[Rose-Hulman Undergraduate Mathematics Journal](https://scholar.rose-hulman.edu/rhumj)

Volume 12
Issue 2

Article 2

A q = -1 Phenomenon for Pattern-Avoiding Permutations

Xin Chen
Carleton College, chenx@carleton.edu

Rose-Hulman Undergraduate Mathematics Journal

A $q = -1$ Phenomenon for Pattern-Avoiding Permutations

Xin Chen[a]

Volume 12, No. 2, Fall 2011

Sponsored by

Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
http://www.rose-hulman.edu/mathjournal

[a]Carleton College, Northfield, MN

Rose-Hulman Undergraduate Mathematics Journal
Volume 12, No. 2, Fall 2011

A $q = -1$ Phenomenon for Pattern-Avoiding Permutations

Xin Chen

Abstract. We give an instance of Stembridge's $q = -1$ phenomenon for pattern-avoiding permutations. In particular, we show that setting $q = -1$ in the generating function for 132-avoiding permutations with respect to the statistic rsg, defined in [4], returns the number of 132-avoiding involutions.

Acknowledgements: Particular thanks to Prof. Eric S. Egge for his suggestion of the topic and his guidance throughout the research. The financial support comes from the Kolenkow Reitz fund of Carleton College. In addition, I would like to thank Daoji Huang and Yiran Robert Guo.

1 Introduction

In [5], Stembridge observed that for certain families of plane partitions, setting $q = -1$ in the generating function for the family returns the number of partitions in the family with a certain symmetry. For example, if $F_n(q)$ is the generating function for cyclically symmetric plane partitions in an $n \times n \times n$ cube with respect to the sum of the parts, then $F_n(-1)$ is the number of self-complementary cyclically symmetric plane partitions in an $n \times n \times n$ cube (See also [1, Ch.6]). More recently, Reiner, Stanton and White have studied this phenomenon in a much more general setting (See [2]).

Similar to the set of plane partitions in an $n \times n \times n$ cube, a set of pattern-avoiding permutations of some fixed length is also a finite set of objects. Moreover, as the Stembridge $q = -1$ phenomenon has an involution (the complement map) and a statistic on the objects (the number of boxes being partitioned), we also have involutions (such as the inverse map and the reverse-complement map) and numerous statistics on pattern-avoiding permutations.

In this paper, we investigate the $q = -1$ phenomenon and identify a permutation statistic that returns the number of involutions when we set $q = -1$ in its generating function. In Section 2, we provide the necessary background material for this paper. In Section 3, we prove that setting $q = -1$ in the generating function for the statistic rsg, introduced in [4], returns the number of 132-avoiding involutions.

2 Background

A *permutation* of a non-empty finite set is a bijection from the set to itself. A permutation π is an *involution* whenever $\pi = \pi^{-1}$. Let S_n denote the set of permutations of $\{1, \ldots, n\}$; we often identify a permutation $\pi \in S_n$ with the sequence $\pi(1)\pi(2)\cdots\pi(n)$. Let $|\pi|$ denote the *length* of the permutation, so that $|\pi| = n$ for $\pi \in S_n$. For $\pi \in S_n$ and $\sigma \in S_k$, we say a subsequence $\pi(i_1)\cdots\pi(i_k)$ has *type* σ whenever $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $\pi(i_l) < \pi(i_r)$ if and only if $\sigma(l) < \sigma(r)$, and we say σ is a *subpermutation* of π whenever π has a subsequence of type σ. For example, the subsequence 1426 of the permutation 145263 has type 1324 so 1324 is a subpermutation of 145263. We say π *avoids* σ whenever σ is not a subpermutation of π. For example, the permutation 214538769 avoids 312 and 2413, but it has 2586 as a subsequence so it does not avoid 1243. In this context σ is sometimes called a *pattern* or a *forbidden subsequence* and π is sometimes called a *restricted permutation* or a *pattern-avoiding permutation*. For all $n \geq 0$ and any set R of permutations, we write $S_n(R)$ to denote the set of permutations of length n which avoid every pattern in R. We also write $S(R)$ to denote the set of all permutations that avoid R.

To state our result, we first need the following definition.

Definition 1. *Suppose π is a permutation, and that we decompose π into a sequence of increasing runs, separated by the descents in the permutation. The statistic* $\mathrm{rsg}(\pi)$ *is the sum of the number of runs of π strictly to the right of each entry i of π which contain elements both larger and smaller than i. Equivalently,* $\mathrm{rsg}(\pi)$ *is the number of* $2{-}13$ *patterns* *in π, where the dash means other numbers may exist between the two entries with pattern* 21 *but the two entries with pattern* 23 *must be consecutive.*

We write $F_n(q)$ to denote the generating function for $S_n(132)$ with respect to rsg*, so that*

$$F_n(q) = \sum_{\pi \in S_n(132)} q^{\mathrm{rsg}(\pi)}.$$

For example, if we let | denote the end of every increasing run in a permutation, then we have

$$\mathrm{rsg}(145|27|6|3) = 0+1+1+0+0+0+0 = 2.$$

Equivalently, we have two $2-13$ patterns in 1452763, namely $4-27$ and $5-27$. If we perform a similar computation for each $\pi \in S_3(132)$ and each $\pi \in S_4(132)$, we find that the generating functions for $S_3(132)$ and $S_4(132)$ with respect to rsg are $F_3(q) = 4+q$ and $F_4(q) = 8+4q+2q^2$, respectively. As an instance of our main result, note that setting $q=-1$ in $F_3(q) = 4+q$ returns 3, the number of involutions in $S_3(132)$. (These involutions are 123, 213 and 321.) In the next section, we will prove the $q=-1$ phenomenon algebraically with a recurrence relation on the generating function of the statistic rsg and a previous result on the number of pattern-avoiding involutions first proved by Simion and Schmidt [3].

3 The Main Result

We can now state our main result.

Theorem 2. *The number of involutions in $S_n(132)$ is $F_n(-1)$.*

To prove our main theorem, we need to first introduce the following definition and lemmas. It is well-known in the pattern-avoidance literature that the permutations in $S(132)$ can be constructed recursively.

Definition 3. *For any permutations α and β, $\alpha \circledast \beta$ is the permutation of length $|\alpha|+|\beta|+1$ whose ith entry is given by*

$$(\alpha \circledast \beta)(i) = \begin{cases} |\beta| + \alpha(i) & \text{if } 1 \le i \le |\alpha|; \\ |\beta| + |\alpha| + 1 & \text{if } i = |\alpha|+1; \\ \beta(i - |\alpha| - 1) & \text{if } |\alpha|+2 \le i \le |\alpha|+|\beta|+1. \end{cases}.$$

The conventional notation in the permutation patterns literature for $\alpha \circledast \beta$ is $(\alpha \oplus 1) \ominus \beta$.

To visualize $\alpha \circledast \beta$, note if we graph the function $\alpha \circledast \beta$, then we obtain the graph in Figure 1, which contains shifted copies of the graphs of α and β.

Lemma 4. *If $\alpha, \beta \in S(132)$, then $\alpha \circledast \beta \in S(132)$.*

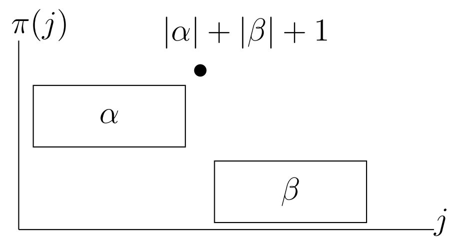

Figure 1: The permutation $\alpha \circledast \beta \in S(132)$.

Proof. It follows from the definition of $\alpha \circledast \beta$ that any entry in α is strictly larger than any entry in β. Thus, if $\alpha \circledast \beta$ contains a subsequence of type 132, then that subsequence is entirely contained in α or it is entirely contained in β. Since $\alpha, \beta \in S(132)$, the result follows. $\square$

Lemma 5. *For any non-empty permutation $\pi \in S(132)$, there exist unique $\alpha, \beta \in S(132)$ such that $\pi = \alpha \circledast \beta$.*

Proof. For any non-empty permutation $\pi \in S(132)$ of length $|\alpha|+|\beta|+1$, we can always pick its largest entry, namely $|\alpha|+|\beta|+1$, and then call the permutation to its left α and the one to its right β. Observe that any entry in α is strictly larger than any entry in β. (If not, then a given entry in α, the entry $|\alpha|+|\beta|+1$, and a given entry in β form a 132 subsequence, which is a contradiction.) It follows from $\pi \in S(132)$ that $\alpha \in S(132)$ and $\beta \in S(132)$. Thus we can always decompose π uniquely into $\alpha \circledast \beta$. $\square$

Lemma 6. *For any non-empty permutation π, let $\mathrm{last}(\pi)$ denote the last entry of π. If π is the empty permutation, then we set $\mathrm{last}(\pi) = 0$. The statistics rsg and last satisfy the following relations:*

$$\mathrm{rsg}(\alpha \circledast \beta) = \mathrm{rsg}(\alpha) + \mathrm{rsg}(\beta) + |\alpha| - \mathrm{last}(\alpha); \tag{1}$$

if β is not empty,

$$\mathrm{last}(\alpha \circledast \beta) = \mathrm{last}(\beta); \tag{2}$$

if β is the empty permutation,

$$\mathrm{last}(\alpha \circledast \beta) = |\alpha| + 1. \tag{3}$$

Proof. Observe that the statistic $\mathrm{rsg}(\alpha \circledast \beta)$ can be decomposed into three parts: the runs in $\alpha(\mathrm{rsg}(\alpha))$, the runs in $\beta(\mathrm{rsg}(\beta))$, and the additional runs after inserting $1+|\alpha|+|\beta|$ between α and β ($|\alpha| - \mathrm{last}(\alpha)$). The largest entry in $\alpha \circledast \beta$ extends the last run in α and it contributes $|\alpha| - \mathrm{last}(\alpha)$ runs towards the statistic rsg, because for every entry between $\mathrm{last}(\alpha)$ and $1+|\alpha|+|\beta|$, it is a legitimate run for $\mathrm{rsg}(\alpha \circledast \beta)$. $\square$

Before proving Theorem 2, we need the following fact about the nth central binomial coefficient $\binom{n}{[n/2]}$, which is the number of length n 132-avoiding involutions (see [3, Thm. 5.6]).

Lemma 7. *The generating function for the central binomial coefficients is*

$$\sum_{n\geq 0}\binom{n}{[n/2]}x^n=\frac{1-2x-\sqrt{1-4x^2}}{4x^2-2x}.$$

Proof. Let $n=2m$ if n is even, and $n=2m+1$ if n is odd. Then we have

$$\sum_{n\geq 0}\binom{n}{[n/2]}x^n=\sum_{m=0}^{\infty}\binom{2m}{m}(x^2)^m+x\sum_{m=0}^{\infty}\binom{2m+1}{m}(x^2)^m.$$

As a special case of the generalized binomial theorem,

$$\frac{1}{\sqrt{1-4x}}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n.$$

Since $\binom{2n+1}{n}=\binom{2n+1}{n+1}$ and the sum of these two binomial coefficients is $\binom{2n+2}{n+1}$, we have

$$\sum_{n=0}^{\infty}\binom{2n+1}{n}x^n=\frac{\frac{1}{\sqrt{1-4x}}-1}{2x}.$$

Thus, our result follows by combining the two generating functions as shown above and replacing x with x^2. □

Now we are ready to prove Theorem 2.

Proof of Theorem 2. We define another generating function for $S_n(132)$ with respect to the statistic rsg:

$$G(q,t,s)=\sum_{\pi\in S(132)}q^{\mathrm{rsg}(\pi)}(-t)^{\mathrm{last}(\pi)}s^{|\pi|};\tag{4}$$

hence,

$$G(-1,-1,s)=\sum_{n\geq 0}F_n(-1)s^n.$$

Therefore, to prove the Theorem, we only need to show that

$$G(-1,-1,s)=\sum_{n\geq 0}\binom{n}{[n/2]}s^n.$$

We divide the permutations in $S(132)$ into three classes and consider the members of each class: π is the empty permutation; $\pi=\alpha\circledast\beta$ is not empty and β is not the empty permutation; π is not empty and β is the empty permutation. Thus,

$$G(q,t,s)=1+\sum_{\substack{\pi\in S(132)\\ \mathrm{last}(\pi)\neq|\pi|}}q^{\mathrm{rsg}(\pi)}(-t)^{\mathrm{last}(\pi)}s^{|\pi|}+\sum_{\substack{\pi\in S(132)\\ \mathrm{last}(\pi)=|\pi|}}q^{\mathrm{rsg}(\pi)}(-t)^{\mathrm{last}(\pi)}s^{|\pi|}.\tag{5}$$

It follows from the recurrence relations (1), (2), (3) in Lemma 6 that

$$G(q,t,s) = 1 + sG(q,-q^{-1},qs)[G(q,t,s)-1] - tsG(q,-q^{-1},-qts). \tag{6}$$

If we let $q=-1$ and $t=-1$ in (6), then we find

$$G(-1,-1,s) = 1 + sG(-1,1,-s)[G(-1,-1,s)-1] + sG(-1,1,-s), \tag{7}$$

from which it follows that

$$G(-1,-1,s)(1-sG(-1,1,-s)) = 1. \tag{8}$$

Similarly, if we let $q=-1$ and $t=1$ in (6), then we find

$$G(-1,1,s) = 1 + sG(-1,1,-s)[G(-1,1,s)-1] - sG(-1,1,s). \tag{9}$$

Now replace s with $-s$ in (9), to obtain

$$G(-1,1,-s) = 1 - sG(-1,1,s)[G(-1,1,-s)-1] + sG(-1,1,-s). \tag{10}$$

Notice equations (9) and (10) form a system of two equations in the unknowns $G(-1,1,-s)$ and $G(-1,1,s)$; when we solve this system we find

$$G(-1,1,-s) = \frac{2s+1 \pm \sqrt{1-4s^2}}{2s}. \tag{11}$$

Since $\lim_{s\to 0^-} \frac{2s+1+\sqrt{1-4s^2}}{2s} = -\infty$,

$$G(-1,1,-s) = \frac{2s+1-\sqrt{1-4s^2}}{2s}. \tag{12}$$

When we substitute (12) into (8), we have

$$G(-1,-1,s) = \frac{1-2s-\sqrt{1-4s^2}}{4s^2-2s}. \tag{13}$$

Now the result follows from Lemma 7. □

4 Conclusions and Future work

We have proved algebraically that setting $q=-1$ in the generating function for 132-avoiding permutations with respect to the statistic rsg returns the number of 132-avoiding involutions. We can use techniques similar to those in the proof of Theorem 2 to find an expression for $G(q^{-1},-q,q^{-1}s)$; we leave this as an exercise for the diligent reader. In addition, we are still looking for a combinatorial proof of our results, in which other Catalan objects may serve as intermediate steps (bracketing sequences, binary trees, non-crossing partitions, to name a few). One can also ask in general whether a $q=-1$ phenomenon exists in permutations that avoid multiple subsequences or a single longer subsequence.

References

[1] D. Bressoud, *"Proofs and Confirmations: The Story of the Alternating-Sign Matrix Conjecture"*, Cambridge University Press, 1999.

[2] V. Reiner, D. Stanton and D. White, *The cyclic sieving phenomenon*, J. Combin. Theory Ser. A **108** (2004), no. 1, 17–50.

[3] R. Simion and F.W. Schmidt, *Restricted permutations*, European J. Combin. **6** (1985), no. 4, 383–406.

[4] R. Simion and D. Stanton, *Octabasic Laguerre polynomials and permutation statistics*, J. Comput. Appl. Math. **68** (1996), no. 1-2, 297–329.

[5] J. Stembridge, *Some hidden relations involving the ten symmetry classes of plane partitions*, J. Combin. Theory Ser. A **68** (1994), no. 2, 372–409.