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The Exponential of a Quaternionic Matrix

Casey Machen

Abstract. The exponential map is important because it provides a map from the Lie algebra of a Lie group into the group itself. We focus on matrix groups over the quaternions and the exponential map from their Lie algebras into the groups. Since quaternionic multiplication is not commutative, the process of calculating the exponential of a matrix over the quaternions is more involved than the process of calculating the exponential of a matrix over the real or complex numbers. We develop processes by which this calculation may be reduced to a simpler problem, and provide some examples.

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1 Introduction

Just as the complex numbers arose as an extension of the real numbers, quaternions arose as an extension of the complex numbers. For $(a,b,c,d) \in \mathbb{R}^4$, we will write $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, add component-wise, and multiply with the following rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$
and
$$\mathbf{i}\cdot\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\cdot\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\cdot\mathbf{i} = \mathbf{j}$$
$$\mathbf{j}\cdot\mathbf{i} = -\mathbf{k}, \quad \mathbf{k}\cdot\mathbf{j} = -\mathbf{i}, \quad \mathbf{i}\cdot\mathbf{k} = -\mathbf{j}.$$

Although the multiplication operation given to the complex numbers turned $\mathbb{R}^2$ into a field, these operations don't quite turn $\mathbb{R}^4$ into a field since the multiplication operation is not commutative (a non-commutative field is called a skew field). In this case, the above operations turn $\mathbb{R}^4$ into a skew-field called the quaternions, $\mathbb{H}$.

We wish to describe a method by which one can calculate the exponential of a matrix with quaternion entries. To this end, we provide a section on background material where we construct an essential map, Ψ_n, on which the material in this paper is based, give a brief overview of matrix Lie groups and their Lie algebras, and introduce the exponential map. Section 3 shows how we use Ψ_n to transform a matrix over $\mathbb{H}$ into a matrix over $\mathbb{C}$. We provide several examples and a few theorems which aim to simplify the calculation of the exponential of a quaternionic matrix. Section 4 concludes the paper and suggests further research.

2 Background Material

In this paper, will be concerned with $n \times n$ matrices with entries in $\mathbb{H}$, denoted $M_n(\mathbb{H})$. Since multiplication in $\mathbb{H}$ is not commutative, there are difficulties in defining a linear algebra over $\mathbb{H}$. Our goal is to turn a quaternionic matrix into a complex matrix and to apply the typical notions of linear algebra over the complex field. In order to do this, we want a map, Ψ_n, that turns an $n \times n$ matrix over $\mathbb{H}$ into a $2n \times 2n$ matrix over $\mathbb{C}$. This map Ψ_n will be described in terms of Ψ_1 (notice that we can write a quaternion as $z + w\mathbf{j}$ for $z, w \in \mathbb{C}$):

$$\Psi_1(z + w\mathbf{j}) = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

To determine $\Psi_n(A)$ for some $A \in M_n(\mathbb{H})$, apply Ψ_1 to each entry of A to create 2×2 blocks.

Observation 1. *The map Ψ_n is a continuous, injective ring homomorphism. In particular, for $A, B \in M_n(\mathbb{H})$ and $c \in \mathbb{R}$, we have the following [1]:*

$$\Psi_n(cA) = c \cdot \Psi_n(A) \tag{1}$$
$$\Psi_n(A + B) = \Psi_n(A) + \Psi_n(B) \tag{2}$$
$$\Psi_n(A \cdot B) = \Psi_n(A) \cdot \Psi_n(B) \tag{3}$$

It turns out that the image under Ψ_n of an invertible matrix is an invertible matrix [2]. Thus, when we talk about the determinant of a quaternionic matrix, we mean the composition $\det \circ \Psi_n$, however, we will just write det. For example, it is clear that $\left(\begin{smallmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{smallmatrix}\right)$ is not invertible, but the ordinary method of calculating the determinant yields $2\mathbf{k} \neq 0$. However,

$$\det(\Psi_2 \left(\begin{smallmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{smallmatrix}\right)) = \det(\left(\begin{smallmatrix} \mathbf{i} & 0 & 0 & 1 \\ 0 & -\mathbf{i} & -1 & 0 \\ \mathbf{i} & 0 & 0 & 1 \\ 0 & -\mathbf{i} & -1 & 0 \end{smallmatrix}\right)) = 0.$$

The notion of a determinant is important in extending the concept of general linear, special linear, and orthogonal groups to the quaternions. Respectively, these are

$$\begin{aligned} GL_n(\mathbb{H}) &= \{A \in M_n(\mathbb{H}) \mid \det(A) \neq 0\} \\ SL_n(\mathbb{H}) &= \{A \in GL_n(\mathbb{H}) \mid \det(A) = 1\} \\ Sp(n) &= \{A \in GL_n(\mathbb{H}) \mid A^{-1} = A^*\}. \end{aligned}$$

We call the group $Sp(n)$ the symplectic group, where A^* is the conjugate transpose of the matrix A, and conjugation in $\mathbb{H}$ is given by $\overline{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$.

Since these groups are all subsets of some Euclidean space, we endow them with the usual Euclidean topology. It is a well-known fact that these groups are actually *differentiable manifolds*. Furthermore, both the group operation of matrix multiplication and the inverse map are smooth. A differentiable manifold with a smooth group operation and inverse map is called a *Lie group*; of course from this discussion, these groups are Lie groups.

The *Lie algebra* $\mathfrak{g}$ of a matrix Lie group G is the tangent space at the identity, along with the bracket operation $[,] : \mathfrak{g} \to \mathfrak{g}$ given by $[A, B] = AB - BA$ for $A, B \in \mathfrak{g}$. It is customary to use the same letters of the Lie group, except lowercase, when talking about the Lie algebra. For example, it is well known that the Lie algebra of $GL_n(\mathbb{H})$ is the set of all $n \times n$ matrices with entries in $\mathbb{H}$:

$$gl_n(\mathbb{H}) = M_n(\mathbb{H})$$

and that the Lie algebra of $Sp(n)$ is the set

$$sp(n) = \{A \in M_n(\mathbb{H}) \mid A^* + A = 0\}.$$

Tapp [2] uses the map Ψ_n to describe the Lie algebra of $SL_n(\mathbb{H})$ as

$$sl_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) \mid trace(\Psi_n(A)) = 0\}.$$

It is not hard to see that $trace(\Psi_n(A)) = 0$ if and only if $\mathrm{Re}(trace(A)) = 0$. We will use the latter characterization when we refer to $sl_n(\mathbb{H})$ (see Theorem 2).

We now define the exponential of a matrix. For any $A \in M_n(\mathbb{H})$,

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

In fact, this converges for any square matrix A [2].

Properties of the Exponential Map

Let $X, Y \in M_n(\mathbb{H})$.

(i) If $XY = YX$, then $e^{X+Y} = e^X e^Y$.

(ii) If Y^{-1} exists, then $e^{YXY^{-1}} = Ye^X Y^{-1}$.

The exponential map is important because it maps the Lie algebra of a matrix group into the group, and if the group is compact, the exponential map is surjective. The exponential maps of real and complex matrices are well documented (see, for example, Curtis [1] or Tapp [2]); yet, we have encountered no work documenting the exponential of a matrix with quaternion entries.

3 Calculating the Exponential of a Quaternionic Matrix

We begin with a theorem which allows us to transform a quaternionic matrix into a complex matrix.

Theorem 1. *For any $A \in M_n(\mathbb{H})$, $e^{\Psi_n(A)} = \Psi_n(e^A)$.*

Proof: We will rely on the fact that Ψ_n is a continuous ring homomorphism.

$$\begin{aligned}
e^{\Psi_n(A)} &= I + \Psi_n(A) + \frac{\Psi_n(A)^2}{2!} + \frac{\Psi_n(A)^3}{3!} + \dots \\
&= \sum_{k=0}^{\infty} \frac{\Psi_n(A)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\Psi_n(A^k)}{k!} \qquad \text{from property (3) of Observation 1} \\
&= \Psi_n\left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right) \qquad \text{from properties (1) and (2) of Observation 1} \\
&= \Psi_n(e^A).
\end{aligned}$$

□

Since Ψ_n is an injective map, this result states that for a matrix $A \in M_n(\mathbb{H})$,

$$e^A = \Psi_n^{-1}(e^{\Psi_n(A)}).$$

In particular, in order to find the exponential of a matrix over the quaternions, we will transform it into a complex matrix, calculate its exponential using linear algebra techniques defined over the complex field, and bring that matrix back into the quaternions. As a first example, we derive the general formula for the exponential of a single quaternion.

Example 1. *Let* $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ *with* $v = \sqrt{b^2 + c^2 + d^2} \neq 0$*, then*

$$e^q = e^a(\cos v + \frac{b\mathbf{i} + c\mathbf{j} + d\mathbf{k}}{v} \sin v).$$

Proof: We can write $b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = b\mathbf{i} + (c + d\mathbf{i})\mathbf{j}$, so that

$$M = \Psi_1(b\mathbf{i} + (c + d\mathbf{i})\mathbf{j}) = \begin{pmatrix} b\mathbf{i} & c + d\mathbf{i} \\ -c + d\mathbf{i} & -b\mathbf{i} \end{pmatrix}.$$

This matrix has characteristic polynomial $\lambda^2 + v^2 = 0$ from which we have eigenvalues $\lambda = v\mathbf{i}$ and $\lambda = -v\mathbf{i}$ with corresponding eigenvectors

$$\begin{pmatrix} c + d\mathbf{i} \\ (v - b)\mathbf{i} \end{pmatrix} \text{ and } \begin{pmatrix} c + d\mathbf{i} \\ -(b + v)\mathbf{i} \end{pmatrix}, \text{ respectively.}$$

We may now diagonalize our matrix M in the standard way:

$$\begin{pmatrix} b\mathbf{i} & c + d\mathbf{i} \\ -c + d\mathbf{i} & -b\mathbf{i} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} c + d\mathbf{i} & c + d\mathbf{i} \\ (v - b)\mathbf{i} & -(b + v)\mathbf{i} \end{pmatrix} \begin{pmatrix} v\mathbf{i} & 0 \\ 0 & -v\mathbf{i} \end{pmatrix} \begin{pmatrix} -(b + v)\mathbf{i} & -c - d\mathbf{i} \\ (b - v)\mathbf{i} & c + d\mathbf{i} \end{pmatrix}.$$

Here, $D = (c + d\mathbf{i})(-2v\mathbf{i})$, which is the determinant of the matrix formed by the eigenvectors of M. We know from property (ii) of the exponential map that $e^{ABA^{-1}} = Ae^B A^{-1}$. Therefore, to calculate the exponential of M, we simply need to calculate the exponential of $diag(v\mathbf{i}, -v\mathbf{i})$ and multiply by the matrix formed by the eigenvalues on the left and on the right by the inverse of the matrix formed by the eigenvectors. But $e^{diag(v\mathbf{i}, -v\mathbf{i})}$ is simply $diag(e^{v\mathbf{i}}, e^{-v\mathbf{i}})$, so we see that

$$e^M = \frac{1}{D} \begin{pmatrix} c + d\mathbf{i} & c + d\mathbf{i} \\ (v - b)\mathbf{i} & -(b + v)\mathbf{i} \end{pmatrix} \begin{pmatrix} e^{v\mathbf{i}} & 0 \\ 0 & e^{-v\mathbf{i}} \end{pmatrix} \begin{pmatrix} -(b + v)\mathbf{i} & -c - d\mathbf{i} \\ (b - v)\mathbf{i} & c + d\mathbf{i} \end{pmatrix}.$$

From this we get

$$\begin{aligned} e^M &= \begin{pmatrix} \frac{e^{v\mathbf{i}} + e^{-v\mathbf{i}}}{2} + \frac{b}{v}(\frac{e^{v\mathbf{i}} - e^{-v\mathbf{i}}}{2}) & \frac{e^{v\mathbf{i}} - e^{-v\mathbf{i}}}{2v\mathbf{i}}(c + d\mathbf{i}) \\ \frac{e^{v\mathbf{i}} - e^{-v\mathbf{i}}}{-2v\mathbf{i}}(c - d\mathbf{i}) & \frac{e^{v\mathbf{i}} + e^{-v\mathbf{i}}}{2} - \frac{b}{v}(\frac{e^{v\mathbf{i}} - e^{-v\mathbf{i}}}{2}) \end{pmatrix} \\ &= \begin{pmatrix} \cos v + \frac{b}{v}\mathbf{i} \sin v & \frac{c}{v} \sin v + \frac{d}{v}\mathbf{i} \sin v \\ -\frac{c}{v} \sin v + \frac{d}{v}\mathbf{i} \sin v & \cos v - \frac{b}{v}\mathbf{i} \sin v \end{pmatrix} \\ &= \Psi_1(\cos v + \frac{b}{v}\mathbf{i} \sin v + \frac{c}{v}\mathbf{j} \sin v + \frac{d}{v}\mathbf{k} \sin v) \\ &= \Psi_1(\cos v + \frac{b\mathbf{i} + c\mathbf{j} + d\mathbf{k}}{v} \sin v). \end{aligned}$$

Since $e^{a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}} = e^a e^{b\mathbf{i}+c\mathbf{j}+d\mathbf{k}}$ by property (i) of the exponential map, we conclude that

$$e^q = e^{a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}} = e^a(\cos v + \frac{b\mathbf{i}+c\mathbf{j}+d\mathbf{k}}{v}\sin v), \text{ for } \sqrt{b^2+c^2+d^2} \neq 0.$$

□

Notice that this agrees with the exponential of a real or complex number. When $a = 0$, we have $q = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. In this case, $q \in sp(1)$, the Lie algebra of $Sp(1)$ (recall that $sp(1) = \{x \in \mathbb{H} : x + \overline{x} = 0\}$). Furthermore, our calculation shows $e^q = \cos|q| + \frac{q}{|q|}\sin|q|$. In particular, the path $\gamma(t) = e^{qt}$ traces a great circle on $Sp(1) \cong \mathbb{S}^3$ (in fact, this circle is the intersection of $\mathbb{S}^3$ with the plane spanned by $\frac{q}{|q|}$ and 1).

At this point we mention that, in general, $e^{X+Y} \neq e^X e^Y$ for $X, Y \in M_n(\mathbb{H})$ (see property (i) of the exponential map).

Counterexample. Consider $\mathbf{i}, \mathbf{j} \in \mathbb{H}$. We already know that $\mathbf{i}\cdot\mathbf{j} \neq \mathbf{j}\cdot\mathbf{i}$. Using the previous example,

$$e^{\mathbf{i}+\mathbf{j}} = \cos\sqrt{2} + \tfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\sin\sqrt{2}.$$

However, $e^{\mathbf{i}}e^{\mathbf{j}} = (\cos 1 + \mathbf{i}\sin 1)(\cos 1 + \mathbf{j}\sin 1) = \cos^2 1 + (\mathbf{i}+\mathbf{j})\cos 1 \sin 1 + \mathbf{k}\sin^2 1$. Clearly these two values are not equal, in particular, the former doesn't even have a $\mathbf{k}$ term. □

In calculating the exponential of a single quaternion, we used the fact that $e^{a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}} = e^a e^{b\mathbf{i}+c\mathbf{j}+d\mathbf{k}}$ from property (i) of the exponential map. We may ask if we can do something similar to simplify the computation when calculating the exponential of an arbitrary element of $M_n(\mathbb{H})$. The following theorem gives rise to a corollary which provides an answer to this question.

Theorem 2. *Every element of $M_n(\mathbb{H})$ can be written uniquely as a sum of a matrix of the form xI, for some $x \in \mathbb{R}$, and a matrix whose trace has no real part (i.e. a matrix in $sl_n(\mathbb{H})$).*

Proof: Let $A \in M_n(\mathbb{H})$ and denote the i,j-entry of A by a_{ij}. Then

$$\mathrm{Re}(trace(A)) = \mathrm{Re}\left(\sum_{k=1}^{n}(a_{kk})\right) = \sum_{k=1}^{n}\mathrm{Re}(a_{kk}).$$

If we set $x = \frac{1}{n}\sum_{k=1}^{n}\mathrm{Re}(a_{kk})$ and $\beta_{ii} = a_{ii} - x$, we see that $x \in \mathbb{R}$ and that

$$\begin{aligned}\mathrm{Re}\left(\sum_{k=1}^{n}\beta_{kk}\right) &= \sum_{k=1}^{n}(\mathrm{Re}(a_{kk}) - x)\\ &= \sum_{k=1}^{n}\left(\mathrm{Re}(a_{kk}) - \frac{1}{n}\sum_{m=1}^{n}\mathrm{Re}(a_{mm})\right)\\ &= \sum_{k=1}^{n}\mathrm{Re}(a_{kk}) - n\cdot\frac{1}{n}\sum_{k=1}^{n}\mathrm{Re}(a_{kk}) = 0.\end{aligned}$$

Let B be the matrix formed by replacing each a_{ii} entry of A by β_{ii} while leaving the other elements of A untouched. Then $\mathrm{Re}(trace(B)) = 0$, as shown above. Furthermore, $A = B + xI$ and this decomposition is unique. $\square$

Corollary 1. *Let $A \in M_n(\mathbb{H})$, then $e^A = e^x e^B$, where x and B are as in the previous proof.*

Proof: Since $x \in \mathbb{R}$, x commutes with all quaternions and both xI and B commute. By property (i) of the exponential map,

$$e^A = e^{xI+B} = e^{xI}e^B.$$

Notice that $e^{xI} = e^x I$, which gives $e^A = e^{xI}e^B = (e^x I)e^B = e^x e^B$. $\square$

The question of determining the exponential of any given matrix over the quaternions becomes a slightly simpler question of determining the exponential of the associated matrix in the Lie algebra $sl_n(\mathbb{H})$.

Example 2. *We will calculate the exponential of the following matrix:*

$$A = \begin{pmatrix} 1+\mathbf{i} & \mathbf{j} \\ \mathbf{j} & 1-\mathbf{i} \end{pmatrix} \in gl_2(\mathbb{H}).$$

Using the notation of Theorem 2, we decompose the matrix A into the form $B + xI$ where $B \in sl_2(\mathbb{H})$. Clearly, $x = 1$, so B is given by the following matrix:

$$B = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{j} & -\mathbf{i} \end{pmatrix} \in sl_2(\mathbb{H}).$$

It is straightforward to find

$$\Psi_2(B) = \begin{pmatrix} \mathbf{i} & 0 & 0 & 1 \\ 0 & -\mathbf{i} & -1 & 0 \\ 0 & 1 & -\mathbf{i} & 0 \\ -1 & 0 & 0 & \mathbf{i} \end{pmatrix}.$$

The characteristic equation is $\lambda^2(\lambda^2+4) = 0$, from which we get the eigenvalues $0, \pm 2\mathbf{i}$.
For $\lambda = 0$, the corresponding eigenvectors are $\begin{pmatrix} \mathbf{i} \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -\mathbf{i} \\ -1 \\ 0 \end{pmatrix}$.
For $\lambda = 2\mathbf{i}$ we get the eigenvector $\begin{pmatrix} \mathbf{i} \\ 0 \\ 0 \\ -1 \end{pmatrix}$.
And for $\lambda = -2\mathbf{i}$ we get the eigenvector $\begin{pmatrix} 0 \\ -\mathbf{i} \\ 1 \\ 0 \end{pmatrix}$.
Let C be the matrix whose columns are formed from the eigenvectors of $\Psi_2(B)$. Thus,

$$C = \begin{pmatrix} \mathbf{i} & 0 & \mathbf{i} & 0 \\ 0 & -\mathbf{i} & 0 & -\mathbf{i} \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad C^{-1} = \frac{1}{2}\begin{pmatrix} -\mathbf{i} & 0 & 0 & 1 \\ 0 & \mathbf{i} & -1 & 0 \\ -\mathbf{i} & 0 & 0 & -1 \\ 0 & \mathbf{i} & 1 & 0 \end{pmatrix}.$$

Therefore, $\Psi_2(B) = C \cdot diag(0, 0, 2\mathbf{i}, -2\mathbf{i}) \cdot C^{-1}$ by diagonalization. Therefore, by property (ii) of the exponential map,

$$e^{\Psi_2(B)} = C \cdot e^{diag(0,0,2\mathbf{i},-2\mathbf{i})} \cdot C^{-1} = C \cdot diag(1, 1, e^{2\mathbf{i}}, e^{-2\mathbf{i}}) \cdot C^{-1}.$$

But we know from Theorem 1 that

$$e^B = \Psi_2^{-1}(e^{\Psi_2(B)}) = \Psi_2^{-1}(C \cdot diag(1, 1, e^{2\mathbf{i}}, e^{-2\mathbf{i}}) \cdot C^{-1}).$$

Direct calculation yields

$$\begin{aligned} e^B &= \Psi_2^{-1}(C \cdot diag(1, 1, e^{2\mathbf{i}}, e^{-2\mathbf{i}}) \cdot C^{-1}) \\ &= \frac{1}{2}\ \Psi_2^{-1}(\begin{pmatrix} 1+e^{2\mathbf{i}} & 0 & 0 & \mathbf{i}(1-e^{2\mathbf{i}}) \\ 0 & 1+e^{-2\mathbf{i}} & \mathbf{i}(1-e^{-2\mathbf{i}}) & 0 \\ 0 & -\mathbf{i}(1-e^{-2\mathbf{i}}) & 1+e^{-2\mathbf{i}} & 0 \\ -\mathbf{i}(1-e^{2\mathbf{i}}) & 0 & 0 & 1+e^{2\mathbf{i}} \end{pmatrix}) \\ &= \frac{1}{2}\begin{pmatrix} 1+e^{2\mathbf{i}} & \mathbf{k}(1-e^{-2\mathbf{i}}) \\ -\mathbf{k}(1-e^{2\mathbf{i}}) & 1+e^{-2\mathbf{i}} \end{pmatrix}. \end{aligned}$$

Finally, by Corollary 1, we conclude that

$$e^A = e^1 e^B = \frac{e}{2}\begin{pmatrix} 1+e^{2\mathbf{i}} & \mathbf{k}(1-e^{-2\mathbf{i}}) \\ -\mathbf{k}(1-e^{2\mathbf{i}}) & 1+e^{-2\mathbf{i}} \end{pmatrix}.$$

We leave it to the reader to verify that this matrix is in $GL_2(\mathbb{H})$.

4 Conclusions

Linear algebra plays an important role in calculating the exponential of a matrix. As we mentioned in the introduction, the non-commutativity of the quaternions creates difficulties in defining a linear algebra over $\mathbb{H}$. We were able to get around this obstacle by transforming a quaternionic matrix into a complex matrix, manipulating it using the well-known linear algebra techniques over $\mathbb{C}$, and turning it back into a matrix over $\mathbb{H}$. Corollary 1 provided a simplification in such calculations, however, it only slightly eased the problem. Further study in this area may find a way to simplify the calculation of a quaternionic matrix even more. There may even be a method which bypasses the transformation into the complex matrices and that doesn't use the map Ψ_n.

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