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# Analytical Solution of the Symmetric Circulant Tridiagonal Linear System

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**Abstract**

A circulant tridiagonal system is a special type of Toeplitz system that appears in a variety of problems in scientific computation. In this paper we give a formula for the inverse of a symmetric circulant tridiagonal matrix as a product of a circulant matrix and its transpose, and discuss the utility of this approach for solving the associated system.

## 1 Introduction

A real $N \times N$ matrix $C$ is said to be *Toeplitz* if $c_{i,j} = c_{i+1,j+1}$ (the matrix is constant along diagonals). A Toeplitz matrix is *circulant* if $c_{i,j} = c_{i+1,j+1}$ where are indices are taken mod $N$ (the matrix is constant along diagonals, with row-wise wrap-around). We write $C = \text{circ}(c_0, ..., c_{N-1})$ to indicate the circulant matrix with first row $c_{1,j} = c_{j-1}$, $j = 1, ..., n$.

Circulant matrices appear in many applications in scientific computing, including computational fluid dynamics [1], numerical solution of integral equations [2], [3], preconditioning Toeplitz matrices [3], and smoothing data [4]. Linear systems involving circulant matrices may be solved efficiently in $O(n \log n)$ operations using three applications of the Fast Fourier Transform (FFT) [3].

Circulant matrices may be banded. The $N \times N$ *circulant tridiagonal* matrix is the matrix $C = \text{circ}(c_0, c_1, 0, ..., 0, c_{N-1})$. If in addition $c_1 = 0$, we say that it is *circulant lower bidiagonal*; if instead $c_{N-1} = 0$, we say that it is *circulant upper bidiagonal.* The eigenvalues of the circulant tridiagonal matrix $\text{circ}(c_0, c_1, 0, ..., 0, c_1)$ are known to be

$$\lambda_i = c_0 + 2c_1 \cos\left(\frac{2\pi i}{N}\right), i = 0, ..., N-1 \quad (1)$$

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[5]. In this paper we will focus on the symmetric circulant tridiagonal matrix in a normalized form that appears in a number of applications, including computational fluid dynamics [1]:

$$\Gamma = \begin{pmatrix} 1 & a & 0 & 0 & a \\ a & 1 & 0 & 0 & a \\ 0 & a & 1 & a & 0 \\ 0 & 0 & a & 1 & a \\ a & 0 & 0 & a & 1 \end{pmatrix} \tag{2}$$

(shown for $N = 5$). In our case,

$$\lambda_i = 1 + 2a\cos\left(\frac{2\pi i}{N}\right), i = 0, ..., N-1 \tag{3}$$

so that $\Gamma$ is singular if $a = -1/2$ ($i = 0$) or if $a = 1/2$ and $N$ is even ($i = N/2$). Note that for $-1/2 < a < 1/2$, $\Gamma$ is strictly diagonally dominant and, from (3), positive definite. Hence we expect it to be well-behaved numerically; in fact, we can easily generate its eigenvalues and use $|\lambda_{\max}| / |\lambda_{\min}|$ as a check on its conditioning [5].

The inverse of a (symmetric) positive definite Toeplitz matrix such as $\Gamma$ may be computed in $O(n^2)$ operations [6]. Although the general circulant linear system $Cx = b$ may be solved in $O(n \log n)$ operations, Chen [5] develops a special LU decomposition for the strictly diagonally dominant symmetric circulant tridiagonal matrix $c_0\Gamma$, in the form $c_0\Gamma = \alpha\hat{L}\hat{U}$ where $\hat{L}$ is lower bidiagonal and $\hat{U}$ is upper bidiagonal, then solves $c_0\Gamma x = b$ as $\alpha\hat{L}\hat{U}x = b$ with the aid of two applications of the Sherman-Morrison formula. The resulting algorithm is $O(n)$ (about $5n$ operations versus about $12n\log_2 n$ for the general FFT-based approach).

We will use a convolution algebra and a $z$-transform [8] idea to develop a formula of the form $\Gamma^{-1} = \gamma MM^T$, with $M$ a circulant matrix that is dependent upon a single parameter. Once $M$ and $\gamma$ are known, $\Gamma x = b$ may be solved as $x = \gamma M(M^T b)$.

## 2 The Convolution Algebra

Consider $\mathbb{Z}_N$, the cyclic group of integers mod $N$, and take the convolution algebra $\mathbb{C}(\mathbb{Z}_N)$ to be the complex vector space of all functions defined on $\mathbb{Z}_N$, with convolution product $*$ defined by

$$f * g(r) = \sum_{k=0}^{N-1} f(k)g(r-k) \mod N$$

giving an associative and commutative $\mathbb{C}$-algebra with multiplicative identity. We use the time sample basis

$$\delta_0, ..., \delta_{N-1} \tag{4}$$

for $\mathbb{C}(\mathbb{Z}_N)$, where $\delta_i(j) = \delta_{i,j}$ (the Kronecker delta function). Given any $f \in \mathbb{C}(\mathbb{Z}_N)$,

$$f = c_0\delta_0 + ... + c_{N-1}\delta_{N-1}$$

where $c_j = f(j)$, and so we may identify $f$ with the column vector $[c_0\ c_1\ \cdots\ c_{N-1}]^T$. Also noting that $\delta_i * \delta_j = \delta_{i+j}$ (indices mod $N$) convolution products are easily calculated using basis expansion above and we see that $\delta_0$ serves as the multiplicative identity $1 \in \mathbb{C}(\mathbb{Z}_N)$.

To relate $\mathbb{C}(\mathbb{Z}_N)$ to circulant matrices, fix an $f \in \mathbb{C}(\mathbb{Z}_N)$ and use it to define a linear transformation

$$L_f : \mathbb{C}(\mathbb{Z}_N) \to \mathbb{C}(\mathbb{Z}_N)$$

by $L_f(g) = f * g$. The matrix of this linear transformation with respect to the basis (4) is

$$C = \mathrm{circ}(c_0, c_{N-1}, c_{N-2}, ..., c_1)$$

(and so by proper choice of $f$ we may arrange for $C$ to be any desired circulant matrix). By associativity,

$$L_{f*g}(h) = (f * g) * h = f * (g * h) = L_f(L_g(h))$$

and hence $f \to L_f$ is an algebra isomorphism onto the subalgebra of circulant matrices. Hence we can find the inverse of the matrix $C$ by finding the inverse of $f$ in the convolution algebra.

# 3 The Symmetric Circulant Tridiagonal Case

We want to invert (2), $\Gamma = \mathrm{circ}(1, a, 0, ..., 0, a)$, when it is nonsingular. The representer polynomial [4] for $\Gamma$ would be $p_\Gamma(z) = 1 + az + az^{N-1}$ (so that $p_\Gamma(1/z)$ is the corresponding $z$-transform), and similarly, the element of $\mathbb{C}(\mathbb{Z}_N)$ corresponding to $\Gamma$ is

$$\begin{aligned} f &= 1\delta_0 + a\delta_1 + a\delta_{N-1} \\ &= 1 + a\delta_1 + a\delta_{N-1} \end{aligned} \tag{5}$$

which we seek to factor as

$$f = c(1 - r\delta_1)(1 - r\delta_{N-1}) \tag{6}$$

i.e. as $f = cf_1f_{-1}$, where $f_1 = 1 - r\delta_1$, $f_{-1} = 1 - r\delta_{N-1}$ (cf. the factorization into a product of circulant bidiagonals in [5]; in particular, $L_{f_1}$ is circulant lower bidiagonal and $L_{f_{-1}}$ is circulant upper bidiagonal). If we can find these factors, then we will have $L_f^{-1} = \gamma L_{f_1}^{-1} L_{f_{-1}}^{-1}$ where $\gamma = 1/c$. Comparing (5) and (6), we see that

$$
\begin{aligned}
c(1+r^2) &= 1 \\
cr &= -a
\end{aligned}
$$

is required. If $a = 0$ then $\Gamma = I_N$; otherwise,

$$
\begin{aligned}
r_{1,2} &= \frac{-1 \pm \sqrt{1-4a^2}}{2a} \\
c_{1,2} &= \frac{1 \pm \sqrt{1-4a^2}}{2}
\end{aligned}
$$

(which are complex when $|a|$ exceeds $1/2$; $c_1$ is Chen's $\alpha$ in $\Gamma = \alpha \hat{L}\hat{U}$). Choose $(r, c) = (r_i, c_i)$ for $i = 1$ or $i = 2$. Since

$$
(1 - r\delta_1)(1 + r\delta_1 + r^2\delta_2 + ... + r^{N-1}\delta_{N-1}) = 1 - r^N
$$

we have

$$
\begin{aligned}
(1 - r\delta_1)^{-1} &= (1 + r\delta_1 + r^2\delta_2 + ... + r^{N-1}\delta_{N-1})/(1 - r^N) \\
&= \frac{1}{1-r^N}\delta_0 + \frac{r}{1-r^N}\delta_1 + \frac{r^2}{1-r^N}\delta_2 + ... + \frac{r^{N-1}}{1-r^N}\delta_{N-1}
\end{aligned}
$$

and so $L_{f_1}^{-1}$ has the matrix representation

$$
M = \frac{1}{1-r^N}\,\mathrm{circ}(1, r^{N-1}, r^{N-2}, ..., r)
$$

and similarly, the matrix representation of $L_{f_{-1}}^{-1}$ is found to be $M^T$. From (6), then,

$$
\Gamma^{-1} = \gamma M M^T \tag{7}
$$

where $\gamma = 1/c$, and $c$ is nonzero when $|a| < 1/2$. Because of the factor $1/(1 - r^N)$, the value of $r_{1,2}$ furthest from unity should usually be chosen (unless the corresponding $c$ value is extremely small).

Solving $Cx = b$ for the general symmetric circulant tridiagonal case $C = \mathrm{circ}(c_0, c_1, 0, ..., 0, c_1)$ is easily handled. We have

$$
\begin{aligned}
C &= c_0\,\mathrm{circ}(1, c_1/c_0, 0, ..., 0, c_1/c_0) \\
&= c_0\Gamma
\end{aligned}
$$

if $c_0$ is nonzero, and from (1) we see that $C$ must have at least one null eigenvalue if $c_0 = 0$.

## 4 Discussion

The method discussed here advances previous work by giving explict formulas for the inverses of the two circulant bidiagonal factors. In addition, for $N$ odd our formula is valid for the weakly diagonally dominant case $a = 1/2$. But because $M$ is dense, solution of $\Gamma x = b$ by the use of (7) in the form

$$x = \gamma M(M^T b) \tag{8}$$

requires two circulant-matrix-by-vector multiplications, each of which requires three FFTs [3]. Hence the method is $O(n \log n)$ once the first row of $M$ is computed. Although we could simplify this somewhat after diagonalizing $M$ by the Fourier matrix [4], it will typically be less efficient than using the the l LU decomposition $\Gamma = \alpha \hat{L}\hat{U}$ in conjunction with the Sherman-Morrison formula, which requires approximately $5n$ operations, or when $\Gamma$ is not strictly diagonally dominant, directly solving $\Gamma x = b$ as a general circulant system using three FFTs.

Significantly, however, our formula applies *whenever* $\Gamma$ *is nonsingular.* It is apparent from (3) that for any fixed $N$ there are up to $N$ values of $a$ that may make $\Gamma$ singular, viz.

$$a = \frac{-1}{2\cos\left(\frac{2\pi i}{N}\right)}$$

for $i = 0, ..., N-1$; in fact, there are $1 + \text{floor}(N/2)$ such distinct values of $a$. If we are willing to use complex arithmetic in (8) then we may solve $\Gamma x = b$ by this formula whenever $\Gamma$ admits an inverse. (Note that (8) and (7) remain correct as written; the transpose does not become the Hermitian transpose when $|a| > 1/2$.) Thus, the choices $(r, c) = (r_i, c_i)$ for $i = 1, 2$ give two distinct (if $a \neq 1/2$) decompositions of $\Gamma^{-1}$ whenever it exists.

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