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Abstract. We present a new proof of the Poisson integral formula for harmonic functions using the methods of representation theory. In doing so, we exhibit the irreducible subspaces and unitary structure of a representation of the group $SU(1,1)$ of 2×2 complex generalized special unitary matrices. Our arguments illustrate a technique that can be used to prove similar reproducing formulas in higher dimensions and for other classes of functions. Our paper should be accessible to readers with minimal knowledge of complex analysis.

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1 Introduction

Recall that a function f of two variables x and y is *harmonic* if it is twice continuously differentiable and satisfies $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Such functions are exactly the solutions to *Laplace's equation*, $\Delta f = 0$. This equation has numerous physical applications. For various interpretations of the function f, it can represent Fick's Law of diffusion, Fourier's law of heat conduction, or Ohm's law of electrical conduction. Moreover, harmonic functions play a role in probabilistic models of Brownian motion [1].

By identifying $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$, a function of a complex variable can be viewed as a function of two real variables and can thus be defined as harmonic in a natural manner. The Poisson integral formula is a fundamental result that enables one to recover all of the values of a harmonic function defined on a disk in the complex plane given only its values on the boundary of the disk:

Theorem 1.1 (The Poisson Integral Formula)**.** *Let f be a complex-valued harmonic function defined on a neighborhood of a closed disk $D(p, R)$ of radius R and center p in the complex plane. Then*

$$f(re^{i\theta} + p) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\phi} + p) \frac{R^2 - r^2}{R^2 + r^2 - 2r\cos(\theta - \phi)} d\phi,$$

where $re^{i\theta}$ is an element of the interior of $D(p, R)$.

By translating and scaling the disk, it is no loss of generality to assume that $R = 1$ and $p = 0$, in which case $D(p, R)$ is the closed unit disk, which we henceforth denote simply by D. In this case, the formula reduces to

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)} d\phi.$$

In this paper, we give a new proof of the Poisson integral formula. Our method makes use of a representation (essentially a group action on a vector space by linear transformations) of the group $SU(1, 1)$ of 2×2 generalized special unitary matrices with complex entries (isomorphic to $SL(2, \mathbb{R})$) over the vector space of harmonic functions on D. The interested reader can find more information on matrix Lie groups like $SU(1, 1)$ and their representations in [7]. We show that as a representation, the space of harmonic functions is generated by the identity function $z \mapsto z$ and its conjugate $z \mapsto \overline{z}$. We then use this fact to reduce the proof of the Poisson integral formula to a few elementary computations, in much the same way that one reduces the study of a linear transformation to the study of its effect on a basis. Along the way, we describe the irreducible subspaces and unitary structure of our representation, properties which the reader may find of independent interest.

Our proof is inspired by a paper by Igor Frenkel and Matvei Libine [2] which uses representation theory to develop analysis over the quaternions. In particular, the authors make use of the theory of the conformal group $SL(2, \mathbb{H})$, the group of 2×2 matrices with quaternion entries and determinant 1. Many of the parallels between complex and quaternionic analysis are made apparent by restating results in complex analysis from the perspective of representations of the complex analogue of $SL(2,\mathbb{H})$, $SL(2,\mathbb{C})$. This lends importance to the question of which results in complex analysis can, in fact, be restated and proven in terms of representations of $SL(2,\mathbb{C})$ and its subgroups, including $SU(1,1)$, for these are the results which can likely be extended to quaternionic analysis.

Although the classical proof of the Poisson integral formula is short and elementary, our proof illustrates a technique which has been successfully used to prove reproducing formulas for other kinds of functions (as is done in [2] and [3]), and which is likely to be used again in the future. For example, methods similar to ours might be used to prove higher dimensional analogues of the Poisson formula in $\mathbb{R}^n$, which are discussed in [11]. The matrix group $SO(n+1,1)$ acts on the vector space of harmonic functions on $\mathbb{R}^n$ and its subgroup $SO(n,1)$ preserves the unit ball, as is explained in [5]. Plausibly, $SO(n,1)$ and its own subgroup $SO(n)$ could play roles similar to those that the groups $SU(1,1)$ and $SO(2)$ play in our paper to engender a proof of the higher dimensional formulas.[1] As another example, in Section 5.4 of [2], the authors conjecture that the Feynman diagrams, which describe the interactions of subatomic particles, correspond to projections onto irreducible components of certain representations of the group $SU(2,2)$. It is quite possible that the technique we illustrate here could be used to prove this conjecture.

We begin with a preliminary section in which we introduce the definitions and concepts we shall use in the remainder of the paper and construct our representation of $SU(1,1)$. This is followed by Section 3, which contains the proof that certain subrepresentations of our representation are in fact irreducible. The only fact from this section that is needed for the proof of the Poisson integral formula is that z and $\overline{z}$ generate the entire vector space, but we give a more detailed exposition of the invariant subspace structure of our representations which the reader might find of independent interest. Section 4 consists of the elementary computations needed to finish the proof of the Poisson integral formula. To complete the description of our representations, we conclude with Section 5, in which we define an $SU(1,1)$-invariant inner product on a modified version of our vector space.

2 Preliminaries

An action of a group G on a set S is a function from $G \times S \to S$, denoted by $(g,x) \mapsto gx$, such that for all $x \in S$ and all $g,h \in G$, $(gh)x = g(hx)$ and $1x = x$, where 1 is the identity in G.

Definition 2.1. A *representation* of a group G over a vector space V is a group homomorphism $\rho : G \to GL(V)$, where $GL(V)$ is the group of invertible linear transformations from V to V.

[1] To complete the analogy, we note that, as real Lie groups, $SL(2,\mathbb{C})/\{\pm 1\}$ is isomorphic to the connected component of the identity of $SO(3,1)$ and $SU(1,1)/\{\pm 1\}$ is isomorphic to the connected component of the identity of $SO(2,1)$.

In effect, a representation is an action of a group on a vector space by linear transformations. Oftentimes, when there is no danger of ambiguity, one refers to the vector space itself, rather than the function ρ, as a representation. A group G that acts on a set U possesses a natural representation over a vector space of functions defined on U given by composition on the right: for $g \in G$, $\rho(g) : f \mapsto f \circ g^{-1}$. The inverse of g is needed so that the representation preserves group multiplication. Representations arise frequently in this context, and it is this sort of representation that we study here. Let us first define our group.

Definition 2.2. The group $SU(1,1)$ is the set of matrices

$$SU(1,1) = \left\{ \gamma = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : a, b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \right\},$$

with group multiplication given by matrix multiplication.

The group $SU(1,1)$ is isomorphic to the group $SL(2,\mathbb{R})$ of 2×2 real matrices with determinant 1.

We let $\mathbb{C}P^1$ denote complex projective space, the set of pairs of complex numbers which are not both equal to zero modulo the equivalence relation of being scalar multiples of one another: $(z, w) \sim (z', w')$ provided $z/w = z'/w'$ or $w = w' = 0$. The group $SL(2,\mathbb{C})$ of 2×2 invertible matrices with complex entries and determinant 1, and hence also its subgroup $SU(1,1)$, acts on $\mathbb{C}P^1$ by matrix-vector multiplication. If we associate with each $z \in \mathbb{C}$ the equivalence class of the tuple $(z, 1) \in \mathbb{C}P^1$ and with ∞ the equivalence class of the tuple $(1, 0) \in \mathbb{C}P^1$, we may think of this as an action of $SL(2,\mathbb{C})$ on the extended complex plane $\mathbb{C} \cup \{\infty\}$. Under this action, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ sends $z \in \mathbb{C} \cup \{\infty\}$ to $\frac{az+b}{cz+d}$.

A function from $\mathbb{C} \cup \{\infty\}$ to itself of the form $z \mapsto \frac{az+b}{cz+d}$ is called a *Möbius transformation.* Thus, we see that each element of $SL(2,\mathbb{C})$ defines a Möbius transformation. We shall henceforth denote the Möbius transformation associated with $\gamma \in SL(2,\mathbb{C})$ by $\tilde{\gamma}$.

Of particular interest for our purposes are those matrices whose Möbius transformations preserve the closed unit disk D. It is easily checked that for each matrix $\gamma \in SU(1,1)$, the Möbius transformation

$$\tilde{\gamma}(z) = \frac{az + b}{cz + d}$$

associated with γ maps each element $z \in D$ to another element of D and does so in a bijective manner[2]. Via these Möbius transformations, the group $SU(1,1)$ acts on D.

We identify the circle group $SO(2)$ with the subgroup of $SU(1,1)$ given by

$$SO(2) = \left\{ k_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi) \right\}.$$

[2] In fact, these Möbius transformations are the *only* complex diffeomorphisms of D [6], but we will not need this fact for our paper.

The group $SO(2)$ is thereby associated with Möbius transformations that merely rotate D, i.e. those of the form $\tilde{k_\theta}(z) = e^{2i\theta}z$.

We next define our vector space.

Definition 2.3. We denote by $\mathcal{V}$ the vector space of complex-valued functions which are continuous on D and harmonic on the interior of D,

$$\mathcal{V} = \left\{ f : D \to \mathbb{C} :\ f \textit{ continuous},\ \frac{\partial^2 f}{\partial x^2}(z) + \frac{\partial^2 f}{\partial y^2}(z) = 0\, \forall z \in \mathrm{int}(D) \right\}.$$

A complex valued function is harmonic if and only if both its real and imaginary parts are harmonic. Recall that for U an open subset of $\mathbb{C}$, a complex valued function $f = u + iv : U \to \mathbb{C}$ is called *holomorphic* if f is complex differentiable. A complex-valued function $f = u + iv : U \to \mathbb{C}$ is *anti-holomorphic* if its complex conjugate $\overline{f} = u - iv$ is holomorphic.

The partial derivatives of a holomorphic function satisfy the Cauchy-Riemann Equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Given these equations, it follows from a simple computation of derivatives that every holomorphic function and every anti-holomorphic function is harmonic. Thus, every function which is either holomorphic or anti-holomorphic on the interior of D and continuous on its boundary is an element of $\mathcal{V}$.

Definition 2.4. We denote by $\mathcal{V}_h$ the subspace of $\mathcal{V}$ consisting of functions which are holomorphic on $\mathrm{int}(D)$, by $\mathcal{V}_{ah}$ the subspace of $\mathcal{V}$ consisting of functions which are anti-homomorphic on $\mathrm{int}(D)$, and by $\mathcal{V}_c$ the subspace of $\mathcal{V}$ consisting of constant functions.

Proposition 2.5. $\mathcal{V} = \mathcal{V}_h + \mathcal{V}_{ah}$.

Proof. Let $f = u + iv : D \to \mathbb{C}$ be in $\mathcal{V}$. We seek to show that f can be expressed as the sum of a holomorphic function and an anti-holomorphic function in $\mathcal{V}$. The real and imaginary parts of f, u and v, are harmonic on $\mathrm{int}(D)$ and continuous on ∂D. So, there exist harmonic conjugates $\tilde{u}$ and $\tilde{v}$ for u and v, respectively, such that the functions $g = u + i\tilde{u}$ and $h = v + i\tilde{v}$ are holomorphic on the interior of D and continuous on its boundary [6]. Then $u = \frac{1}{2}(g + \overline{g})$ and $v = \frac{1}{2}(h + \overline{h})$, so $f = \frac{1}{2}(g + \overline{g}) + \frac{i}{2}(h + \overline{h})$ expresses f as the sum of holomorphic and antiholomorphic functions $f_1 = \frac{1}{2}(g + ih)$ and $f_2 = \frac{1}{2}(\overline{g} + i\overline{h})$. □

This is almost, but not quite a direct sum: harmonic functions cannot be expressed as a sum of holomorphic and antiholomorphic functions in a unique way, since constant functions are both holomorphic and anti-holomorphic: $\mathcal{V}_h \cap \mathcal{V}_{ah} = \mathcal{V}_c$.

It can be shown that every holomorphic function is analytic in the sense of being equal to a convergent power series in any open disk contained in its domain. Typically this fact is proven using the Cauchy integral formula [6], but it can also be proven using elliptic operator theory [4]. As is always the case for analytic functions, the series representation centered at any given point is unique. From the fact that any holomorphic function can be expressed as a convergent series in powers of $z - p$ on any open disc with center p contained in its domain, it follows immediately that any anti-holomorphic function can be expressed as a convergent series in powers of $\overline{z-p}$ on any open disk with center p contained in its domain.

Since the interior of D is an open disk centered at the origin, Proposition 2.5 implies that each function in $f \in \mathcal{V}$ can be expressed as $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n$ on $\mathrm{int}(D)$. We thus have the following alternative characterization of $\mathcal{V}$:

$$\mathcal{V} = \left\{ f : D \to \mathbb{C} : f \textit{ continuous},\ f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n \textit{ on } \mathrm{int}(D) \right\}.$$

When there is no danger of ambiguity, we sometimes write $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n$ for functions $f \in \mathcal{V}$, keeping in mind that this series representation is only valid on the interior of D.

Define a norm on $\mathcal{V}$ by

$$\|f\| = \max\{|f(z)| : z \in D\}.$$

Like any norm, $\|.\|$ induces a topology on $\mathcal{V}$. Henceforth when we speak of subspaces of $\mathcal{V}$ being closed or open, we mean with respect to this topology.

Recall that a sequence of functions $\{f_n\}$ on a common domain X converges uniformly to a function f on a set $S \subset X$ if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$ for all $x \in S$. This is contrasted with pointwise convergence, under which N may vary for different choices of x. If the domain X is open, one often replaces uniform convergence on X with the requirement that $\{f_n\}$ converge uniformly on any compact subset K of X, as we do in Section 4. It is easily seen that convergence with respect to the maximum norm we defined above is equivalent to uniform convergence on D. In introductory analysis texts, it is proven that the integrals of a uniformly convergent sequence of functions converge to the integral of the limit function [9].

A metric space M is said to be *complete* if every Cauchy sequence in M converges to an element of M, and a normed linear space is said to be a *Banach space* if it is complete with respect to its norm. Once the Poisson formula is established, it is easy to show that $\mathcal{V}$ is complete, and is hence a Banach space. This fact is, of course, not needed in our proof.

We are now ready to define our representation.

Definition 2.6. Define a map $\rho : SU(1,1) \to GL(\mathcal{V})$ by $\rho(\gamma)f = (f \circ \tilde{\gamma}^{-1})$, where $\tilde{\gamma}$ is the Möbius transformation induced by γ by its action on D.

Before we prove that this is indeed a representation of $SU(1,1)$, we note that for a representation over an infinite-dimensional vector space, most authors require that the representation function ρ be continuous. There are a variety of notions of continuity, discussed, for example, in [10]. It can be shown that our representation satisfies this requirement under most standard definitions.

Proposition 2.7. *ρ is a representation of $SU(1,1)$.*

Proof. In light of the discussion after Definition 2.1, we need only show that for all $f \in \mathcal{V}$, for all $\gamma \in SU(1,1)$, $\rho(\gamma)f \in \mathcal{V}$. The Möbius transformation $\tilde{\gamma}^{-1}$ associated with γ^{-1} is holomorphic on D, and the composition of holomorphic functions is holomorphic. Since the composition of continuous functions is continuous, if $f_1 \in \mathcal{V}_h$, then $\rho(\gamma)f_1 \in \mathcal{V}_h$. If $c : z \mapsto \overline{z}$ denotes the conjugate function, then $f_2 \in \mathcal{V}_{ah}$ if and only if $(c \circ f_2)$ is holomorhpic on $\mathrm{int}(D)$, in which case $\rho(\gamma)(c \circ f_2) = (c \circ f_2 \circ \tilde{\gamma}^{-1})$ is holomorphic on $\mathrm{int}(D)$. So, $(f_2 \circ \tilde{\gamma}^{-1})$ is anti-holomorphic on $\mathrm{int}(D)$ and $\rho(\gamma)f_2 \in \mathcal{V}_{ah}$. A function is in $\mathcal{V}$ if and only if it is the sum of a function in $\mathcal{V}_h$ and a function in $\mathcal{V}_{ah}$. Therefore $\rho(\gamma)f \in \mathcal{V}$ for all $f \in \mathcal{V}$. □

This representation of $SU(1,1)$ induces a representation of the subgroup $SO(2)$ of $SU(1,1)$ in the obvious way: by restricting ρ to $SO(2)$.

The question of which subspaces of a vector space are preserved under the action of a group is of great importance in representation theory. We devote the remainder of this section and most of the next to exploring which subspaces of $\mathcal{V}$ are preserved under the action of $SU(1,1)$.

Definition 2.8. Let ρ be a representation of a group G over a vector space V. We say that a subspace W of V is *G-invariant* if $\rho(g)w \in W$ for all $g \in G$ and all $w \in W$. A G-invariant subspace W of V is a *subrepresentation* if W is closed, and is a *proper subrepresentation* if W is neither the zero subspace nor all of V. A subrepresentation W of V is *irreducible* if W itself has no proper subrepresentations.

Note here the requirement that a subrepresentation W be closed, in the topological sense. This is important in the context of infinite dimensional representations in that it implies that W must contain the limit of any convergent series of its elements, as well as any finite linear combination of them. For example, although the set of rational functions with no singularities on D is an $SU(1,1)$-invariant subspace of $\mathcal{V}$, it is not closed because there exist sequences of rational functions (even polynomials) which converge uniformly to elements of $\mathcal{V}$ which are not themselves rational functions, such as the exponential function e^z. So, this subspace is not a subrepresentation.

Proposition 2.9. *$\mathcal{V}_c$, $\mathcal{V}_h$, and $\mathcal{V}_{ah}$ are $SU(1,1)$ subrepresentations of $\mathcal{V}$.*

Proof. We have established that these subspaces are $SU(1,1)$-invariant in the proof of Proposition 2.7. It remains to show that they are closed. Let $\{f_n\}$ be a sequence in $\mathcal{V}_h$ which converges uniformly on D to $f \in \mathcal{V}$. For each n, express f_n as a power series on $\mathrm{int}(D)$ as $f_n(z) = \sum_{k=0}^{\infty} a_{nk} z^k$. Write $f(z) = \sum_{k=0}^{\infty} c_k z^k + \sum_{k=1}^{\infty} d_k \overline{z}^k$ for $z \in \mathrm{int}(D)$. To show that $f \in \mathcal{V}_h$, we must show that $d_m = 0$ for all $m = 1, 2, 3, \ldots$.

Fix $r \in (0,1)$. Since the series for f is valid on $\mathrm{int}(D)$ and $r < 1$, $f_r(z) = f(rz)$ is equal to a power series on all of D, including its boundary. Because power series converge uniformly on compact subsets of their domain [9], they can be integrated term by term. For each positive integer m, we thus have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f_r(e^{i\theta})e^{im\theta}d\theta &= \frac{1}{2\pi}\sum_{k=0}^{\infty} c_k \int_0^{2\pi} e^{im\theta}(re^{i\theta})^k d\theta + \frac{1}{2\pi}\sum_{k=1}^{\infty} d_k \int_0^{2\pi} e^{im\theta}\overline{(re^{i\theta})}^k d\theta \\ &= \frac{1}{2\pi}\sum_{k=0}^{\infty} r^k c_k \int_0^{2\pi} e^{i(m+k)\theta}d\theta + \frac{1}{2\pi}\sum_{k=1}^{\infty} r^k d_k \int_0^{2\pi} e^{i(m-k)\theta}d\theta.\end{aligned}$$

We evaluate the terms of this sum individually. For each non-negative integer k,

$$\int_0^{2\pi} e^{i(m+k)\theta}d\theta = -i(m+k)^{-1}e^{i(m+k)\theta}|_0^{2\pi} = 0,$$

since the period of $e^{i(m+k)\theta}$ is an integer fraction of 2π. Similarly, for each $k \neq m$, $\int_0^{2\pi} e^{i(m-k)\theta}d\theta = 0$. Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} f_r(e^{i\theta})e^{im\theta}d\theta = \frac{r^m d_m}{2\pi}\int_0^{2\pi} e^{i(m-m)\theta}d\theta = r^m d_m.$$

Because $\{f_n(rz)\} \longrightarrow f(rz)$ uniformly as $n \longrightarrow \infty$ and integrals commute with uniform limits, we can integrate the series for $f_n(rz)$ on ∂D term by term to obtain

$$r^m d_m = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{im\theta}d\theta = \lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f_n(re^{i\theta})e^{im\theta}d\theta = 0,$$

by the same computations as above and the fact that the series for the holomorphic functions f_n contain no $\overline{z}^m$ term. Therefore $d_m = 0$ for every m, so $f \in \mathcal{V}_h$ and $\mathcal{V}_h$ is closed. By an identical argument, $\mathcal{V}_{ah}$ is closed. As a consequence, $\mathcal{V}_c = \mathcal{V}_h \cap \mathcal{V}_{ah}$, as the intersection of closed sets, is also closed. Thus, these three $SU(1,1)$-invariant subspaces are indeed subrepresentations. □

We end this section with a lemma about $\mathcal{V}$ which we shall need in Section 3.

Lemma 2.10. *Let $f \in \mathcal{V}$ be expressed as a convergent series on* $\mathrm{int}(D)$ *as $f(z) = \sum_{k=0}^{\infty} a_n z^k + \sum_{m=1}^{\infty} b_m \overline{z}^m$. Then there exists a sequence of polynomials in z^k and $\overline{z}^m$ (finite linear combinations of powers of z and $\overline{z}$) which converges uniformly to f. We may choose this sequence so that for each k with $a_k = 0$ and each m with $b_m = 0$, the coefficients on z^k and $\overline{z}^m$ for the polynomials in the sequence are zero.*

Proof. For each $r \in (0,1)$, define $f_r(z) = f(rz)$. Since f is uniformly continuous on D, f_r converges uniformly to f on D as $r \longrightarrow 1$ from the left. As in the proof of Proposition 2.9, the series representation of f is valid on the interior of D, so since each r is less than 1, f_r is equal to a series on all of D. Since power series converge uniformly on compact subsets of their domain,

$$f_r(z) = \lim_{n\to\infty}\sum_{k=0}^{n} r^k a_k z^k + \sum_{m=1}^{n} r^m b_m \overline{z}^m \equiv \lim_{n\to\infty} g_{r,n}(z),$$

and this limit is uniform on all of D. Each $g_{r,n}$ is a polynomial in z^k and $\overline{z}^m$, and for each k or m with a_k or b_m equal to zero, the coefficients on z^k and $\overline{z}^m$ in the formula for $g_{r,n}$ are zero. Moreover, for a suitable choice of $\{r_j\} \longrightarrow 1^-$ and $\{n_j\} \longrightarrow \infty$, a diagonal subsequence $\{g_{r_j,n_j}\}$ converges uniformly to f, as desired. □

In particular, this lemma, along with Proposition 2.9, implies that $\mathcal{V}_h = \overline{\text{span}\{1, z, z^2, ...\}}$ and $\mathcal{V}_{ah} = \overline{\text{span}\{1, \overline{z}, \overline{z}^2, ...\}}$, where the bar denotes the closure.

3 Invariant Subspaces

Our aim in this section is to show that $\mathcal{V}_h$, $\mathcal{V}_{ah}$, and $\mathcal{V}_c$ are in fact the *only* proper $SU(1,1)$ subrepresentations of $\mathcal{V}$. In particular, this will imply that the identity function and its conjugate *generate* $\mathcal{V}$ as a representation of $SU(1,1)$, in the sense that the smallest subrepresentation of $\mathcal{V}$ which contains these two functions is is all of $\mathcal{V}$. This fact will play a key role in our proof of the Poisson integral formula in Section 4. We shall first consider subspaces of $\mathcal{V}$ which are invariant under the action of the subgroup $SO(2)$ of $SU(1,1)$.

Lemma 3.1. *Let $\alpha \in \mathbb{R}$ and define an operator $A_\alpha : \mathcal{V} \to \mathcal{V}$ by*

$$A_\alpha(f)(z) = \frac{1}{\pi}\int_0^\pi e^{i\alpha\theta}\rho(k_\theta)f(z)d\theta = \frac{1}{\pi}\int_0^\pi e^{i\alpha\theta}f(e^{-2i\theta}z)d\theta,$$

where each $k_\theta \in SO(2)$. If W is a closed $SO(2)$-invariant subspace of $\mathcal{V}$ and $f \in W$, then $A_\alpha(f) \in W$.

Proof. It is clear that $A_\alpha(f)$ is continuous on D, and it follows from differentiation under the integral sign that $A_\alpha(f)$ is harmonic on $\text{int}(D)$, so $A_\alpha(f)$ is an element of $\mathcal{V}$. It remains to show that $A_\alpha(f) \in W$. The integral expression for $A_\alpha(f)$ is given by a limit of Riemann sums: for each $z \in D$,

$$A_\alpha(f)(z) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n e^{i\alpha\theta_j}f(e^{-2i\theta_j}z),$$

where for each n, $0 \leq \theta_1 < ... < \theta_n \leq \pi$ are test points in a partition of the interval $[0,\pi]$ which becomes arbitrarily fine as $n \longrightarrow \infty$. Since W is $SO(2)$-invariant, the function

$$f_n(z) \equiv \frac{1}{n}\sum_{j=1}^n e^{i\alpha\theta_j}f(e^{-2i\theta_j}z)$$

is in W for each n. By definition, $\{f_n\} \longrightarrow A_\alpha(f)$ pointwise. We claim that this convergence is uniform.

By the Arzelá-Ascoli Theorem [8], it suffices to show that $\{f_n\}$ is *equicontinuous*[3]. That is, given $\epsilon > 0$, there is a single $\delta > 0$ such that for all n, for all $z, w \in D$ with $|z - w| < \delta$,

[3] The Arzelá-Ascoli Theorem states that any bounded sequence of equicontinuous functions on a compact set has a uniformly convergent subsequence. It is an immediate consequence (and indeed is usually proven in the course of proving the theorem itself) that an equicontinuous sequence which converges poitwise on a compact set also converges uniformly.

we have $|f_n(z) - f_n(w)| < \epsilon$. The function f is uniformly continuous, so we can choose $\delta > 0$ such that $|f(z) - f(w)| < \epsilon$ for all $z, w \in D$ with $|z - w| < \delta$. Since $|e^{-2i\theta_j}| = 1$ for all j, $|z - w| < \delta$ implies that $|e^{-2i\theta_j}z - e^{-2i\theta_j}w| < \delta$. For all n we thus have

$$\begin{aligned} |f_n(z) - f_n(w)| &= \left| \frac{1}{n}\sum_{j=1}^{n} e^{i\alpha\theta_j} f(e^{-2i\theta_j}z) - \frac{1}{n}\sum_{j=1}^{n} e^{i\alpha\theta_j} f(e^{-2i\theta_j}w) \right| \\ &\leq \frac{1}{n}\sum_{j=1}^{n} |f(e^{-2i\theta_j}z) - f(e^{-2i\theta_j}w)| < \frac{n\epsilon}{n} = \epsilon. \end{aligned}$$

So, δ satisfies the ϵ requirement for equicontinuity of $\{f_n\}$. Thus, $\{f_n\} \longrightarrow A_\alpha(f)$ uniformly, so since W is closed, $A_\alpha(f) \in W$. □

Lemma 3.2. *Let W be a closed $SO(2)$-invariant subspace of $\mathcal{V}$ and let $f \in W$, $f(z) = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \overline{z}^k$ on $\operatorname{int}(D)$. For each n, if $a_n \neq 0$ then $z^n \in W$, and if $b_n \neq 0$ then $\overline{z}^n \in W$.*

Proof. We consider only z^n. The case for its conjugate is identical. By the Lemma 3.1, the function $A_{2n}(f)$ is in W. We claim that this function is a constant multiple of z^n. Indeed, let $z \in \operatorname{int}(D)$. Then since power series can be integrated term by term,

$$\begin{aligned} A_{2n}(f)(z) &= \frac{1}{\pi}\sum_{k=0}^{\infty}\int_0^{\pi} a_k e^{2ni\theta}(e^{-2i\theta}z)^k d\theta + \frac{1}{\pi}\sum_{k=1}^{\infty}\int_0^{\pi} b_k e^{2ni\theta}\overline{(e^{-2i\theta}z)}^k d\theta \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty} a_k \int_0^{\pi} e^{2(n-k)i\theta} d\theta z^k + \frac{1}{\pi}\sum_{k=1}^{\infty} b_k \int_0^{\pi} e^{2(n+k)i} d\theta \overline{z}^k. \end{aligned}$$

As in the proof of Proposition 2.9, for each non-negative integer $k \neq n$,

$$\int_0^{\pi} e^{2(n-k)i\theta} d\theta = (2(n-k)i)^{-1} e^{2(n-k)i\theta}|_0^{\pi} = 0,$$

and if $k > 0$, $\int_0^{\pi} e^{2(n+k)i\theta} d\theta = 0$. Therefore

$$A_{2n}(f)(z) = \frac{a_n}{\pi}\int_0^{\pi} e^{2(n-n)i\theta} d\theta z^n = a_n z^n.$$

By continuity, $A_{2n}(f)(z) = a_n z^n$ on ∂D as well. So, since $a_n \neq 0$, W contains z^n. □

We can use these two lemmas to completely characterize the closed $SO(2)$-invariant subspaces of $\mathcal{V}$.

Proposition 3.3. *Let W be a closed $SO(2)$-invariant subspace of $\mathcal{V}$. Then*

$$W = \overline{\operatorname{span}\{z^{n_1}, \overline{z}^{m_1}, z^{n_2}, \overline{z}^{m_2} ...\}},$$

where the n_j are the non-negative integers such that $z^{n_j} \in W$, the m_j are the positive integers such that $\overline{z}^{m_j} \in W$, and the bar denotes the closure.

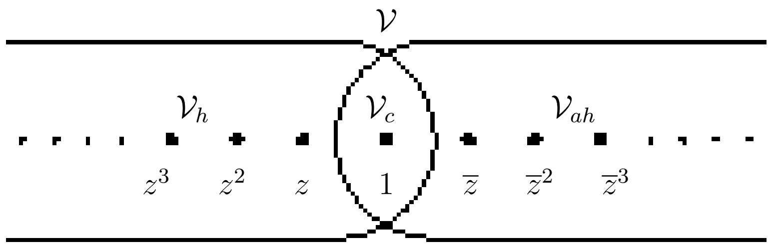

Figure 1: Subrepresentations of $\mathcal{V}$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{m=1}^{\infty} b_m \overline{z}^m \in W$ be expressed as a power series valid on $\mathrm{int}(D)$. By Lemma 3.2, for each n such that $a_n \neq 0$, $z^n \in W$ and for each m such that $b_m \neq 0$, $\overline{z}^m \in W$. By Lemma 2.10, f is equal to the uniform limit of a sequence of polynomials in only those z^n and $\overline{z}^m$ that occur with non-zero coefficients in the power series for f, i.e. only a subset of the z^{n_j} and the $\overline{z}^{m_j}$. Thus, $W = \overline{\mathrm{span}\{z^{n_1}, \overline{z}^{m_1}, z^{n_2}, \overline{z}^{m_2}\}}$, as required. $\square$

Before we prove the climactic theorem of this section, we take a moment to think about our representation visually, and, perhaps, to reflect on why we decided to study mathematics rather than, say, art. The subrepresentation structure of our representation is shown in Figure 1. The subspaces $\mathcal{V}_h$ and $\mathcal{V}_{ah}$ of functions which are holomorphic or anti-holomorphic on the interior of D are subrepresentations, and intersect in a third subrepresentation: the constant functions, $\mathcal{V}_c$. We are about to prove that $\mathcal{V}$ cannot be decomposed any further than shown in the diagram. Note that our claim is only the quotient spaces $\mathcal{V}_h/\mathcal{V}_c$ and $\mathcal{V}_{ah}/\mathcal{V}_c$, not $\mathcal{V}_h$ and $\mathcal{V}_{ah}$ themselves, are irreducible[4]. Indeed, this is the best that could be hoped for, given that the space of constant functions $\mathcal{V}_c$ is an $SU(1,1)$-invariant subspace of both $\mathcal{V}_h$ and $\mathcal{V}_{ah}$.

Theorem 3.4. *The only proper $SU(1,1)$ subrepresentations of $\mathcal{V}$ are $\mathcal{V}_h$, $\mathcal{V}_{ah}$, and $\mathcal{V}_c$. In particular, $\mathcal{V}_h/\mathcal{V}_c$ and $\mathcal{V}_{ah}/\mathcal{V}_c$ are irreducible.*

Proof. Let W be a proper $SU(1,1)$ subrepresentation of $\mathcal{V}$, i.e. a closed $SU(1,1)$-invariant subspace not equal to $\mathcal{V}$ or $\{0\}$. Assume that W does not consist solely of constant functions. Since W is invariant under the action of $SU(1,1)$, W is also invariant under the action of its subgroup $SO(2)$. Because W contains a non-constant function, Lemma 3.2 implies that W must contain either z^n or $\overline{z}^n$ for some $n > 0$.

Suppose first that W contains z^n. Because W is $SU(1,1)$-invariant, for each Möbius transformation $\tilde{\gamma}(z) = \frac{az+b}{\overline{b}z+\overline{a}}$ corresponding to an element of $SU(1,1)$, W contains the function $\tilde{\gamma}(z)^n$. Choose one such $\tilde{\gamma}$ with a and b non-zero. Since $\tilde{\gamma}(z)^n$ is holomorphic on D, it equals its Taylor series about the origin. The linear coefficient of this series is the derivative of $\tilde{\gamma}(z)^n$ evaluated at $z = 0$,

$$n\tilde{\gamma}(0)^{n-1}\tilde{\gamma}'(0) = n\left(\frac{b}{\overline{a}}\right)^{n-1}\left(\frac{1}{\overline{a}^2}\right) = \frac{b^{n-1}}{\overline{a}^{n+1}} \neq 0.$$

[4]Since $\mathcal{V}_c$ is $SU(1,1)$-invariant, the representation ρ gives rise to a well-defined representation of $SU(1,1)$ on the quotient space $\mathcal{V}/\mathcal{V}_c$.

Thus, by Lemma 3.2, the identity function z must be in W. Therefore, again by invariance, W contains the function $\tilde{\gamma}$ itself. Using partial fractions and the properties of geometric series, we find that for all $z \in D$,

$$\tilde{\gamma}(z) = \frac{az+b}{\bar{b}z+\bar{a}} = \frac{b}{\bar{a}} + \frac{z}{\bar{a}^2}\left(\frac{1}{1+(\bar{b}/\bar{a})z}\right) = \frac{b}{\bar{a}} + \frac{1}{\bar{a}^2}\sum_{k=1}^{\infty}(-\bar{b}/\bar{a})^{k-1}z^k.$$

In particular, for each non-negative integer k, the coefficient on z^k in this series is non-zero. Thus, again by Lemma 3.2, W contains z^k for every non-negative integer k. Since W is closed, the comment following Lemma 2.10 implies that W must contain all of $\mathcal{V}_h$. This same argument works for a subspace of $\mathcal{V}_h$ that contains a non-constant function. Hence, $\mathcal{V}_h/\mathcal{V}_c$ is irreducible.

By an identical argument, if $\bar{z}^n \in W$ for some n, then W contains $\bar{z}$, and hence $\bar{z}^k$ for all k. Therefore $\mathcal{V}_{ah} \subset W$ and $\mathcal{V}_{ah}/\mathcal{V}$ is irreducible. Likewise, if W contains both z^n and $\bar{z}^m$ for some m and n, then W contains both $\mathcal{V}_h$ and $\mathcal{V}_{ah}$, and since $\mathcal{V} = \mathcal{V}_h + \mathcal{V}_{ah}$, W is all of $\mathcal{V}$, contrary to the assumption that W is proper. We are thus left with only two possibilities: W must be either $\mathcal{V}_h$ or $\mathcal{V}_{ah}$. This completes the proof. $\square$

Let W be a closed $SU(1,1)$-invariant subspace of $\mathcal{V}$ which contains h and $\bar{h}$. Theorem 3.4 implies that W must be all of $\mathcal{V}$. In other words,

Corollary 3.5. *The identity function $h : z \mapsto z$ and its conjugate $\bar{h} : z \mapsto \bar{z}$ generate $\mathcal{V}$ as a representation of $SU(1,1)$.*

In particular, each $f \in \mathcal{V}$ must be the uniform limit of a sequence of finite linear combinations of the images of h and $\bar{h}$ under the action of elements of $SU(1,1)$.

4 The Poisson Integral Formula

Having established the necessary representation theoretic background for our proof of the Poisson integral formula (Theorem 1.1 in the introduction), we are almost ready to prove the formula itself. We start with a definition.

Definition 4.1. Let ρ be a representation of a group G over a vector space V. An operator T on V is *G-equivariant* or *G-invariant* if $\rho(g)T(v) = T(\rho(g)v)$ for all $g \in G$ and all $v \in V$.

The proof we finish in this section is based on the following general result:

Proposition 4.2. *Let ρ be a representation of a group G over a topological vector space V. Suppose that the set $S \subset V$ generates V as a representation of V (in the terminology given at the beginning of Section 3). Then a continuous, linear G-equivariant operator T on V is completely determined by its values on S.*

Proof. Since S generates V, any $v \in V$ can be expressed as $v = \lim_{n\to\infty} v_n$, where each $v_n = \sum_{k=1}^{N_n} a_{nk}\rho(g_{nk})s_{nk}$ for scalars a_{nk} and for some $s_{nk} \in S$ and $g_{nk} \in G$. Since T is continuous and linear, it commutes with limits and sums. Thus, G-invariance gives

$$T(v) = \lim_{n\to\infty} T(v_n) = \lim_{n\to\infty} \sum_{k=1}^{N_n} a_{nk}\rho(g_{nk})T(s_{nk}).$$

In particular, $T(v)$ depends only on the values of T on the s_{nk}. □

The idea of Proposition 4.1 is similar to that of a linear transformation being completely determined by its values on a basis for a vector space, or a group homomorphism being completely determined by the images of a set of generators. In our case, we have already shown that our generating set consists of the identity function h and its conjugate $\overline{h}$ (Corollary 3.5). We must now show that the Poisson integral operator is in fact continuous and equivariant under the action of the group $SU(1,1)$, then evaluate the operator at the identity function and its conjugate. The necessary computations require no more analysis than is typically encountered in introductory courses.

Before we define the Poisson integral operator, we need to define its target space. The formula is only defined for z on the interior of the disk D, so properly speaking it does not output an element of $\mathcal{V}$, which consists of functions defined on the boundary of D as well. However, since the integrand in the formula is continuous, by the properties of the integral we do know that the output function is continuous as well. We therefore define $\mathcal{V}'$ to be the vector space of all continuous complex-valued functions defined on the interior of D, with topology such that convergence in $\mathcal{V}'$ is equivalent to uniform convergence on compact subsets of the interior of D. We note that by restricting elements of $\mathcal{V}$ to the interior of D, $\mathcal{V}$ can be viewed as a subspace of $\mathcal{V}'$. Since convergence in $\mathcal{V}$ is uniform convergence, convergence in $\mathcal{V}$ implies convergence in $\mathcal{V}'$. Consequently, this inclusion of $\mathcal{V}$ in $\mathcal{V}'$ is continuous.

Definition 4.3. Let $P : \mathcal{V} \to \mathcal{V}'$ be the Poisson integral operator,

$$P(f)(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\phi})\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}d\phi$$

for $z = re^{i\theta} \in \operatorname{int}(D)$.

The term multiplied by f in the integrand is called the *Poisson Kernel*. Routine computation shows that for $z = re^{i\theta}$, this term equals

$$\frac{1-r^2}{r^2+1-2r\cos(\theta-\phi)} = \frac{1-|z|^2}{|z-e^{i\phi}|^2} = \frac{e^{i\phi}}{e^{i\phi}-z} + \frac{e^{-i\phi}}{e^{-i\phi}-\overline{z}} - 1. \tag{1}$$

Proposition 4.4. *P is a continuous operator on $\mathcal{V}$.*

Proof. Let $K \subset \text{int}(D)$ be compact. Then since the Poisson kernel is continuous on the compact set $K \times [0, 2\pi]$, there exists $B > 0$ such that for all $re^{i\theta} \in K$, for all $\phi \in [0, 2\pi]$,

$$\left|\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\right| < B.$$

Now, suppose that $\{f_n\} \longrightarrow f$ in $\mathcal{V}$. Then $\{f_n - f\} \longrightarrow 0$ uniformly. So by the properties of integration under uniform limits,

$$\begin{aligned} |P(f_n)(re^{i\theta}) - P(f)(re^{i\theta})| &= \left|\frac{1}{2\pi}\int_0^{2\pi}(f_n(e^{i\phi}) - f(e^{i\phi}))\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}d\phi\right| \\ &\leq \frac{B}{2\pi}\int_0^{2\pi}\left|f_n(e^{i\phi}) - f(e^{i\phi})\right| d\phi \longrightarrow 0. \end{aligned}$$

Since the term tending to zero does not depend on $re^{i\theta} \in K$, this convergence is uniform on all compact subsets K of D. □

To show the $SU(1,1)$-equivariance of P, we shall need two simple lemmas.

Lemma 4.5. *The operator P is $SO(2)$-equivariant, i.e. if $\psi \in [0, 2\pi)$ and $k_\psi \in SO(2)$, then for all $f \in \mathcal{V}$, $\rho(k_\psi)P(f) = P(\rho(k_\psi)f)$.*

Proof. Writing $z = re^{i\theta} \in \text{int}(D)$, we have

$$\rho(k_\psi)P(f)(re^{i\theta}) = P(f \circ \tilde{k}_{-\psi})(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i(\phi-2\psi)})\frac{1-r^2}{r^2+1-2r\cos(\theta-\phi)}d\phi.$$

Substituting $\sigma = \phi - 2\psi$ and $d\sigma = d\phi$ gives

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i(\sigma)})\frac{1-r^2}{r^2+1-2r\cos(\theta-2\psi-\sigma)}d\sigma = P(f)(re^{i(\theta-2\psi)}) = P(\rho(k_\psi)f)(re^{i\theta}).$$

□

Lemma 4.6. *Each $\gamma \in SU(1,1)$ can be decomposed as $\gamma = k_s\gamma_r k_t$, where*

$$k_s = \begin{pmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{pmatrix}, \quad \gamma_r = \begin{pmatrix} \cosh(r) & \sinh(r) \\ \sinh(r) & \cosh(r) \end{pmatrix}, \quad \textit{and} \quad k_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

for $s, r, t \in \mathbb{R}$.

Proof. For $a, b \in \mathbb{R}$, we may write

$$\gamma = \begin{pmatrix} ae^{i\psi} & be^{i\theta} \\ be^{-i\theta} & ae^{-i\psi} \end{pmatrix} = \begin{pmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix},$$

where $s = \frac{\psi+\theta}{2}$ and $t = \frac{\psi-\theta}{2}$. Since $\det(\gamma) = a^2 - b^2 = 1$, $a = \cosh(r)$ and $b = \sinh(r)$ for some $r \in \mathbb{R}$. □

Proposition 4.7. *The operator P is $SU(1,1)$-equivariant, i.e. if $\gamma \in SU(1,1)$, then for all $f \in \mathcal{V}$, $P(\rho(\gamma)f) = \rho(\gamma)P(f)$.*

Proof. It suffices to check this fact only for the generators of $SU(1,1)$. The matrices shown to generate $SU(1,1)$ in Lemma 4.6 correspond to Möbius transformations of the forms

$$\tilde{k}_t(z) = e^{2it}z \qquad \text{and} \qquad \tilde{\gamma}_r(z) = \frac{\cosh(r)z + \sinh(r)}{\sinh(r)z + \cosh(r)}.$$

The verification that $P(\rho(k_t)f) = \rho(k_t)P(f)$ for $k_t \in SO(2)$ was completed in Lemma 4.5. Therefore we can restrict our attention to Möbius transformations of the latter form. Moreover, by rotating the unit disk so that the point in question lies on the real axis, we may assume that $z \in \text{int}(D) \cap \mathbb{R}$. Applying equation (1), for all $f \in \mathcal{V}$, we have

$$\begin{aligned}
\rho(\gamma_r^{-1})(P(\rho(\gamma_r)f)(z) &= P(f \circ \tilde{\gamma}_r^{-1})(\tilde{\gamma}_r(z)) \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\tilde{\gamma}_r^{-1}(e^{i\phi}))\frac{1 - |\tilde{\gamma}_r(z)|^2}{|\tilde{\gamma}_r(z) - e^{i\phi}|^2}d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\tilde{\gamma}_r^{-1}(e^{i\phi}))\frac{1 - \tilde{\gamma}_r(z)^2}{|\tilde{\gamma}_r(z) - e^{i\phi}|^2}d\phi,
\end{aligned}$$

where the last equality follows from the fact that z is real. We must show that this quantity equals $P(f)(z)$. Let $e^{i\sigma} = \tilde{\gamma}_r^{-1}(e^{i\phi}) = \frac{\cosh(r)e^{i\phi} - \sinh(r)}{-\sinh(r)e^{i\phi} + \cosh(r)}$. Then $e^{i\phi} = \tilde{\gamma}_r(e^{i\sigma})$. Differentiating gives

$$ie^{i\sigma}d\sigma = \frac{ie^{i\phi}d\phi}{(-\sinh(r)e^{i\phi} + \cosh(r))^2} = \frac{i\tilde{\gamma}_r(e^{i\sigma})d\phi}{(-\sinh(r)\tilde{\gamma}_r(e^{i\sigma}) + \cosh(r))^2}.$$

Plugging in the expression for γ_r, solving for $d\phi$ and simplifying yields

$$\begin{aligned}
d\phi &= e^{i\sigma}\left(\frac{\cosh(r)e^{i\sigma} + \sinh(r)}{\sinh(r)e^{i\sigma} + \cosh(r)}\right)^{-1}\left(-\sinh(r)\left(\frac{\cosh(r)e^{i\sigma} + \sinh(r)}{\sinh(r)e^{i\sigma} + \cosh(r)}\right) + \cosh(r)\right)^2 d\sigma \\
&= \left(\sinh(r)^2 + \cosh(r)^2 + \sinh(r)\cosh(r)(e^{i\sigma} + e^{-i\sigma})\right)^{-1} d\sigma.
\end{aligned}$$

Substitute $e^{i\sigma} = \tilde{\gamma}_r^{-1}(e^{i\phi})$ and $d\phi = (\sinh(r)^2 + \cosh(r)^2 + \sinh(r)\cosh(r)(e^{i\sigma} + e^{-i\sigma}))^{-1} d\sigma$. Since $\gamma_r(1) = 1$, the limits of integration remain unchanged and we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\sigma})\left(\frac{1 - \tilde{\gamma}_r(z)^2}{|\tilde{\gamma}_r(z) - \tilde{\gamma}_r(e^{i\sigma})|^2}\right)\left(\sinh(r)^2 + \cosh(r)^2 + \sinh(r)\cosh(r)(e^{i\sigma} + e^{-i\sigma})\right)^{-1} d\sigma. \tag{2}$$

For the numerator of the middle term, we have

$$\begin{aligned}
1 - \tilde{\gamma}_r(z)^2 &= 1 - \left(\frac{\cosh(r)z + \sinh(r)}{\sinh(r)z + \cosh(r)}\right)^2 \\
&= \frac{(\sinh(r)z + \cosh(r))^2 - (\cosh(r)z + \sinh(r))^2}{(\sinh(r)z + \cosh(r))^2} \\
&= \frac{1 - z^2}{(\sinh(r)z + \cosh(r))^2}.
\end{aligned}$$

The denominator equals

$$
\begin{aligned}
&|\tilde{\gamma}_r(z) - \tilde{\gamma}_r(e^{i\sigma})|^2 = \left| \left(\frac{\cosh(r)z + \sinh(r)}{\sinh(r)z + \cosh(r)} \right) - \left(\frac{\cosh(r)e^{i\sigma} + \sinh(r)}{\sinh(r)e^{i\sigma} + \cosh(r)} \right) \right|^2 \\
&= \left| \frac{(\cosh(r)z + \sinh(r))(\sinh(r)e^{i\sigma} + \cosh(r)) - (\cosh(r)e^{i\sigma} + \sinh(r))(\sinh(r)z + \cosh(r))}{(\sinh(r)z + \cosh(r))(\sinh(r)e^{i\sigma} + \cosh(r))} \right|^2 \\
&= \frac{|z - e^{i\sigma}|^2}{(\sinh(r)z + \cosh(r))^2 \, (\sinh(r)^2 + \cosh(r)^2 + \sinh(r)\cosh(r)(e^{i\sigma} + e^{-i\sigma}))},
\end{aligned}
$$

where we can drop the absolute value on the denominator since z is real. Substituting into equation (2) and cancelling terms gives

$$
(\tilde{\gamma}_r \circ P(f) \circ \tilde{\gamma}_r^{-1})(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\sigma}) \frac{1 - z^2}{|z - e^{i\sigma}|^2} d\sigma = P(f)(z),
$$

as required. We thus see that P is an $SU(1,1)$-equivariant operator. $\square$

Proposition 4.8. *Let $h(z) = z$ be the identity function on D and let $\overline{h}(z) = \overline{z}$ be its conjugate. Then $P(h) = h$ and $P(\overline{h}) = \overline{h}$.*

Proof. Let $z \in \mathrm{int}(D)$. Equation (1) and partial fraction decomposition give

$$
\begin{aligned}
P(h)(z) &= \frac{1}{2\pi} \int_0^{2\pi} e^{i\phi} \left(\frac{e^{i\phi}}{e^{i\phi} - z} + \frac{e^{-i\phi}}{e^{-i\phi} - \overline{z}} - 1 \right) d\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{2i\phi}}{e^{i\phi} - z} + \frac{1}{e^{-i\phi} - \overline{z}} - e^{i\phi} d\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} z + \frac{z^2}{e^{i\phi} - z} + \frac{1}{e^{-i\phi} - \overline{z}} d\phi \\
&= z + \frac{1}{2\pi} \int_0^{2\pi} \frac{z^2}{e^{i\phi} - z} + \frac{1}{e^{-i\phi} - \overline{z}} d\phi.
\end{aligned}
$$

To show that the remaining term of this integral equals 0, substitute $u = e^{i\phi}$ and $du = ie^{i\phi} d\phi = iud\phi$. Letting a circled integral sign indicate integration once counter-clockwise around the unit circle, this gives

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \frac{z^2}{e^{i\phi} - z} + \frac{1}{e^{-i\phi} - z} d\phi &= \frac{1}{2\pi i} \oint \frac{z^2}{u(u - z)} + \frac{1}{1 - \overline{z}u} du \\
&= \frac{1}{2\pi i} \oint \frac{z}{u - z} - \frac{z}{u} + \frac{1}{1 - \overline{z}u} du \\
&= \frac{1}{2\pi i} \left(z \log(e^{i\phi} - z) - z \log(e^{i\phi}) - \frac{1}{\overline{z}} \log(1 - \overline{z}e^{i\phi}) \right) \Big|_0^{2\pi} = 0,
\end{aligned}
$$

since $e^{0i} = e^{2\pi i}$. Thus, the entire integral equals z, as desired.

It remains to consider $\overline{h}$. By Lemma 4.5, P is invariant under the action of $SU(1,1)$. So, given $z = re^{i\theta} \in \mathrm{int}(D)$, we can apply the rotation $z \mapsto e^{-i\theta}z$ without changing the value of $P(h)(z)$. This rotation maps z to r, so it suffices to consider only the case where z is real. In this case, $z = \overline{z}$, and so

$$P(\overline{h})(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\phi}\left(\frac{e^{i\phi}}{e^{i\phi}-z} + \frac{e^{-i\phi}}{e^{-i\phi}-z} - 1\right)d\phi.$$

Substitute $\sigma = -\phi$ and $d\sigma = -d\phi$ and use the periodicity of $e^{i\phi}$ to obtain

$$P(\overline{h})(z) = \frac{1}{2\pi}\int_{-2\pi}^{0} e^{i\sigma}\left(\frac{e^{-i\sigma}}{e^{-i\sigma}-z} + \frac{e^{i\sigma}}{e^{i\sigma}-z} - 1\right)d\phi = P(h)(z) = z = \overline{z},$$

since z is assumed real. □

We are now ready to prove the Poisson integral formula.

Proof of Theorem 1.1. We must show that for any function $f \in \mathcal{V}$, $P(f) = f$ on $\mathrm{int}(D)$. But, for h the identity function and $\overline{h}$ its conjugate, $P(h) = h$ and $P(\overline{h}) = \overline{h}$. Since h and $\overline{h}$ generate $\mathcal{V}$ as a representation of $SU(1,1)$ and P is continuous, linear, and $SU(1,1)$-equivariant, Proposition 4.1 implies that $P(f)$ does indeed equal f for all $f \in \mathcal{V}$. □

The astute reader will have noticed that we did not invoke the full force of Theorem 3.4 in our proof. We used only the implication that h and $\overline{h}$ generate all of $\mathcal{V}$, rather than the stronger assertion, which we proved, that $\mathcal{V}_c$, $\mathcal{V}_h$ and $\mathcal{V}_{ah}$ are the only proper subrepresentations of $\mathcal{V}$. The stronger result was proven to shed more light on the structure of our representations.

5 An SU(1,1)-Invariant Inner Product

Although it is not a necessary part of our proof of the Poisson integral formula, for the sake of completeness of our description of our representations of $SU(1,1)$, we end this paper by constructing an inner product which is invariant under the action of $SU(1,1)$: that is, one satisfying $\langle\rho(\gamma)f, \rho(\gamma)g\rangle = \langle f, g\rangle$ for all $\gamma \in SU(1,1)$. However, $\mathcal{V}$ itself cannot possess an $SU(1,1)$-invariant inner product, for if it did, the orthogonal complement of a subrepresentation would also be a subrepresentation. But, as we proved in Section 3, the only subrepresentations of $\mathcal{V}$ are $\mathcal{V}_h$, $\mathcal{V}_{ah}$ and their intersection $\mathcal{V}_c$. In particular, every subrepresentation intersects the proper subrepresentation $\mathcal{V}_h$ non-trivially, so $\mathcal{V}_h$ cannot have a closed $SU(1,1)$-invariant orthogonal complement. Consequently, we must define a slightly modified version of our vector space for use in this section.

Definition 5.1. We denote by $\tilde{\mathcal{V}}$ the vector space of complex-valued functions f which are harmonic on some neighborhood U_f of the closed unit disk $D \subset \mathbb{C}$ modulo the equivalence relation of differing by a constant:

$$\tilde{\mathcal{V}} = \left\{f : U_f \to \mathbb{C} : D \subset U_f,\ \frac{\partial^2 f}{\partial x^2}(z) + \frac{\partial^2 f}{\partial y^2}(z) = 0\ \forall z \in U_f\right\} / \{\text{constant functions}\}.$$

For each $\gamma \in SU(1,1)$, $\rho(\gamma)$ preserves the subspace of constant functions in $\mathcal{V}$, and the set of functions which are harmonic on a neighborhood of D is a subset of $\mathcal{V}$. Therefore our representation restricts to a well defined representation of $SU(1,1)$ on $\tilde{\mathcal{V}}$.

Define the degree operator from $\tilde{\mathcal{V}}$ to $\tilde{\mathcal{V}}$ by $\deg(f)(x+iy) = \left(x\frac{\partial f}{\partial x}(x+iy) + y\frac{\partial f}{\partial y}(x+iy)\right)$. Since the partial derivatives of a constant are zero, deg is well-defined on $\tilde{\mathcal{V}}$. Moreover, if p_n is the equivalence class of $z \mapsto z^n$ in $\tilde{\mathcal{V}}$, for each $n = 1, 2, 3, \ldots$ we have

$$\deg(p_n)(x+iy) = nx(x+iy)^{n-1} + niy(x+iy)^{n-1} = n(x+iy)^n = np_n$$

and, similarly,

$$\deg(\overline{p}_n)(x+iy) = n\overline{p}_n.$$

Proposition 5.2. *Define an operator* $\langle .,. \rangle$ *on* $\tilde{\mathcal{V}} \times \tilde{\mathcal{V}}$ *by*

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \deg(f)(e^{i\phi})\overline{g(e^{i\phi})}d\phi.$$

Then $\langle .,. \rangle$ *is a hermitian inner product on* $\tilde{\mathcal{V}}$*, and* $\{p_1, \overline{p}_1, p_2, \overline{p}_2, \ldots\}$ *is orthogonal with respect to* $\langle .,. \rangle$*.*

Proof. Linearity in the first argument and anti-linearity in the second are obvious. To show that $\langle .,. \rangle$ is an inner product, we must show that it is well defined on $\tilde{\mathcal{V}}$ (i.e. adding a constant to one or the other input function doesn't change the operator's value), and we must check positive definiteness and symmetry. Since any element of $\tilde{\mathcal{V}}$ can be written as a power series in z^n and $\overline{z}^n$, $n = 1, 2, \ldots$, valid on all of D, it suffices to verify these properties only for elements of $\{p_1, \overline{p}_1, p_2, \overline{p}_2, \ldots\}$. The relevant properties can then be generalized to arbitrary elements of $\tilde{\mathcal{V}}$ using the fact that power series can be integrated term by term.

From the computations given after the definition of the degree operator, we obtain that if $f(z) = c$ is constant, then for any positive integer n,

$$\langle p_n, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} ne^{ni\phi}\overline{c}d\phi = -i\overline{c}e^{ni\phi}\Big|_0^{2\pi} = 0,$$

since the period of $e^{i\phi}$ is an integer fraction of 2π. Similarly, $\langle \overline{p}_n, f\rangle = 0$. Since $\deg(f) = 0$, $\langle f, g\rangle = 0$ for any $g \in \tilde{\mathcal{V}}$. Therefore $\langle .,. \rangle$ is well defined on $\tilde{\mathcal{V}}$: it does not matter which representatives of the equivalence classes of functions which differ by a constant we choose in computing the inner product.

Moreover, for any positive integers n and m,

$$\langle p_n, p_m\rangle = \frac{1}{2\pi}\int_0^{2\pi} ne^{ni\phi}e^{-mi\phi}d\phi = \frac{n}{2\pi}\int_0^{2\pi} e^{i(n-m)\phi}d\phi = \begin{cases} n & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

Similar computations show that $\langle \overline{p}_n, \overline{p}_m\rangle = n\delta_{n,m}$ and $\langle p_n, \overline{p}_m\rangle = \langle \overline{p}_n, p_m\rangle = 0$. This implies that $\{p_1, \overline{p}_1, p_2, \overline{p}_2, \ldots\}$ is orthogonal with respect to $\langle .,. \rangle$, and that $\langle .,. \rangle$ is both symmetric and positive definite on powers of z and $\overline{z}$.

Thus, if $f = \sum_{n=1}^{\infty} a_n p_n + \sum_{n=1}^{\infty} b_n \overline{p}_n$ and $g(z) = \sum_{n=1}^{\infty} c_n p_n + \sum_{n=1}^{\infty} d_n \overline{p}_n$ are elements of $\tilde{\mathcal{V}}$,

$$\begin{aligned}\langle f, g\rangle &= \langle\sum_{n=1}^{\infty} a_n p_n, \sum_{n=1}^{\infty} c_n p_n\rangle + \langle\sum_{n=1}^{\infty} a_n p_n, \sum_{n=1}^{\infty} d_n \overline{p}_n\rangle + \langle\sum_{n=1}^{\infty} b_n \overline{p}_n, \sum_{n=1}^{\infty} c_n p_n\rangle + \langle\sum_{n=1}^{\infty} b_n \overline{p}_n, \sum_{n=1}^{\infty} d_n \overline{p}_n\rangle \\ &= \sum_{n=1}^{\infty} a_n \overline{c}_n \langle p_n, p_n\rangle + \sum_{n=1}^{\infty} b_n \overline{d}_n \langle p_n, p_n\rangle = \sum_{n=1}^{\infty} n a_n \overline{c}_n + \sum_{n=1}^{\infty} n b_n \overline{d}_n = \overline{\langle g, f\rangle}\end{aligned}$$

and

$$\langle f, f\rangle = \sum_{n=1}^{\infty} n|a_n|^2 + \sum_{n=1}^{\infty} n|b_n|^2 \geq 0,$$

so $\langle ., .\rangle$ is an inner product. The fact that $\{p_1, \overline{p}_1, p_2, \overline{p}_2, ...\}$ is orthogonal was established above. □

Unlike $\mathcal{V}$ under the maximum norm, $\tilde{\mathcal{V}}$ is not complete with respect to the norm induced by this inner product. However, $\tilde{\mathcal{V}}$ can be completed to a *Hilbert space,* or a complete inner product space with Hilbert basis $\{p_1, \overline{p}_1, p_2, \overline{p}_2, ...\}$.

Proposition 5.3. *The inner product $\langle ., .\rangle$ is $SU(1,1)$-invariant, i.e. for all $f, g \in \tilde{\mathcal{V}}$, for all $\gamma \in SU(1,1)$, $\langle f, g\rangle = \langle \rho(\gamma) f, \rho(\gamma) g\rangle$.*

Proof. Let $\tilde{\gamma} = \frac{az+b}{\overline{b}z+\overline{a}}$ be the Möbius transformation associated with $\gamma \in SU(1,1)$. We must show that $\langle f \circ \tilde{\gamma}, g \circ \tilde{\gamma}\rangle = \langle f, g\rangle$ for all $f, g \in \tilde{\mathcal{V}}$. By linearity and term-by-term integration of power series, it suffices to show this only for the cases where $f(z) = z^n$ and $f(z) = \overline{z}^n$, $n \in \mathbb{N}$. Define $f(z) = z^n$. We first compute $\deg(f \circ \tilde{\gamma})$. Writing $z = x + iy$,

$$\frac{\partial}{\partial x}\left(\frac{a(x+iy)+b}{\overline{b}(x+iy)+\overline{a}}\right)^n = n\left(\frac{a(x+iy)+b}{\overline{b}(x+iy)+\overline{a}}\right)^{n-1}\left(\frac{1}{(\overline{b}(x+iy)+\overline{a})^2}\right)$$

and

$$\frac{\partial}{\partial y}\left(\frac{a(x+iy)+b}{\overline{b}(x+iy)+\overline{a}}\right)^n = n\left(\frac{a(x+iy)+b}{\overline{b}(x+iy)+\overline{a}}\right)^{n-1}\left(\frac{i}{(\overline{b}(x+iy)+\overline{a})^2}\right).$$

Thus,

$$\deg(f \circ \tilde{\gamma})(z) = n\left(\frac{a(x+iy)+b}{\overline{b}(x+iy)+\overline{a}}\right)^{n-1}\left(\frac{x}{(\overline{b}(x+iy)+\overline{a})^2} + \frac{iy}{(\overline{b}(x+iy)+\overline{a})^2}\right) = \frac{n\tilde{\gamma}(z)^{n-1}z}{(\overline{b}z+\overline{a})^2}.$$

For all $g \in \tilde{\mathcal{V}}$, we therefore have

$$\langle f \circ \tilde{\gamma}, g \circ \tilde{\gamma}\rangle = \frac{1}{2\pi}\int_0^{2\pi} n\tilde{\gamma}(e^{i\phi})^{n-1}\left(\frac{e^{i\phi}}{(\overline{b}e^{i\phi}+\overline{a})^2}\right)\overline{g(\tilde{\gamma}(e^{i\phi}))}d\phi.$$

Let $e^{i\sigma} = \tilde{\gamma}(e^{i\phi})$. Then $e^{i\sigma} d\sigma = \left(\frac{e^{i\phi}}{(\bar{b}e^{i\phi}+\bar{a})^2}\right) d\phi$, so substitution gives

$$\langle f \circ \tilde{\gamma}, g \circ \tilde{\gamma} \rangle = \frac{1}{2\pi} \int_{\psi}^{2\pi+\psi} n(e^{i\sigma})^n \overline{g(e^{i\sigma})} d\sigma = \frac{1}{2\pi} \int_0^{2\pi} \deg(f)(e^{i\sigma}) \overline{g(e^{i\sigma})} d\sigma = \langle f, g \rangle,$$

where $e^{i\psi} = \tilde{\gamma}(1)$ and the equivalence of the different limits of integration follows because both correspond to integrating exactly once around the unit circle in the same direction. A similar computation shows that $\langle f \circ \tilde{\gamma}, g \circ \tilde{\gamma} \rangle = \langle f, g \rangle$ for $f(z) = \bar{z}^n$, and we thus conclude that $\langle f \circ \tilde{\gamma}, g \circ \tilde{\gamma} \rangle = \langle f, g \rangle$ for all $f, g \in \tilde{\mathcal{V}}$. $\square$

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