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A Beckman-Quarles type theorem for linear fractional transformations of the extended double plane

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Abstract. In this presentation, we consider the problem of characterizing maps that preserve pairs of right hyperbolas or lines in the extended double plane whose hyperbolic angle of intersection is zero. We consider two disjoint spaces of right hyperbolas and lines in the extended double plane $\mathscr{H}^+$ and $\mathscr{H}^-$ and prove that bijective mappings on the respective spaces that preserve tangency between pairs of hyperbolas or lines must be induced by a linear fractional transformation.

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1 Introduction

Beckman and Quarles proved that any mapping from $\mathbb{R}^n$ to itself, $n \geq 2$, that preserves a fixed distance between two points is necessarily a rigid motion [1]. This theorem was among the first results of which are now called *characterizations of geometrical mappings under mild hypotheses* [4]. Theorems of this sort require no regularity conditions, such as differentiability or continuity. Today there are many theorems that belong to this area.

Another example is Lester's characterization of Möbius transformations of the *extended complex plane* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. It is well known in complex analysis that Möbius transformations preserve the space of circles and lines $\mathscr{C}$ as well as the angle of intersection θ_{AB} for any pair $A, B \in \mathscr{C}$. She showed that the converse is also true. Lester proved the following Beckman-Quarles type theorem.

Theorem 1 (Lester [7])**.** *For a fixed real $0 \leq \alpha < \pi$, let $X \to \bar{X}$ be a bijective mapping from $\mathscr{C}$ to itself such that, for all A, B in $\mathscr{C}$,*

$$\theta_{AB} = \alpha \text{ if and only if } \theta_{\bar{A}\bar{B}} = \alpha.$$

Then the mapping is induced on $\mathscr{C}$ by a Möbius transformation of $\hat{\mathbb{C}}$.

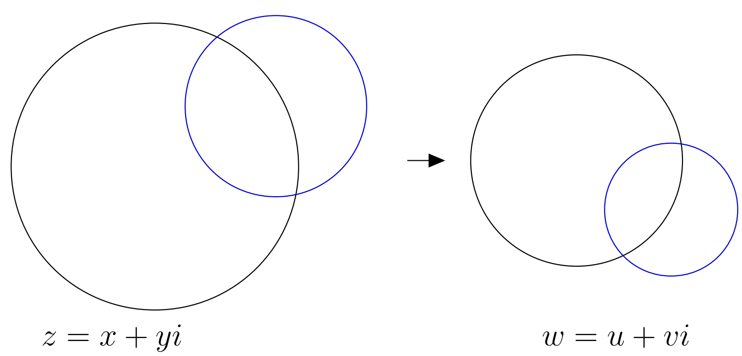

Figure 1: Möbius transformations preserve the space of circles and lines and angles of intersection.

We consider the analogous problem for the *extended double plane* $\hat{\mathbb{P}} = \mathbb{P} \cup H_\infty$, where $\mathbb{P} = \{x + yk : x, y \in \mathbb{R}, k^2 = 1\}$ and $H_\infty = \{(\alpha \pm \alpha k)^{-1} : \alpha \in \mathbb{R} \cup \{\infty\}\}$ (see Section 2.2). As in complex analysis, we found that in the extended double plane, linear fractional transformations preserve both the space of vertical right hyperbolas and lines with slopes greater than -1 and less than 1, denoted $\mathscr{H}^+$, and the space of horizontal right hyperbolas and lines with slopes less than -1 or greater than 1, denoted $\mathscr{H}^-$. Furthermore, the *hyperbolic angle of intersection* $\varphi_{h_1 h_2}$ for any pair right hyperbolas or lines h_1, h_2 in $\mathscr{H}^+$ or $\mathscr{H}^-$ is preserved by linear fractional transformations. Following Lester's model, we prove the following Beckman-Quarles type theorem.

Theorem 2. *Let T be a bijective mapping from $\mathscr{H}^+$ (resp. $\mathscr{H}^-$) to itself such that, for all $h_1, h_2 \in \mathscr{H}^+$ (resp. $\mathscr{H}^-$),*

$$\varphi_{h_1 h_2} = 0 \text{ if and only if } \varphi_{T(h_1)T(h_2)} = 0.$$

Then T is induced on $\mathscr{H}^+$ (resp. $\mathscr{H}^-$) by a linear fractional transformation of $\hat{\mathbb{P}}$.

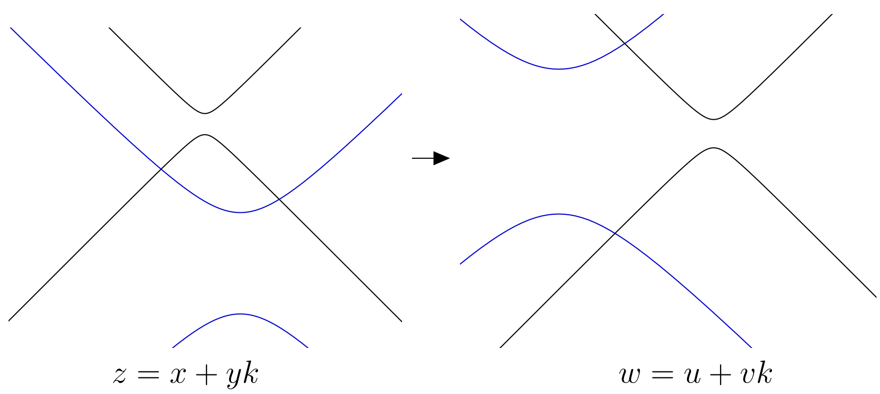

Figure 2: Linear fractional transformations of the extended double plane preserve vertical and horizontal right hyperbolas and lines and hyperbolic angles of intersection.

2 Geometry in the Extended Double Plane $\hat{\mathbb{P}}$

2.1 The Double Plane $\mathbb{P}$

Definition 1. *A* ***double number***[1] *is a formal expression $x + yk$ where $x, y \in \mathbb{R}$ and $k^2 = 1$, but $k \notin \mathbb{R}$ (k is known as a unipotent). Every double number $z = x + yk$ has a* ***real component*** *$Re(z) = x$, a* ***double component*** *$Im(z) = y$, and* ***conjugate*** *$\bar{z} = x - yk$.*

The *double plane* $\mathbb{P}$ is the set of all double numbers, which we call *points*. This may be thought of as a two-dimensional vector space over $\mathbb{R}$, where each double number $x + yk$ corresponds to the vector (x, y) in $\mathbb{R}^2$ with

Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$;
Scalar Multiplication: $c(x_1, y_1) = (cx_1, cy_1)$; and
Multiplication: $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 + y_1y_2, x_1y_2 + x_2y_1)$.

[1] The name *double number* was used by Yaglom. Others have called them *perplex, split-complex, spacetime, or hyperbolic numbers.*

This way we can see that double numbers form a commutative algebra over $\mathbb{R}$.

It is straightforward to verify that for $z = x + yk$,

$$z^{-1} \stackrel{def}{=} \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - yk}{x^2 - y^2}$$

is the *multiplicative inverse of* z whenever $x \neq \pm y$. Therefore, while $\mathbb{P}$ is a commutative ring, it is neither a field nor even an integral domain, because every nonzero number with form $\alpha \pm \alpha k, \alpha \in \mathbb{R}$, is a zero-divisor.

The *double modulus* of $z = x + yk$ is

$$|z|_{\mathbb{P}} \stackrel{def}{=} \sqrt{|z\bar{z}|} = \sqrt{|x^2 - y^2|}.$$

This is considered the *double distance* of the point z from the origin.

Numbers with form $\alpha \pm \alpha k$ are in a sense isotropic, since $|\alpha \pm \alpha k|_{\mathbb{P}} = 0$, even when $\alpha \neq 0$. So there are points $z_1, z_2 \in \mathbb{P}$, where $z_1 \neq z_2$, such that $|z_1 - z_2|_{\mathbb{P}} = 0$. Therefore, strictly speaking, the double distance given by the modulus is not a metric. But it gives a geometry on $\mathbb{R}^2$ that is quite different from the Euclidean geometry of the complex plane, where $|z|_{\mathbb{C}} = \sqrt{x^2 + y^2} = 0$ if and only if $z = 0$.

In the complex plane, the set of all points $z = x + yi \in \mathbb{C}$ satisfying $|z|_{\mathbb{C}} = r > 0$ forms a circle with radius r. Similarly, we have that the set of all points $z \in \mathbb{P}$ with $|z|_{\mathbb{P}} = \rho > 0$ forms a four-branched right hyperbola with semi-axes of length ρ, whose asymptotes are the isotropic lines $y = \pm x$. As the parameter φ increases, $-\infty < \varphi < \infty$, each branch is drawn exactly once. See Figure 3.

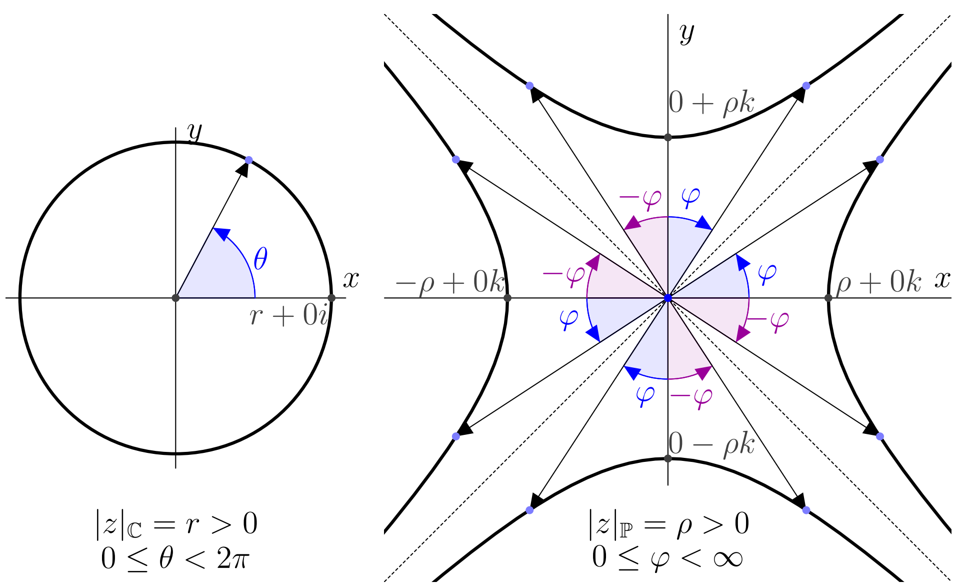

Figure 3: The r-circle and the ρ-right hyperbola

The *double argument* of $z = x + yk$ is $\arg(z) \overset{def}{=} \varphi$ where

$$\varphi = \begin{cases} \tanh^{-1}\frac{y}{x} & : |y| < |x| \\ \tanh^{-1}\frac{x}{y} & : |y| > |x| \\ \text{undefined} & : |y| = |x| \end{cases}.$$

It is interesting to note that $r > 0$ and fixed $0 \leq \theta < \pi$ determine a unique point in $\mathbb{C}$, whereas $\rho > 0$ and $\varphi \in \mathbb{R}$ determine four points in $\mathbb{P}$. Therefore, the double number version of polar form does not allow for unique representation of points. For more on double numbers, see [9, 8, 10].

2.2 Hyperbolas in the Extended Double Plane $\hat{\mathbb{P}}$.

The *extended double plane* $\hat{\mathbb{P}}$ is the union $\mathbb{P} \cup H_\infty$, where $H_\infty = \{(\alpha \pm \alpha k)^{-1} : \alpha \in \mathbb{R} \cup \{\infty\}\}$. We sometimes refer to the *points* in H_∞ as the *points at infinity*. This set may be thought of as two lines at infinity that intersect at $(0 + 0k)^{-1}$.

A *hyperbola* h in $\mathscr{H}^+$ or $\mathscr{H}^-$ is the subset of $\hat{\mathbb{P}}$ that includes every *point* $z = x + yk \in \mathbb{P}$ satisfying

$$Az\bar{z} + Re[(B + Ck)z] + D = 0, \tag{1}$$

where $A, B, C, D \in \mathbb{R}$, $4AD + C^2 - B^2 \neq 0$. For convenience, we denote h by $[A : B : C : D]$. Notice that for any nonzero $\lambda \in \mathbb{R}$, $[\lambda A : \lambda B : \lambda C : \lambda D]$ defines the same hyperbola. This means that there are many choices $A, B, C, D \in \mathbb{R}$, which give us the same hyperbola. For the remainder of this article—unless otherwise stated—the reader may assume that $A, B, C, D \in \mathbb{R}$ are chosen such that $4AD + C^2 - B^2 = \pm 1$. This normalization makes calculations later much easier and neither affects $\mathscr{H}^+$ nor $\mathscr{H}^-$.

Proposition 1. *Let $h = [A : B : C : D]$, where $A, B, C, D \in \mathbb{R}$. Then*

(i) $h \in \mathscr{H}^+$ if and only if $4AD + C^2 - B^2 = 1$;

(ii) $h \in \mathscr{H}^-$ if and only if $4AD + C^2 - B^2 = -1$.

A hyperbola $h = [A : B : C : D]$ also includes point(s) at infinity. Using linear fractional transformations, we can verify the point(s) in H_∞ which h intersects.

(i) $(\alpha_1 + \alpha_1 k)^{-1}$ where $\alpha_1 = \begin{cases} \frac{-A}{B-C} & : B \neq C \\ \infty & : B = C \end{cases}$

(ii) $(\alpha_2 - \alpha_2 k)^{-1}$ where $\alpha_2 = \begin{cases} \frac{-A}{B+C} & : B \neq -C \\ \infty & : B = -C \end{cases}$

(Notice that $(0 + 0k)^{-1} = (0 - 0k)^{-1}$, but $(\infty + \infty k)^{-1} \neq (\infty - \infty k)^{-1}$.) If $A \neq 0$, then h is a vertical or horizontal right hyperbola and includes exactly two points at infinity. But if $A = 0$, then h is a line and only includes one point at infinity. Moreover, h is a line if and only if $h \ni (0 + 0k)^{-1}$.

Using stereographic projection, the extended double plane can be viewed as an infinite hyperboloid. Consider the hyperboloid $x^2 - y^2 + (z-1)^2 = 1$, where the xy-plane is the double plane, and take the point $(0, 0, 2)$ as the projection point. Any line drawn through a point on the extended double plane and the projection point intersects the hyperboloid at a second point. Hyperbolas in $\mathscr{H}^+$ correspond with planar cross sections of the hyperboloid that take the form of ellipses and hyperbolas. A second hyperboloid $-x^2 + y^2 + (z-1)^2 = 1$ corresponds with hyperbolas in $\mathscr{H}^-$.

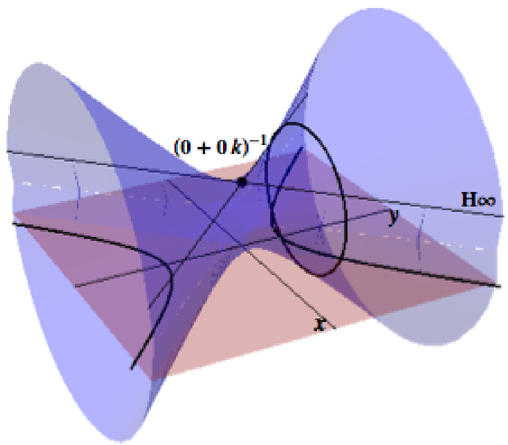

Figure 4: Projection of a hyperbola onto the xy-plane.

2.3 Linear Fractional Transformations of $\hat{\mathbb{P}}$

Definition 2. ***Direct*** *and* ***indirect linear fractional transformations*** *of the extended double plane are mappings* $\mu : \hat{\mathbb{P}} \to \hat{\mathbb{P}}$ *with forms*

$$\mu(z) = \frac{az+b}{cz+d} \ \text{ and } \ \mu(z) = \frac{a\bar{z}+b}{c\bar{z}+d},$$

respectively, where $a, b, c, d \in \mathbb{P}$ *and* $ad - bc \neq \alpha \pm \alpha k$ *for* $\alpha \in \mathbb{R}$.

The restriction $ad - bc \neq \alpha \pm \alpha k$ guarantees that we avoid transformations which take the entire double plane to a single point or to lines with slopes ± 1. The set of direct and indirect linear fractional transformations of the extended double number plane form a group under composition. We denote this group by $\mathcal{LFT}(\hat{\mathbb{P}})$.

Any direct linear fractional transformation is composed of at most four of the following "simple" transformations.

- Translation: $\mu(z) = z + b$, for $b \in \mathbb{P}$
- Rotation and Dilation: $\mu(z) = az$, for $a \in \mathbb{P}, a \neq \alpha \pm \alpha k$, where $\alpha \in \mathbb{R}$
- Inversion: $\mu(z) = \dfrac{1}{z}$

The full group is obtained by including conjugation $z \mapsto \bar{z}$. By further restricting $ad - bc = \pm 1$, we find that $\mathcal{LFT}(\hat{\mathbb{P}})$ is homomorphic to the group $SL(2,\mathbb{P})$ with the two-to-one correspondence $\frac{az+b}{cz+d}, \frac{a\bar{z}+b}{c\bar{z}+d} \leftrightarrow \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. This restriction is for normalization purposes only and does not affect the group in any way.

Proposition 2. *Let $[A:B:C:D]$ be a hyperbola in $\mathscr{H}^+$, and let μ be a "simple" linear fractional transformation.*

(i) (Translation) If $\mu(z) = z + b$, where $b = x_0 + y_0 k$ for $x_0, y_0 \in \mathbb{R}$, then

$$[A:B:C:D] \xrightarrow{\mu} [A : B - 2Ax_0 : C + 2Ay_0 : A(x_0^2 - y_0^2) - Bx_0 - Cy_0 + D].$$

(ii) (Rotation or Dilation) If $\mu(z) = az$, where $a = x_0 + y_0 k$ for $x_0, y_0 \in \mathbb{R}$ but $y_0 \neq \pm x_0$, then

$$[A:B:C:D] \xrightarrow{\mu} [A : Bx_0 - Cy_0 : Cx_0 - By_0 : D(x_0^2 - y_0^2)].$$

(iii) (Inversion) If $\mu(z) = \frac{1}{z}$, then

$$[A:B:C:D] \xrightarrow{\mu} [D : B : -C : A].$$

(iv) (Conjugation) If $\mu(z) = \bar{z}$, then

$$[A:B:C:D] \xrightarrow{\mu} [A : B : -C : D].$$

As mentioned prior, every $\mu \in \mathcal{LFT}(\hat{\mathbb{P}})$ maps hyperbolas in $\mathscr{H}^+$ or $\mathscr{H}^-$ onto hyperbolas in $\mathscr{H}^+$ or $\mathscr{H}^-$. One may verify this by showing that each of the four "simple" transformations performs this.

A pair of hyperbolas in $\mathscr{H}^+$ (resp. $\mathscr{H}^-$) are said to be *disjoint*, *tangent* or *intersecting* if they share, respectively, 0, 1 or 2 point(s) in $\hat{\mathbb{P}}$ (note that *intersecting* excludes *tangent*). Because linear fractional transformations are bijections, they preserve the number of intersection points.

2.4 The Hyperbolic Angle of Intersection in $\hat{\mathbb{P}}$

Definition 3. *Let $h_1 = [A:B:C:D]$ and $h_2 = [E:F:G:H]$ be distinct hyperbolas in $\mathscr{H}^+$ (resp. $\mathscr{H}^-$). Then the **hyperbolic angle of intersection** $\varphi_{h_1h_2}$ is defined by*

$$\cosh^2 \varphi_{h_1h_2} = \frac{(2AH + 2DE + CG - BF)^2}{(4AD + C^2 - B^2)(4EH + G^2 - F^2)}. \tag{2}$$

It is straightforward to verify that for any $\mu \in \mathcal{LFT}(\hat{\mathbb{P}})$, $\varphi_{h_1h_2} = \varphi_{\mu(h_1)\mu(h_2)}$. In particular, one need only check that this holds for the four "simple" transformations described in Proposition 2. Since μ is composed of finitely many "simple" transformations, it follows the hyperbolic angle of intersection for a pair of hyperbolas is preserved by μ.

We find that the value of $\varphi_{h_1h_2}$ is particularly helpful for classifying the relationship between two hyperbolas h_1 and h_2.

Proposition 3. *If $h_1, h_2 \in \mathscr{H}^+$ (resp. $\mathscr{H}^-$), then*

(i) h_1 and h_2 are disjoint if and only if $\varphi_{h_1h_2}$ is undefined;

(ii) h_1 and h_2 are tangent if and only if $\varphi_{h_1h_2} = 0$; and

(iii) h_1 and h_2 are intersecting if and only if $\varphi_{h_1h_2} > 0$.

2.5 Canonical Forms

In our argument, we use *canonical* representation of hyperbola pairs. The conception is that after a suitable linear fractional transformation, a pair of hyperbolas may be assumed to have a simplified form. See Figure 5.

To demonstrate this, we let h_1, h_2 be distinct hyperbolas in $\mathscr{H}^+$. Next, we select three distinct points $p_0, p_1, p_\infty \in h_1$ and define μ_0 by

$$\mu_0(z) = \frac{(z - p_0)(p_1 - p_\infty)}{(z - p_\infty)(p_1 - p_0)}.$$

This takes h_1 to the line $[0 : 0 : 1 : 0]$. We wish to use a subsequent linear fractional transformation to simplify $\mu_0(h_2) = [E : F : G : H]$.

Note that the only $\mu \in \mathcal{LFT}(\hat{\mathbb{P}})$ which preserve $\mu_0(h_1) = [0 : 0 : 1 : 0]$ are $\mu(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = \pm 1$. (It is straightforward to verify that any such μ takes the line $y = 0$ to itself; moreover, a linear fractional transformation that preserves $[0 : 0 : 1 : 0]$ must have this form.)

We assume next that $G \geq 0$—if necessary, multiply E, F, G, and H by -1, as this has no effect on the hyperbola. At this point, our simplification is divided into three cases.

Case 1. If $G < 1$, then $4EH + G^2 - F^2 = 1$ implies that $F^2 - 4EH < 0$, thus $E, H \neq 0$ lest $F^2 < 0$. Define μ_1 and μ_2 by

$$\mu_1(z) = z + \frac{F}{2E} \quad \text{and} \quad \mu_2(z) = \frac{2E}{\sqrt{4EH - F^2}} z.$$

Then $\mu_2 \circ \mu_1 \circ \mu_0(h_1) = [0 : 0 : 1 : 0]$ and $\mu_2 \circ \mu_1 \circ \mu_0(h_2) = [1 : 0 : \lambda : 1]$, where $\lambda = \frac{2G}{\sqrt{4EH - F^2}} \geq 0$. We call these a *canonical pair of disjoint hyperbolas in $\mathscr{H}^+$*.

Case 2. If $G = 1$, then $4EH + G^2 - F^2 = 1$ implies that $F^2 - 4EH = 0$. If $E = 0$, then $F = 0$, so $\mu_0(h_2) = [0 : 0 : 1 : H]$, $H \neq 0$. Let $\mu_1(z) = \frac{1}{H} z$, then $\mu_1 \circ \mu_0(h_1) = [0 : 0 : 1 : 0]$ and $\mu_1 \circ \mu_0(h_2) = [0 : 0 : 1 : 1]$.

If $E \neq 0$, then solving the equations $y = 0$ and $E(x^2 - y^2) + Fx + y + H = 0$ for x yields $x = -\frac{F}{2E}$. Let

$$\mu_1(z) = \frac{-\frac{1}{E}}{z + \frac{F}{2E}},$$

then $\mu_1 \circ \mu_0(h_1) = [0 : 0 : 1 : 0]$ and $\mu_1 \circ \mu_0(h_2) = [0 : 0 : 1 : 1]$.

Follow these with $\mu_2(z) = 2z+k$ to get $\mu_2\circ\mu_1\circ\mu_0(h_1) = [0:0:1:-1]$ and $\mu_2\circ\mu_1\circ\mu_0(h_2) = [0:0:1:1]$. We call these a *canonical pair of tangent hyperbolas in* $\mathscr{H}^+$.

Case 3. If $G > 1$, then $4EH + G^2 - F^2 = 1$ implies that $F^2 - 4EH > 0$. If $E = 0$, then $F \neq 0$, so $\mu_0(h_2) = [0:F:G:H]$, $F, G \neq 0$. Let $\mu_1(z) = z + \frac{H}{F}$, then $\mu_1\circ\mu_0(h_1) = [0:0:1:0]$ and $\mu_1\circ\mu_0(h_2) = [0:\lambda:1:0]$, where $\lambda = \frac{F}{G} \neq 0$.

If $E \neq 0$, then $\mu_0(h_1)$ and $\mu_0(h_2)$ intersect at $z^{\pm} = \frac{-F\pm\sqrt{F^2-4EH}}{2E}$. Let

$$\mu_1(z) = \frac{z - z^+}{z - z^-},$$

then $\mu_1\circ\mu_0(h_1) = [0:0:1:0]$ and $\mu_1\circ\mu_0(h_2) = [0:\lambda:1:0]$, where $\lambda = \frac{\sqrt{F^2-4EH}}{G}$. We call these a *canonical pair of intersecting hyperbolas in* $\mathscr{H}^+$.

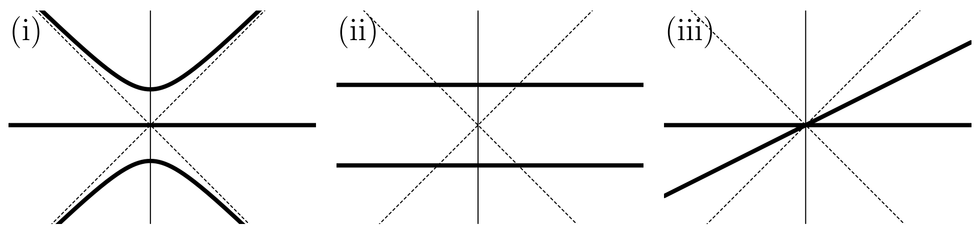

Figure 5: Canonical forms for (i) disjoint; (ii) tangent, and; (iii) intersecting hyperbolas in $\mathscr{H}^+$.

3 Adaptation of the Hays-Mitchell Theorem

In 2009, Hays and Mitchell extended research on geometrical mappings on the extended double plane. They showed that injective mappings that are restricted to *closed middle regions* and send hyperbolas in $\mathscr{H}^+$ or $\mathscr{H}^-$ to other hyperbolas in $\mathscr{H}^+$ or $\mathscr{H}^-$ are linear fractional transformations. In their article, they proved the following theorem.

Theorem 3 (Hays-Mitchell [5]). *Every injective mapping from a closed middle region bounded by a horizontal or vertical right hyperbola that sends hyperbolas in* $\mathscr{H}^+\cup\mathscr{H}^-$ *to hyperbolas in* $\mathscr{H}^+\cup\mathscr{H}^-$ *is a linear fractional transformation.*

In order to make a precise understanding of the term *closed middle region* used by Hays and Mitchell, we give the following definition.

Definition 4. *Let* $h = [A:B:C:D]$ *be a vertical or horizontal right hyperbola and let* P *denote the proper subset of points in* $\mathbb{P}$

$$\{x + yk : A(x^2 - y^2) + Bx + Cy + D \geq 0\}.$$

We call P *the* ***closed middle region bounded by*** h*.*

During our research, we anticipated that Theorem 3 would be a powerful tool for our proof of Theorem 2 and aspired to apply the work of Hays and Mitchell to our own—however, we found that we could not directly apply their result to our statement in Theorem 2. At the pith of our proof for Theorem 2, we will define a certain injection on $\hat{\mathbb{P}}$ that consequently maps hyperbolas in $\mathscr{H}^+$ to hyperbolas $\mathscr{H}^+$ (see section 4.2). But the hypothesis in Theorem 3 requires that hyperbolas in $\mathscr{H}^+ \cup \mathscr{H}^-$ are sent to hyperbolas in $\mathscr{H}^+ \cup \mathscr{H}^-$, and the injective mapping in our argument has no control over hyperbolas in $\mathscr{H}^-$. Therefore, we give a following slightly modified statement of Theorem 3, which we will later invoke in section 4.

Lemma 1 (The Modified Hays-Mitchell Theorem). *If $f : \hat{\mathbb{P}} \to \hat{\mathbb{P}}$ is an injective mapping that sends hyperbolas in $\mathscr{H}^+$ to hyperbolas in $\mathscr{H}^+$, and P is a closed middle region bounded by a vertical right hyperbola, then the restriction $f|_P$ is a linear fractional transformation.*

To prove Lemma 1, we will adopt the majority of the proof for Theorem 3 given in [5] and make changes in areas which are not true for our injective mapping f. This may seem a bit confusing on the surface, but as the argument proceeds reasoning will become more clear. The proof in [5] is constructive, and in one of the steps a hyperbola in $\mathscr{H}^-$ is used, to which we do not have access; however, we have an advantage, since f is injective on all $\hat{\mathbb{P}}$ and not just on a closed middle region, so we may use hyperbolas which extended beyond the boundary of P, whereas Hays and Mitchell may only use hyperbolas which P includes.

In our situation, two modifications to the proof in [5] are needed. First, where Hays and Mitchell use the fact that $[1:0:0:-4]$ is preserved in order to arrive at a special preserved point of $[1:0:0:1]$ ([5, §3.7]), we use the preservation of $[1:-2:0:2]$ to arrive at a different preserved point that plays the same role. Second, where Hays and Mitchell again use a horizontal hyperbola to argue for the preservation of $\left[1:0:0:\frac{4}{9}\right]$ ([5, §3.7]), we give a new argument for the preservation of this hyperbola. With these modifications, the remainder of the Hays-Mitchell arguments applies to our situation, and we may conclude that f is a linear fractional transformation when restricted to P.

Proof of Lemma 1. An immediate result of the hypothesis is that f preserves the number of intersection points shared by a pair of hyperbolas h_1, h_2, for injectivity implies that a point $z \in h_1 \cap h_2$ if and only if $f(z) \in f(h_1) \cap f(h_2)$.

Now, consider $h_0 = [1:0:0:1]$ and the closed middle region bounded by it,

$$P = \left\{x + yk \in \mathbb{P} : x^2 - y^2 + 1 \geq 0\right\}.$$

To further clarify the two modifications, we revisit the proof of Theorem 3 at the stage of [5, §3.7], where the following may be assumed of f after having been pre- and post-composed with the suitable linear fractional transformations.

1. f preserves $h_0 = [1:0:0:1]$ and the points k and $(0+0k)^{-1}$ ([5, §3.2]).

2. f maps parallel lines to parallel lines ([5, §3.3]).

3. f preserves $l_1 = [0 : 0 : -1 : 1], l_2 = [0 : 0 : 1 : 1]$ and $l_3 = [0 : 0 : 1 : 0]$ ([5, §3.4]).

4. f preserves the points $1 + k$, $1 - k$, $-1 + k$, and $-1 - k$ ([5, §3.5-6]).

The hyperbola $h_1 = [1 : -2 : 0 : 2]$ is tangent with l_1 at $1 + k$ and tangent with l_2 at $1 - k$. It follows that f maps h_1 to itself, and consequently,

$$f(h_0) \cap f(h_1) = h_0 \cap h_1 = \left\{ \frac{1}{2} + \frac{\sqrt{5}}{2}k, \frac{1}{2} - \frac{\sqrt{5}}{2}k \right\}.$$

We will show that f preserves not only this set, but each point in it.

Construct the line which passes through the points $\frac{1}{2} + \frac{\sqrt{5}}{2}k$ and $1 + k$,

$$l_4 = \left[0 : 2 - \sqrt{5} : -1 : \sqrt{5} - 1\right] \in \mathscr{H}^+.$$

Since $f(l_4)$ must be a line with slope in $(-1, 1)$, we conclude that $f(\frac{1}{2} + \frac{\sqrt{5}}{2}k) = \frac{1}{2} + \frac{\sqrt{5}}{2}k$. See Figure 6. (Moreover, because f is injective, we also conclude that $f(\frac{1}{2} - \frac{\sqrt{5}}{2}k) = \frac{1}{2} - \frac{\sqrt{5}}{2}k$.) Therefore, we have found a point on h_0 which is preserved by f, thus fulfilling our first goal.

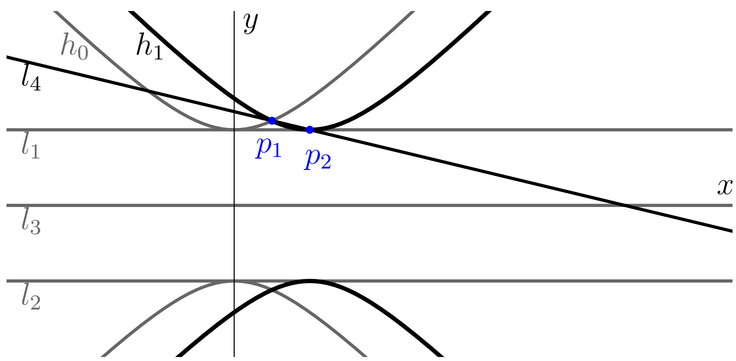

Figure 6: The line l_4, which passes through the points $p_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}k$ and $p_2 = 1 + k$.

Next, we will show that $\left[1 : 0 : 0 : \frac{4}{9}\right]$ is preserved by f.

By a similar argument as before, we claim that $-\frac{1}{2} + \frac{\sqrt{5}}{2}k$ and $-\frac{1}{2} - \frac{\sqrt{5}}{2}k$ are preserved by f. (These are the points of intersection of h_0 and $[1 : 2 : 0 : 2]$). Proceed to construct the line tangent to h_0 at $\frac{1}{2} + \frac{\sqrt{5}}{2}k$ and the line tangent to h_0 at $-\frac{1}{2} + \frac{\sqrt{5}}{2}k$. They are

$$l_5 = \left[0 : \frac{1}{\sqrt{5}} : -1 : \frac{2}{\sqrt{5}}\right] \text{ and } l_6 = \left[0 : -\frac{1}{\sqrt{5}} : -1 : \frac{2}{\sqrt{5}}\right],$$

respectfully. Because f preserves tangency, maps lines to lines, and preserves the points $\frac{1}{2} + \frac{\sqrt{5}}{2}k$ and $-\frac{1}{2} + \frac{\sqrt{5}}{2}k$, it follows that $f(l_5) = l_5$ and $f(l_6) = l_6$. Since $l_3 \cap l_5 \ni -2 + 0k$ and $l_3 \cap l_6 \ni 2 + 0k$, it follows that $f\left(-2 + 0k\right) = -2 + 0k$ and $f\left(2 + 0k\right) = 2 + 0k$.

Next, construct two lines: one through the points $-2+0k$ and $1+k$ and the other through the points $2+0k$ and $-1+k$. These lines intersect at $0+\frac{2}{3}k$. Each of these lines is preserved, which implies that $f\left(0+\frac{2}{3}k\right) = 0+\frac{2}{3}k$. Similarly, by constructing the line through $-2+0k$ and $1-k$ and the line through $2+0k$ and $-1-k$, we find that their intersection point $0-\frac{2}{3}k$ is preserved. See Figure 7.

Consequently, the horizontal lines $[0:0:1:-\frac{2}{3}]$ and $[0:0:1:\frac{2}{3}]$ are preserved by f. These are tangent to $[1:0:0:\frac{4}{9}]$ at $0+\frac{2}{3}k$ and $0-\frac{2}{3}k$, respectfully. It follows that $f\left(\left[1:0:0:\frac{4}{9}\right]\right) = \left[1:0:0:\frac{4}{9}\right]$, thus fulfilling our second goal.

As mentioned prior, the remaining portions of the Hays-Mitchell argument follow. Therefore, we may conclude that f is a direct or indirect linear fractional transformation when restricted to a closed middle region bounded by a vertical right hyperbola. $\square$

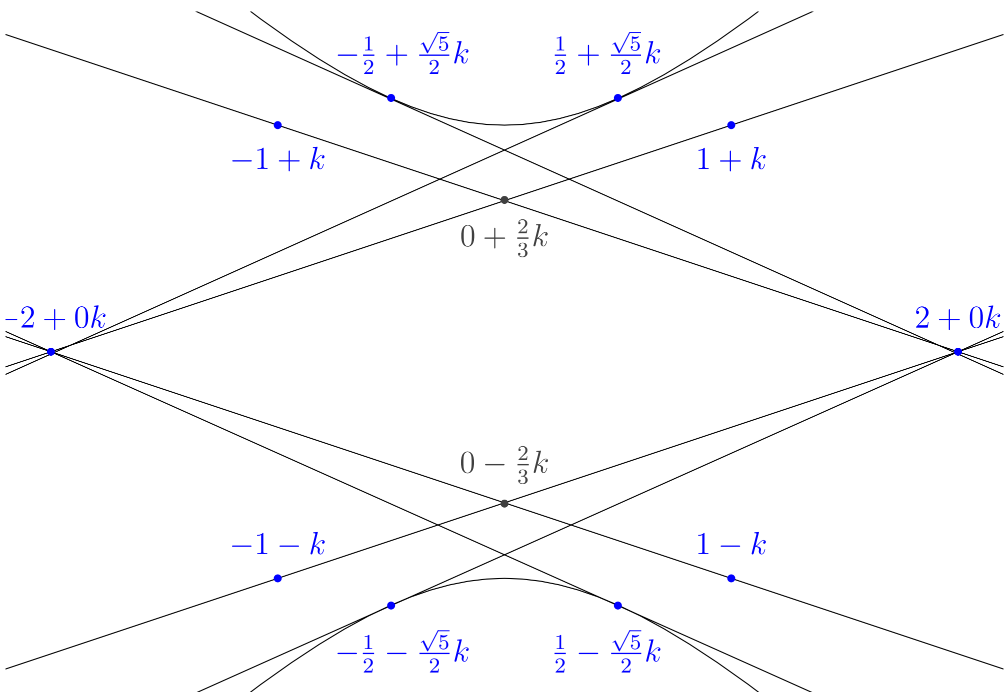

Figure 7: Lines and points preserved by f.

4 Proof of Theorem 2

4.1 T preserves families of tangent hyperbolas

The hypothesis of Theorem 2 says that T is a bijection on the space of hyperbolas $\mathscr{H}^+$ that preserves hyperbolic angle of intersection zero.[2] This is equivalent to saying that *a pair of hyperbolas in $\mathscr{H}^+$ are tangent if and only if their images under T are tangent.* This means that neither T nor T^{-1} can map a pair of tangent hyperbolas to a pair of disjoint or intersecting hyperbolas. This further implies that given any collection of mutually tangent hyperbolas, their images will remain mutually tangent.

Lemma 2. *Any four pairwise tangent hyperbolas in $\mathscr{H}^+$ share exactly one, mutual point.*

Proof. Let $h_1, h_2, h_3, h_4 \in \mathscr{H}^+$ be pairwise tangent. In canonical form, we take $h_1 = [0 : 0 : 1 : -1]$ and $h_2 = [0 : 0 : 1 : 1]$, and characterize every hyperbola $[A : B : C : D]$ in $\mathscr{H}^+$ that is tangent to both h_1 and h_2. From (2) in section 2.4 and the normalization on $[A : B : C : D]$, we get the following system equations.

$$(-2A + C)^2 = 1 \tag{3}$$
$$(2A + C)^2 = 1 \tag{4}$$
$$4AD + C^2 - B^2 = 1 \tag{5}$$

Equations (3) and (4) tell us that either $A = 0$ or $C = 0$. If $A = 0$, then we get a one-parameter family of horizontal lines $[0 : 0 : 1 : D]$, where $D \neq \pm 1$. If $C = 0$, then we get a one-parameter family of hyperbolas $\left[1 : \lambda : 0 : 1 + \frac{\lambda^2}{4}\right]$, where λ is any real number. At this point our argument is divided into three cases.

Case 1. Recall that both h_3 and h_4 are each tangent to h_1 and h_2 and that they are also tangent to each other. If h_3 and h_4 belong to the first family, then $h_1, h_2, h_3,$ are h_4 are all horizontal lines, and thus $\bigcap_{i=1}^{4} h_i = \{(0 + 0k)^{-1}\}$, which contains exactly one member.

Case 2. Now suppose that h_3 and h_4 both belong to the latter family. We let $h_3 = \left[1 : \lambda : 0 : 1 + \frac{\lambda^2}{4}\right]$ and $h_4 = \left[1 : \eta : 0 : 1 + \frac{\eta^2}{4}\right]$, where $\lambda, \eta \in \mathbb{R}$, and find the intersection point(s) using the following system of equations.

$$x^2 - y^2 + \lambda x + 1 + \frac{\lambda^2}{4} = 0 \tag{6}$$
$$x^2 - y^2 + \eta x + 1 + \frac{\eta^2}{4} = 0 \tag{7}$$

Solving for (6) and (7) for x and y tells us that $h_3 \cap h_4 = \left\{-\frac{\lambda+\eta}{4} \pm \frac{\sqrt{16+(\lambda-\eta)^2}}{4}k\right\}$. This contradicts that h_3 and h_4 are tangent.

[2] The reader should take note that we will only prove the $\mathscr{H}^+$ version of Theorem 2. However, the argument for the $\mathscr{H}^-$ version is completely analogous. Therefore, one should be able to draw further analogous statements and definitions for the $\mathscr{H}^-$ version. The reader should also take note that when we use the term *hyperbola* it is understood that object under discussion is either a vertical right hyperbola or a line with slope greater than -1 and less than 1—that is, a member of $\mathscr{H}^+$.

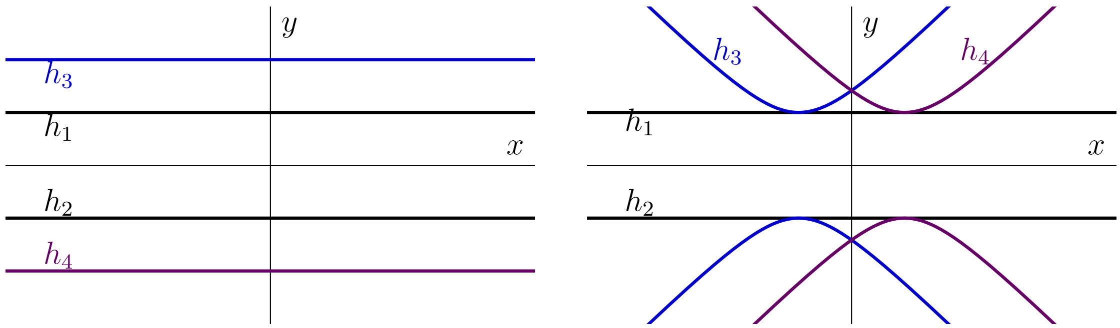

Figure 8: (*Case 1.*) Hyperbolas h_1, h_2, h_3, and h_4 are pairwise tangent at exactly one mutual point (left); (*Case 2.*) Hyperbolas h_3 and h_4 are intersecting (right).

Case 3. Finally, suppose that h_3 and h_4 each belong to different familes. We let $h_3 = [0 : 0 : 1 : D]$ and $h_4 = \left[1 : \lambda : 0 : 1 + \frac{\lambda^2}{4}\right]$, and find the intersection point(s) using the following system of equations.

$$y + D = 0 \tag{8}$$

$$x^2 - y^2 + \lambda x + 1 + \frac{\lambda^2}{4} = 0 \tag{9}$$

Solving for (8) and (9) we find that if $|D| < 1$, then $h_3 \cap h_4 = \varnothing$, whereas if $|D| > 1$, then $h_3 \cap h_4 = \left\{(\frac{-\lambda}{2} \pm \sqrt{D^2 - 1}) + Dk\right\}$. These both contradict that h_3 and h_4 are tangent.

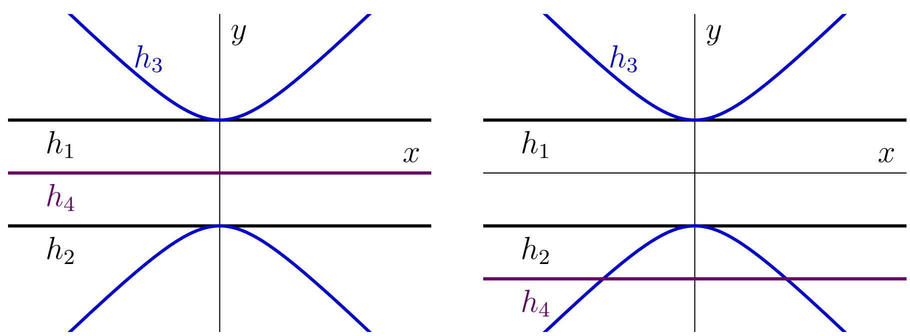

Figure 9: (*Case 3.*) Hyperbolas h_3 and h_4 are either disjoint, or intersecting.

So *Case 1* is the only valid generalization of h_1, h_2, h_3 and h_4. Therefore, we may conclude that four distinct pairwise tangent hyperbolas must all meet at exactly one mutual point. $\square$

It immediately follows from Lemma 2 that any arbitrary collection of four or more pairwise tangent hyperbolas all share exactly one, mutual point. We extend this concept of a collection of pairwise tangent hyperbolas and classify special families of infinitely many pairwise tangent hyperbolas. If we pick any point p in $\hat{\mathbb{P}}$ and a real number m with $|m| < 1$, then we can construct a family of tangent hyperbolas in $\mathscr{H}^+$ that we denote $\mathscr{T}_{p,m}$. We divide families into four types.

First, whenever p is finite (i.e. $p \in \mathbb{P}$), we call $\mathscr{T}_{p,m}$ the *family of hyperbolas whose slope is m at point p.*[3] In particular, if $p = x_0 + y_0 k$, then

$$\mathscr{T}_{p,m} \overset{def}{=} \{[A : -m - 2Ax_0 : 1 + 2Ay_0 : A({x_0}^2 - {y_0}^2) + mx_0 - y_0] : A \in \mathbb{R}\}.$$

Next, if $p \in H_\infty \backslash \{(\infty \pm \infty k)^{-1}\}$, then p^{-1} is finite. Whenever this is the case we define $\mathscr{T}_{p,m}$ as the *family of hyperbolas obtained as images of $\mathscr{T}_{p^{-1},m}$ under the inversion mapping $z \to \frac{1}{z}$.* In this case, m does not represent a 'slope', but it does allow us to identify a specific set of hyperbolas at p. In particular, if $p = (x_0 \pm x_0 k)^{-1}$, with $x_0 \neq \infty$, then

$$\mathscr{T}_{p,m} \overset{def}{=} \{[mx_0 \mp x_0 : -m - 2Ax_0 : -1 \mp 2Ax_0 : A] : A \in \mathbb{R}\}.$$

We have left to construct a family of hyperbolas for $p = (\infty \pm \infty k)^{-1}$. To accomplish this, we begin with the point $(1 + k)^{-1}$ and construct the set $\mathscr{T}_{(1+k)^{-1},m}$, and then use a subsequent $\mu \in \mathcal{LFT}(\hat{\mathbb{P}})$ so that $\mu((1 + k)^{-1}) = (\infty + \infty k)^{-1}$. The approach is somewhat indirect; we claim that since a point in H_∞ corresponds to the asymptote of the hyperbolas containing it, if we know where the asymptote on $\mathbb{P}$ goes, then we can identify where points at infinity go. The μ we choose to use is $\mu(z) = z - \frac{1}{2}$. So when $p = (\infty + \infty k)^{-1}$, then

$$\begin{aligned} \mathscr{T}_{p,m} &\overset{def}{=} \mu(\mathscr{T}_{(1+k)^{-1},m}) \\ &= \left\{\left[m - 1 : -1 - 2A : -1 - 2A : -\frac{m+1}{4}\right] : A \in \mathbb{R}\right\}. \end{aligned}$$

Similarly, when $p = (\infty - \infty k)^{-1}$, we begin with $(1 - k)^{-1}$ and use the same μ to get

$$\begin{aligned} \mathscr{T}_{p,m} &\overset{def}{=} \mu(\mathscr{T}_{(1-k)^{-1},m}) \\ &= \left\{\left[m + 1 : 1 - 2A : -1 + 2A : -\frac{m-1}{4}\right] : A \in \mathbb{R}\right\}. \end{aligned}$$

It is straightforward to verify that for any of these types, hyperbolas in the family $\mathscr{T}_{p,m}$ are pairwise tangent—namely, at the point p.

Lemma 3. *Let $p_1, p_2 \in \hat{\mathbb{P}}$ and let $m_1, m_2 \in \mathbb{R}$ such that $|m_j| < 1$. Then $\mathscr{T}_{p_1,m_1} = \mathscr{T}_{p_2,m_2}$ if and only if $p_1 = p_2$ and $m_1 = m_2$.*

Proof. If $p_1 = p_2$ and $m_1 = m_2$, then obviously $\mathscr{T}_{p_1,m_1} = \mathscr{T}_{p_2,m_2}$.

[3] In the argument for the $\mathscr{H}^-$ version, we permit $m = \infty$ in order to account for vertical lines and the hyperbolas tangent to them, in which case the slope is undefined.

Conversely, suppose we begin with $\mathscr{T}_{p_1,m_1} = \mathscr{T}_{p_2,m_2}$. Then $h_1, h_2 \in \mathscr{T}_{p_1,m_1}$ implies that $h_1, h_2 \in \mathscr{T}_{p_2,m_2}$. If $p_1 \neq p_2$, then h_1 and h_2 share two points—namely, p_1 and p_2. This is a contradiction since h_1 and h_2 are tangent. Thus $p_1 = p_2 \overset{set}{=} p$.

By pre-composing with an appropriate linear fractional transformation, we may assume that p is finite (i.e. $p \in \mathbb{P}$). Since all hyperbolas in $\mathscr{T}_{p,m_1} = \mathscr{T}_{p,m_2}$ are tangent at one mutual point, they must share the same slope at p. Hence $m_1 = m_2$.

Therefore, we conclude that $\mathscr{T}_{p_1,m_1} = \mathscr{T}_{p_2,m_2}$ if and only if $p_1 = p_2$ and $m_1 = m_2$. $\square$

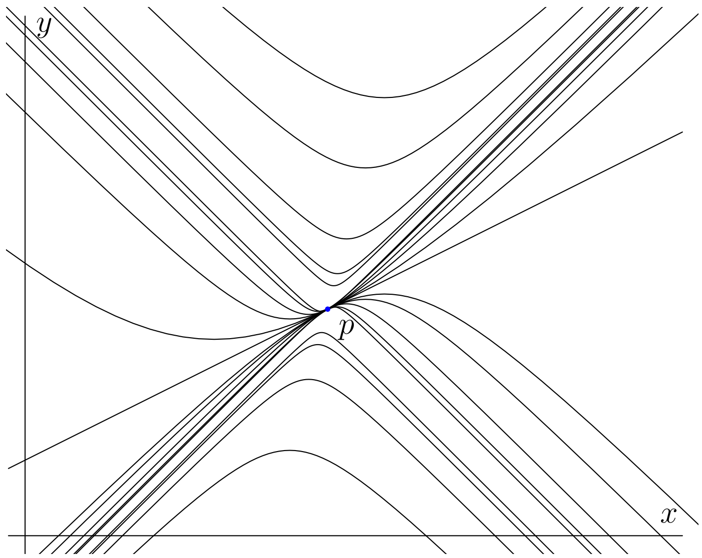

Figure 10: The family of hyperbolas $\mathscr{T}_{p,m}$.

Lemma 4. *If $p \in \hat{\mathbb{P}}$ and $m \in \mathbb{R}$ such that $|m| < 1$, then there exist $p' \in \hat{\mathbb{P}}$ and $m' \in \mathbb{R}, |m'| < 1$ so that*

$$T(\mathscr{T}_{p,m}) = \mathscr{T}_{p',m'}$$

Proof. If $h_1, h_2, h_3, h_4 \in \mathscr{T}^+_{p,m}$, then $T(h_1)$, $T(h_2)$, $T(h_3)$, and $T(h_4)$ are pairwise tangent. Moreover, by Lemma 2, they are tangent a mutual point $p' \in \hat{\mathbb{P}}$. Next, we post-compose T with a suitable linear fractional transformation in order that we may assume that p' is finite, and let $m' \in \mathbb{R}$, with $|m'| < 1$, be the slope of $T(h_j)$ at p', $1 \leq j \leq 4$. Then $T(h_j) \in \mathscr{T}^+_{p',m'}$.

We show that $T(\mathscr{T}^+_{p,m}) = \mathscr{T}^+_{p',m'}$.

Suppose that $h_5 \in \mathscr{T}_{p,m}$, but $h_5 \notin \{h_1, h_2, h_3, h_4\}$. Then $T(h_5) \in T(\mathscr{T}_{p,m})$ and, moreover, $T(h_5)$ is tangent to $T(h_j), 1 \leq j \leq 4$. It follows that $T(h_5) \in \mathscr{T}_{p',m'}$, thus $T(\mathscr{T}_{p,m}) \subseteq \mathscr{T}_{p',m'}$.

Now let $h'_5 \in \mathscr{T}_{p',m'}$, but $h'_5 \notin \{T(h_1), T(h_2), T(h_3), T(h_4)\}$. Because T is surjective, there is an $h_5 \in \mathscr{H}^+$ such that $h'_5 = T(h_5)$; furthermore, because T is injective, $h_5 \notin \{h_1, h_2, h_3, h_4\}$. So $T(h_5)$ is tangent to $T(h_j), 1 \leq j \leq 4$, thus $T^{-1}(T(h_5)) = h_5$ is tangent to $T^{-1}(T(h_j)) = h_j \in \mathscr{T}_{p,m}$. Then $h_5 \in \mathscr{T}_{p,m}$ and therefore, $h'_5 = T(h_5) \in T(\mathscr{T}_{p,m})$. Hence $\mathscr{T}_{p',m'} \subseteq T(\mathscr{T}_{p,m})$. $\square$

4.2 $T : \mathscr{H}^+ \to \mathscr{H}^+$ is induced via $p \mapsto p'$

We have just shown that a family of mutually tangent hyperbolas $\mathscr{T}_{p,m}$ maps onto a family of mutually tangent of hyperbolas $\mathscr{T}_{p',m'}$. This suggests the existence of a mapping $p \mapsto p'$. We show that this mapping is a well-defined—that is, p is always mapped to p', regardless the choice of m.

Lemma 5. *The mapping $p \mapsto p'$ is well-defined on $\hat{\mathbb{P}}$*

Proof. We begin by selecting two distinct real numbers m_1 and m_2 with $|m_j| < 1$, and construct the two families $\mathscr{T}_{p,m_1}$ and $\mathscr{T}_{p,m_2}$. By Lemma 4, there exist $p_1', p_2' \in \hat{\mathbb{P}}$ and m_1', m_2' with $|m_j'| < 1$ such that

$$T(\mathscr{T}_{p,m_1}) = \mathscr{T}_{p_1',m_1'} \text{ and } T(\mathscr{T}_{p,m_2}) = \mathscr{T}_{p_2',m_2'}.$$

We must show that $p_1' = p_2'$. Our reasoning is as follows.

If $p_1' \neq p_2'$ we will construct hyperbolas $h_1 \in \mathscr{T}_{p_1',m_1'}$ and $h_2 \in \mathscr{T}_{p_2',m_2'}$ that are tangent to one another. Since T is a bijection that preserves tangency, this would mean that hyperbolas $T^{-1}(h_1) \in \mathscr{T}_{p,m_1}$ and $T^{-1}(h_2) \in \mathscr{T}_{p,m_2}$ exist and are tangent to each other where $m_1 \neq m_2$, which is clearly false.

To simplify matters, we post-compose T with a suitable linear fractional transformation μ in order that we may assume that $p_1' = 0$, $m_1' = 0$ and p_2' is finite. This μ can be written as a composition $\mu_1 \circ \mu_2$ where μ_2 sends p_1' to 0 and p_2' to some finite point; and μ_1 is a suitable rotation $\mu_2(z) = az$, where a is some real number.

The hyperbola $h_1 \in \mathscr{T}_{p_1',m_1'} = \mathscr{T}_{0,0}$ can then be written $[\lambda : 0 : 1 : 0]$ for some $\lambda \in \mathbb{R}$. At this point our argument is divided into two cases.

Case 1. We assume that $p_2' = x_0 + y_0 k$ where $y_0 \neq 0$. To simplify notation we set $m_2' = m$. We choose $\lambda = 0$ so that $h_1 = [0 : 0 : 1 : 0]$, and will look for numbers $E, F, G, H \in \mathbb{R}$ so that $h_2 = [E : F : G : H]$ is tangent to h_1 and belongs to $\mathscr{T}_{p_2',m_2'} = \mathscr{T}_{x_0+y_0k,m}$.

The normalization on h_2 and the tangency with h_1 then require

$$4EH + G^2 - F^2 = 1 \tag{10}$$

$$G^2 = 1 \tag{11}$$

In addition, as h_2 can be expressed in coordinates by $E(x^2 - y^2) + Fx + Gy + H = 0$, we have that $h_2 \in \mathscr{T}_{x_0+y_0k,m}$ requires

$$E(x_0^2 - y_0^2) + Fx_0 + Gy_0 + H = 0 \tag{12}$$

$$E(2x_0 - 2y_0 m) + F + Gm = 0. \tag{13}$$

(The latter equation is obtained by implicit differentiation. Recall that m is the slope of h_2 at $x_0 + y_0 k$.)

From (11), we choose $G = 1$ and substitute into the remaining equations. (The choice of $+1$ or -1 makes no difference since multiplying all components of h_2 by -1 has no effect on h_2.) From (13), we find $F = -2E(x_0 - y_0 m) - m$ and substitute into the remaining equations. Then from (12), we also find $H = E(x_0^2 + y_0^2 - 2x_0 y_0 m) + x_0 m - y_0$ and substitute into the remaining equations. Thus from (10) we get the following equation.

$$E^2 - \frac{1}{y_0}E - \frac{m^2}{4{y_0}^2(1-m^2)} = 0$$

This is quadratic in E and has real solutions. So h_2 exists.

Case 2. We assume that $p_2' = x_0 + 0k$ with $x_0 \neq 0$ and we again let $m_2' = m$. By post-composing T with a further dilation we may assume that $x_0 = 1$. We choose $\lambda = 1$ and will look for numbers $E, F, G, H \in \mathbb{R}$ so that $h_2 = [E : F : G : H]$ is tangent to h_1 and belongs to $\mathscr{T}_{p_2', m_2'} = \mathscr{T}_{1,m}$.

The normalization on h_2 and the tangency with h_1 then require that

$$4EH + G^2 - F^2 = 1 \tag{14}$$
$$(2H + G)^2 = 1. \tag{15}$$

In addition, as h_2 can be expressed in coordinates by $E(x^2 - y^2) + Fx + Gy + H = 0$, we have that $h_2 \in \mathscr{T}_{1,m}$ requires that

$$E + F + H = 0 \tag{16}$$
$$2E + F + Gm = 0. \tag{17}$$

(The latter equation is obtained by implicit differentiation.)

From (15), we choose $2H + G = 1$ and substitute $G = 1 - 2H$ into the remaining equations. (The choice of $+1$ or -1 makes no difference since multiplying all components of h_2 by -1 has no effect on h_2.) From (16), we also find $F = -(E + H)$ and substitute into the remaining equations. We then have two equations

$$4H^2 - 4H - (E - H)^2 = 0 \tag{18}$$
$$(E - H) + (1 - 2H)m = 0. \tag{19}$$

By solving (19) for $E - H$ and substituting into (18), and simplifying, we obtain the following equation for H.

$$4(1 - m^2)H^2 - 4(1 - m^2)H - m^2 = 0.$$

This is quadratic in H and has real solutions. So h_2 exists.

Therefore, regardless of the choice of slope m for the family $\mathscr{T}_{p,m}$, the point p will always be mapped the same image point p'—that is, $p \mapsto p'$ is well-defined. □

Definition 5. *Following the well-defined mapping $p \mapsto p'$ in Lemma 5, we define $\hat{T} : \hat{\mathbb{P}} \to \hat{\mathbb{P}}$ by*

$$\hat{T}(p) = p'.$$

We call this the ***mapping inducing*** $T : \mathscr{H}^+ \to \mathscr{H}^+$.

This next lemma will show that the pointwise mapping $\hat{T}$ actually determines the hyperbola mapping T.

Lemma 6. *If $h \in \mathscr{H}^+$, then $T(h) = \{\hat{T}(p) : p \in h\}$.*

Proof. Let $p \in h$ and let $m \in \mathbb{R}$, such that $|m| < 1$, and construct $\mathscr{T}_{p,m}$. Then $h \in \mathscr{T}_{p,m}$. By Lemma 4, there is a $p' \in \hat{\mathbb{P}}$ and an $m' \in \mathbb{R}$, with $|m'| < 1$, so that $T(\mathscr{T}_{p,m}) = \mathscr{T}_{p',m'}$. Then $T(h) \in \mathscr{T}_{p',m'} = \mathscr{T}_{\hat{T}(p),m'}$ and $\hat{T}(p) \in T(h)$. Thus $\{\hat{T}(p) : p \in h\} \subseteq T(h)$.

Now let $p' \in T(h)$ and suppose that $T(h) \in \mathscr{T}_{p',m'}$, for some m'. Then for T^{-1}, by Lemma 4, there is $p \in \hat{\mathbb{P}}$ and $m \in \mathbb{R}, |m| < 1$, so that $T(\mathscr{T}_{p,m}) = \mathscr{T}_{p',m'}$. Furthermore, because T^{-1} is bijective, there is a unique $h \in \mathscr{H}^+$ such that $h = T^{-1}(T(h)) \in T^{-1}(\mathscr{T}_{p',m'}) = \mathscr{T}_{p,m}$, and thus $p \in h$. By definition of $\hat{T}$, we also have that $\hat{T}(p) = p'$, which means $p' \in \{\hat{T}(p) : p \in h\}$. Hence $T(h) \subseteq \{\hat{T}(p) : p \in h\}$. □

The second part of the proof of Lemma 6 gives us that $\hat{T}$ is surjective, since every $p' \in T(h)$ has a preimage $p \in h$ for any $h \in \mathscr{H}^+$. Surjectivity tells us that intersection points cannot be created by $\hat{T}$; in order to show that intersection points cannot be destroyed, we must show that $\hat{T}$ is also injective.

Lemma 7. *$\hat{T}$ is injective.*

Proof. Let $p_1, p_2 \in \hat{\mathbb{P}}$ and suppose that $\hat{T}(p_1) = \hat{T}(p_2)$. Let $m' \in \mathbb{R}, |m'| < 1$, and construct the families $\mathscr{T}_{\hat{T}(p_1),m'}$ and $\mathscr{T}_{\hat{T}(p_2),m'}$. By Lemma 3, $\mathscr{T}_{\hat{T}(p_1),m'} = \mathscr{T}_{\hat{T}(p_2),m'}$. Then applying T^{-1}, there is an $m \in \mathbb{R}$ with $|m| < 1$, such that

$$\mathscr{T}_{p_1,m} = T^{-1}(\mathscr{T}_{\hat{T}(p_1),m'}) = T^{-1}(\mathscr{T}_{\hat{T}(p_2),m'}) = \mathscr{T}_{p_2,m}.$$

By Lemma 3, $p_1 = p_2$. Hence $\hat{T}$ is injective. □

We now have that $\hat{T}$ is an injective mapping on $\hat{\mathbb{P}}$ which sends hyperbolas in $\mathscr{H}^+$ to hyperbolas in $\mathscr{H}^+$. Therefore, by Lemma 1 from section 3, we know that $\hat{T}$ is a linear fractional transformation when restricted to a closed middle region. We have left to show that $\hat{T}$ is linear fractional on the entire extended double number plane—and not only on some closed middle region.

4.3 $\hat{T}$ is a linear fractional transformation of $\hat{\mathbb{P}}$

Let P be a closed middle region bounded by some vertical right hyperbola. Define $\mu \in \mathcal{LFT}(\hat{\mathbb{P}})$ such that $\mu = \hat{T}|_P$ and let $P' = \hat{T}(P)$. Then $\mu^{-1}(P') = P$ and, furthermore, $\mu^{-1} \circ \hat{T}$ fixes every point in the region P. See Figure 11. We will show that $\mu^{-1} \circ \hat{T}$ fixes every point *outside* of P as well.

Completion of the Proof of Theorem 2. Suppose that p is a finite number. Then we can construct two distinct lines which intersect at p. We claim that every line in $\mathscr{H}^+$ must intersect the closed middle region P at least twice (in fact, infinitely many times). Then

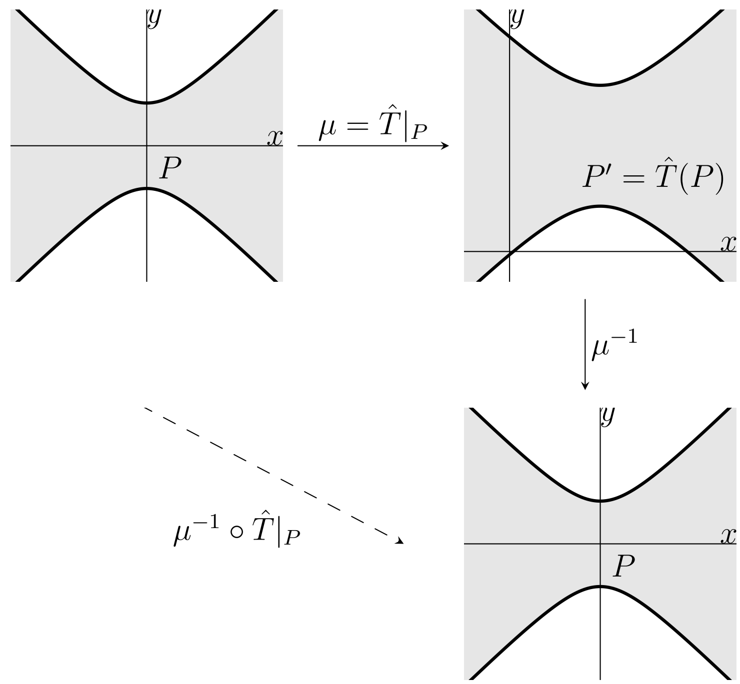

Figure 11: The mapping $\mu^{-1} \circ \hat{T}|_P$ is the identity on the region P.

every line is preserved by $\mu^{-1} \circ \hat{T}$, and thus intersection points of any pair of lines are preserved. Hence $\mu^{-1} \circ \hat{T}(p) = p$.

Now suppose $p \in H_\infty$. Since a hyperbola is uniquely determined by its points in $\mathbb{P}$, every hyperbola in $\mathscr{H}^+$ is preserved by $\mu^{-1} \circ \hat{T}$. Then it follows that

$$\mathscr{T}_{p,m} = \mu^{-1} \circ \hat{T}(\mathscr{T}_{p,m}) = \mathscr{T}_{\mu^{-1} \circ \hat{T}(p), m'}.$$

Therefore, by Lemma 3, $\mu^{-1} \circ \hat{T}(p) = p$.

Since $\mu^{-1} \circ \hat{T}$ fixes every point on the extended double plane, it follows that it is the identity mapping, and hence a linear fractional transformation. Thus, by the group structure of $\mathcal{LFT}(\hat{\mathbb{P}})$,

$$\hat{T} = \mu \circ \mu^{-1} \circ \hat{T} \in \mathcal{LFT}(\hat{\mathbb{P}}).$$

Therefore, bijective mappings that sends tangent hyperbolas in $\mathscr{H}^+$ to tangent hyperbolas in $\mathscr{H}^+$ are induced by a linear fractional transformation of the extended double plane. $\square$

5 Conclusion

This result furthers a connection between the *complex numbers, dual numbers*[4] and *double numbers*. Ferdinands and Kavlie showed in [3] that a bijection on the space of parabolas that preserves a fixed distance 1 between intersecting parabolas is induced by a linear fractional transformation of the dual plane $\mathbb{D}$. This result along with Lester's and our own show how intrinsic the linear fractional transformations are with the geometrical spaces in which they act. We further wonder if there is a more *unified* way to set up Theorem 2—that is, what are the necessary and sufficient conditions for T if we take into consideration the entire space of right hyperbolas and lines $\mathscr{H} = \mathscr{H}^+ \cup \mathscr{H}^-$? This question came into mind while determining whether or not $\hat{T}$ is well-defined when only assuming that $T : \mathscr{H} \to \mathscr{H}$ is a bijection. It also stands to show whether or not a stronger version of Theorem 2 is true by assuming a fixed angle > 0 is preserved.

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[4]The dual numbers are of form $x + yj$ where $j^2 = 0$.