = (b_1, 0, \ldots, 0)$ and $\beta = (0, c_2, 0, \ldots, 0)$. Notice $\alpha, \beta \in A$ by Corollary 3.4 since $b, c \in A$. This implies $b_1$ and $c_2$ are not units, since otherwise we reach the contradiction shown in the previous paragraph.

Since $\alpha \in A$, there exists $y = (y_1, y_2, \ldots, y_n) \in Y \setminus A$ such that $y \sim \alpha$. Clearly $\text{ann}(y) \setminus \{0\} \subseteq Y$. Consider the element $y' = (y_1', y_2', \ldots, y_n')$ where $y_1' = 1$ and $y_i' = y_i$ for all $i \neq 2$. By Lemma 3.5, $\text{ann}(y') \setminus \{0\} \subseteq \text{ann}(y) \setminus \{0\}$. Also, $\text{ann}(y') \setminus \{0\} \subseteq A$ since any element which annihilates $y'$ must have a zero in the second position. Therefore, $\text{ann}(y') \setminus \{0\} \subseteq A$ since $\beta \notin \text{ann}(y') \setminus \{0\}$. Thus $\text{ann}(y') \setminus \{0\}$ forms a cut-set, a contradiction of the minimality of $A$. Therefore, all elements in $A$ must be of the form $(0, \ldots, 0, a_i, 0, \ldots, 0)$. □
Theorem 3.7. A set $A$ of $V\left(\Gamma\left(\prod_{i=1}^{m}Z_{n_i}\right)\right)$ with $m \geq 2$ is a cut-set if and only if $A = \{(0,0,\ldots,a_i,\ldots,0),(0,0,\ldots,a_{i2},\ldots,0),\ldots,(0,0,\ldots,a_i,\ldots,0)\}$ where $\{a_i,a_{i2},\ldots,a_{ik}\} = \text{ann}(p)\setminus\{0\}$ for some prime $p \in \mathbb{Z}$ such that $p|n_i$ in $\mathbb{Z}_{n_i}$.

Proof. ($\Leftarrow$) First note that the case where $n_i = p$ is included in this theorem because the nonzero annihilators of $p$ are exactly the nonzero elements of $\mathbb{Z}_{n_i}$. Without loss of generality, let $p$ be a prime such that $p$ divides $n_1$ and let $\text{ann}(p) = \{0,a_{i1},a_{i2},\ldots,a_{ik}\}$ in $\mathbb{Z}_{n_1}$. Because $\text{ann}(p,1,\ldots,1) = \{(0,0,\ldots,0),(a_{i1},0,\ldots,0),(a_{i2},0,\ldots,0),\ldots,(a_{ik},0,\ldots,0)\}$, then by Theorem 3.1 we know that for any nonzero $x \in \mathbb{Z}_{n_1}$, $y \in \mathbb{Z}_{n_1}$, and primes $p_1, p_2, \ldots, p_k$ dividing $n_1$, $\text{ann}(p_1) = \{0,a_{i1},a_{i2},\ldots,a_{ik}\}$ in $\mathbb{Z}_{n_1}$, implying that no proper subset of $A$ will act as a cut-set.

($\Rightarrow$) Let $A$ be a cut-set of $\Gamma(\prod_{i=1}^{m}Z_{n_i})$ and let $X$ and $Y$ be two subgraphs created when $A$ is removed. Take any $(x_1,x_2,\ldots,x_m) \in X \setminus A$ and $(y_1,y_2,\ldots,y_m) \in Y \setminus A$ such that $(x_1,x_2,\ldots,x_m) - (a_{i1},a_{i2},\ldots,a_{ik})$ and $(y_1,y_2,\ldots,y_m) - (b_1,b_2,\ldots,b_m)$ for some $(a_{i1},a_{i2},\ldots,a_{ik}), (b_1,b_2,\ldots,b_m) \in A$. Because any $(u_1,u_2,\ldots,u_m)$ where each $u_i$ is a unit contains only the zero element in its annihilator, we know that at least one of the $x_i$ must be a zero-divisor of the corresponding $\mathbb{Z}_{n_i}$, and similarly for the $y_i$. Since we also know that all elements of $A$ are zero in every component position but, say, the $i$th position by Theorem 3.6, the $i$th component position of both $(x_1,x_2,\ldots,x_m)$ and $(y_1,y_2,\ldots,y_m)$ must contain a zero-divisor. We may therefore assume without loss of generality that both $x_1$ and $y_1$ are zero-divisors of $\mathbb{Z}_{n_1}$. Since this is the case we can rewrite $x_1 = rp_{x_1}$ and $y_1 = rp_{y_1}$ for $r,q \in \mathbb{Z}_{n_1}$ and primes $p_{x_1}, p_{y_1} \in \mathbb{Z}$ dividing $n_1$.

First assume that $p_{x_1} \neq p_{y_1}$, where $p_{x_1}$ does not divide $y_1$ and $p_{y_1}$ does not divide $x_1$. Then by Theorem 3.1 we know that for any nonzero $\beta \in \text{ann}(p_{y_1})$ and any nonzero $c(n/p_{x_1})$, $y_1 - \beta - c(n/p_{x_1}) - x_1$ in $\Gamma(\mathbb{Z}_{n_1})$. Therefore we have that $(y_1,y_2,\ldots,y_m) - (\beta,0,\ldots,0) - (c(n/p_{x_1}),0,\ldots,0) - (x_1,x_2,\ldots,x_m)$. This implies that $(\beta,0,\ldots,0) \in A$, which would imply inclusion of all such $(\beta,0,\ldots,0)$ in $A$, or $(c(n/p_{x_1}),0,\ldots,0) \in A$, which would imply a similar inclusion. Since $p_{x_1}$ does not divide $y_1$ and $p_{y_1}$ does not divide $x_1$, $(x_1,x_2,\ldots,x_m)$ is not connected to elements of the form $(\beta,0,\ldots,0)$ and $(y_1,y_2,\ldots,y_m)$ is not connected to elements of the form $(c(n/p_{x_1}),0,\ldots,0)$. However, since we know that both $(x_1,x_2,\ldots,x_m)$ and $(y_1,y_2,\ldots,y_m)$ are connected to elements in $A$ and that the set of all nonzero $(\beta,0,\ldots,0)$ and the set of all nonzero $(c(n/p_{x_1}),0,\ldots,0)$ are cut-sets by the first direction, we see that $A$ contains a proper subset that is a cut-set, a contradiction.

We may therefore assume that $p_{x_1} = p_{y_1}$. Then since $\text{ann}(p_{x_1}) \subseteq \text{ann}(x_1)$ and $\text{ann}(p_{x_1}) \subseteq \text{ann}(y_1)$, then for any nonzero $\alpha \in \text{ann}(p_{x_1})$, $(x_1,x_2,\ldots,x_m) - (\alpha,0,\ldots,0) - (y_1,y_2,\ldots,y_m)$. We then have that $(\alpha,0,\ldots,0) \subseteq \text{ann}(x_1,x_2,\ldots,x_m)$ and $(\alpha,0,\ldots,0) \subseteq \text{ann}(y_1,y_2,\ldots,y_m)$, meaning that $(\alpha,0,\ldots,0) \subseteq A$, but since the set of all nonzero $(\alpha,0,\ldots,0)$ where $\alpha \in \text{ann}(p_{x_1})$ is a cut-set by the first direction, then $A = \{(\alpha,0,\ldots,0) \mid \alpha \neq 0, \alpha \in \text{ann}(p_{x_1})\}$. \qed
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