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Zero-Divisor Graphs and Lattices of Finite Commutative Rings

Darrin Weber

Abstract. In this paper we consider, for a finite commutative ring R, the well-studied zero-divisor graph $\Gamma(R)$ and the compressed zero-divisor graph $\Gamma_c(R)$ of R and a newly-defined graphical structure — the zero-divisor lattice $\Lambda(R)$ of R. We give results which provide information when $\Gamma(R) \cong \Gamma(S)$, $\Gamma_c(R) \cong \Gamma_c(S)$, and $\Lambda(R) \cong \Lambda(S)$ for two finite commutative rings R and S. We also provide a theorem which says that $\Lambda(R)$ is almost always connected.

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1 Introduction

The notion of a zero-divisor graph was introduced in [7] and was studied further in [4]. However, the definition we will provide of the zero-divisor graph of a ring is more generally accepted and was first used in [3]. Much research has been done on zero-divisor graphs over the past ten years, and many of the papers can be found in the reference section of [2]. The hope is that the graph-theoretic properties of the zero-divisior graph will help us better understand the ring-theoretic properties of R.

Not only have zero-divisor graphs been studied by professional mathematicians, but they have also been the focus of master's theses and doctorial dissertations. Further, many undergraduates have researched zero-divisor graphs extensively providing a number of substantial results in the field. In particular, the Wabash Summer Institute in Mathematics at Wabash College has, in part, focused on the interplay between ring structure and zero-divisor graph structure. Some tools used to aid in the research process in this program were Mathematica notebooks that displayed the zero-divisor graph of certain rings. These notebooks have been rewritten as well as additional notebooks added, and they can be found at [14]. All of the graphs displayed in this paper were generated using those notebooks.

To help study large zero-divisor graphs, we introduce another related definition to the zero-divisor graph, called the compressed zero-divisor graph, which first appeared in a similar form in [12] and was further studied in [13]. We also introduce the definition for the zero-divisor lattice suggested by Dr. Nicholas Baeth of the University of Central Missouri.

In section 2, we provide the necessary definitions for this paper. In section 3, we look at the connections between the zero-divisor graph and the compressed zero-divisor graph and prove that isomorphic graphs yield isomorphic compressed graphs in Theorem 3.1. In Theorem 3.3, we also demonstrate that in most cases cut-sets are preserved when looking at the compressed zero-divisor graph. In section 4, we explore the connections between all three graphical structures and show in Theorems 4.1 and 4.2 that isomorphic graphs or isomorphic compressed graphs gives us isomorphic lattices. In section 5, we show that the zero-divisor lattice is connected in almost every case.

2 Definitions

Throughout this paper, R denotes a finite commutative ring with identity. An element a is a *zero-divisor* if there exists a nonzero $r \in R$ such that $ar = 0$. We denote the set of all zero-divisors in R as $Z(R)$. The set of *annihilators* of a ring element x is $\text{ann}(x) = \{a \mid ax = 0\}$. A ring is called *local* if it has one unique maximal ideal. (A *maximal ideal* is an ideal A of a ring R such that if $A \subseteq B \subseteq R$, where B is also an ideal, then either $A = B$ or $B = R$.) A *field* is a commutative ring with identity in which every nonzero element has a multiplicative inverse.

For a graph G, we denote the set of vertices of G as $V(G)$ and the set of edges as $E(G)$. We define a *path* between two elements $a_1, a_n \in V(G)$ to be an ordered sequence of distinct vertices and edges $\{a_1, e_1, a_2, \ldots, e_{n-1}, a_n\}$ of G such that edge e_i, denoted by a_i

— a_{i+1}, is incident to vertices a_i and a_{i+1}, for each $i \in \{1, \ldots, n-1\}$. For $x, y \in V(G)$, the minimum length of all paths from x to y, if it exists, is called the *distance* from x to y and is denoted $d(x, y)$. If no path from x to y exists, then $d(x, y) = \infty$. The *diameter* of a graph is $diam(G) = \sup\{d(x, y) \mid x, y \in V(G)\}$. The *neighborhood* of a vertex is the set $nbd(x) = \{z \in V(G) \mid x — z\}$. A graph is *connected* if a path exists between any two distinct vertices. A *complete r-partite* graph is the disjoint union of r nonempty vertex sets in which two distinct vertices are adjacent if and only if they are in distinct vertex sets. In the case where $r = 2$, we call the graph *complete bipartite.*

Two graphs G and H are said to be *isomorphic* if there exists a one-to-one and onto function $\phi : V(G) \to V(H)$ such that if $x, y \in V(G)$, then $x — y$ if and only if $\phi(x) — \phi(y)$. Lemma 2.1 shows that graph isomorphism preserves neighborhoods. We provide a proof for completeness.

Lemma 2.1. *Let $G \cong H$. If $\phi(x) = y$, then $\phi(nbd(x)) = nbd(y)$.*

Proof. Let $\phi : G \to H$ be a graph isomorphism. Let $x \in V(G)$ and $\phi(x) = y \in V(H)$. Then $\phi(nbd(x)) = \{\phi(z) \mid x — z\} = \{\phi(z) \mid \phi(x) — \phi(z)\} = \{\phi(z) \mid y — \phi(z)\} = nbd(y)$. ☐

The *zero-divisor graph* of a ring R, denoted $\Gamma(R)$, is a graph with $V(\Gamma(R)) = Z(R)\backslash\{0\}$ and $E(\Gamma(R)) = \{a — b \mid ab = 0\}$. By [3], we know that $\Gamma(R)$ is always connected and $diam(\Gamma(R)) \leq 3$ for any ring R. Notice that $nbd(x) = \text{ann}(x)$ in a zero-divisor graph.

A *cut-set* of a graph G is a set $A \subset V(G)$ minimal among all subsets of $V(G)$ such that there exist distinct vertices $c, d \in V(G)\backslash A$ such that every path from c to d involves at least one element of A.

Example 2.2. *In $\Gamma(\mathbb{Z}_{30})$, shown in Figure 1(a), there are three cut-sets: $\{15\}$, $\{10, 20\}$, and $\{6, 12, 18, 24\}$. The cut-sets in $\Gamma_c(R)$ are $\{\overline{15}\}$, $\{\overline{10}\}$, and $\{\overline{6}\}$ and are shown in Figure 1(b).*

A *cut vertex* is a cut-set of size 1. The study of cut vertices of zero-divisor graphs began in [6] and was continued in [9]. In [6], it was shown that a cut vertex along with zero form an ideal in the ring. In [9], the cut vertex was generalized to cut-sets, and cut-sets were classified for finite, nonlocal commutative rings. In addition, it was shown that the cut-set along with zero form an ideal in the ring.

For algebraic definitions and concepts not listed here, see [10], and for graph theory definitions and concepts, see [8].

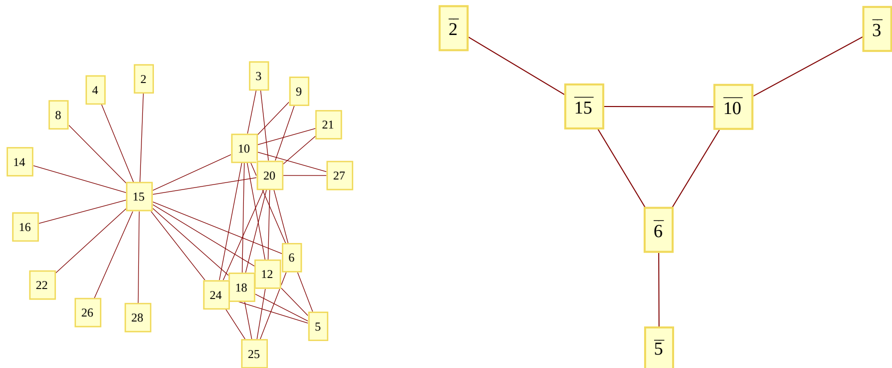
To define both the compressed zero-divisor graph and the zero-divisor lattice, we first need to define an equivalence relation on the zero-divisors of R.

Definition 2.1. *Let R be a commutative ring. Define a relation $\equiv$ on R by $x \equiv y$ if and only if $\text{ann}(x) = \text{ann}(y)$.*

It is easy to see that $\equiv$ is an equivalence relation on R. We denote the equivalence class of x by $\bar{x}$. Notice that $\text{ann}(0) = R$ and $\bar{x} = \bar{y}$ for all $x, y \in R\backslash Z(R)$. However, we will only focus on the equivalence classes of the nonzero zero-divisors.

Definition 2.2. *For a ring R, the* compressed zero-divisor graph*, denoted $\Gamma_c(R)$, is a graph whose vertices are the equivalence classes of the nonzero zero-divisors, and two vertices $\bar{a}$ and $\bar{b}$ are connected by an edge if and only if $ab = 0$.*

Example 2.3. *For $\mathbb{Z}_{30}$, Figure 1 shows the difference between the zero-divisor graph, (a), and the compressed zero-divisor graph, (b).*



(a) $\Gamma(\mathbb{Z}_{30})$ (b) $\Gamma_c(\mathbb{Z}_{30})$

Figure 1: The zero-divisor graph and compressed zero-divisor graph of $\mathbb{Z}_{30}$

Notice that by Theorem 2.8 in [3], when the zero-divisor graph is a complete graph, either the ring is $\mathbb{Z}_2 \times \mathbb{Z}_2$, or every vertex loops to itself. In the case that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\Gamma(R) \cong \Gamma_c(R)$. Every other complete zero-divisor graph compresses into a single vertex in the compressed zero-divisor graph.

To get the definition of a zero-divisor lattice, note that we can put a partial order on $V(\Gamma_c(R))$ by defining $\bar{x} < \bar{y}$ if $\mathrm{ann}(x) \subsetneq \mathrm{ann}(y)$.

Definition 2.3. *For a ring R, the* zero-divisor lattice*, denoted $\Lambda(R)$, is a lattice where the vertices are the equivalence classes of $V(\Gamma(R))$ and there is an edge $\bar{y} \to \bar{x}$ if and only if $\bar{x} < \bar{y}$ and there does not exist $\bar{z}$ with $\bar{x} < \bar{z} < \bar{y}$.*

Example 2.4. *Figure 2 displays the zero-divisor lattice of $\mathbb{Z}_{30}$.*

A zero-divisor lattice is said to be *connected* if, when the edges are considered to be undirected, you can reach any vertex y from any other vertex x. A *root* of the zero-divisor lattice is a vertex x such that for every other vertex y, either $y < x$ or x and y are incomparable. The root can also be thought of as a maximal element in the lattice. Notice that a zero-divisor lattice can have multiple roots.

For many of the results in this paper, we will need a common theorem about Noetherian rings, which is restated here. Although this theorem applies to all Noetherian rings, we will only focus on Corollary 2.6, which deals with finite rings.

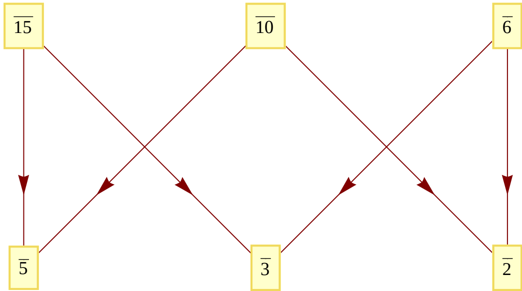


Figure 2: $\Lambda(\mathbb{Z}_{30})$

Theorem 2.5. *[11, Theorem 80] Let R be a Noetherian ring, and let A be a finitely generated non-zero R-module. Then there are only a finite number of maximal primes of A, and each is the annihilator of a non-zero element of A.*

Corollary 2.6. *Let R be a finite commutative ring with identity. Then the maximal ideals of R can be realized as the annihilator sets of single elements.*

Note that this theorem only applies to the maximal ideals of the ring. For example, in the ring $\mathbb{Z}_4[x]/(x^2 + 2x)$, the set $\{0, 2x\}$ forms an ideal in the ring, but this ideal cannot be realized by the annihilator of a single element.

The following lemma is well-known and will be used in Section 5.

Lemma 2.7. *If $R \cong R_1 \times \cdots \times R_i \times \cdots \times R_n$, then the maximal ideals of R have the form $M = R_1 \times \cdots \times M_i \times \cdots \times R_n$ where M_i is a maximal ideal in R_i.*

The next lemma is also a well-known result, and we provide a proof for completeness.

Lemma 2.8. *Let $R \cong R_1 \times \cdots \times R_i \times \cdots \times R_n$ and let $M = R_1 \times \cdots \times M_i \times \cdots \times R_n$ be a maximal ideal. Then $M = \operatorname{ann}((0, 0, \ldots, l_i, \ldots, 0))$ where $M_i = \operatorname{ann}(l_i)$ for some $l_i \in R_i$.*

Proof. It suffices to show $M = \operatorname{ann}((0, \ldots, l_i, \ldots, 0))$. Let $(r_1, \ldots, r_n) \in M$. Then $r_i \in \operatorname{ann}(l_i)$, so $(r_1, \ldots, r_n)(0, \ldots, l_i, \ldots, 0) = (0, 0, \ldots, 0)$. Hence $(r_1, \ldots, r_n) \in \operatorname{ann}((0, \ldots, l_i, \ldots, 0))$. Let $(r_1, \ldots, r_n) \in \operatorname{ann}((0, \ldots, l_i, \ldots, 0))$. Then $(r_1, \ldots, r_n)(0, \ldots, l_i, \ldots, 0) = (0, 0, \ldots, 0)$, which implies $r_i l_i = 0$. Thus, $r_i \in \operatorname{ann}(l_i)$, and $(r_1, \ldots, r_n) \in M$. $\square$

3 Connections Between $\Gamma(R)$ and $\Gamma_c(R)$

To start, we show that the compressed zero-divisor graph preserves certain properties of the zero-divisor graph.

Theorem 3.1. *Let R and S be finite commutative rings. If $\Gamma(R) \cong \Gamma(S)$, then $\Gamma_c(R) \cong \Gamma_c(S)$.*

Proof. Suppose $V(\Gamma(R)) = \{r_1, r_2, \ldots, r_n\}$ and $V(\Gamma(S)) = \{s_1, s_2, \ldots, s_n\}$ such that the isomorphism $\phi : \Gamma(R) \to \Gamma(S)$ satisfies $\phi(r_i) = s_i$ for each $i \in \{1, 2, \ldots, n\}$. By Lemma 2.1, $\phi(\text{ann}(r_i)) = \text{ann}(s_i)$ for each i, and the mapping of edges $\phi : E(\Gamma_c(R)) \to E(\Gamma_c(S))$ which sends the edge $\bar{r}_i — \bar{r}_j$ in $\Gamma(R)$ to the edge $\bar{s}_i — \bar{s}_j$ in $\Gamma(S)$ is a well-defined bijection. Thus, $\Gamma_c(R) \cong \Gamma_c(S)$. □

The converse of this theorem is false as illustrated in Example 3.2. Take $\mathbb{Z}_{2p}$ and $\mathbb{Z}_{2q}$, where p, q are distinct primes. We have that $\Gamma_c(\mathbb{Z}_{2p}) \cong \Gamma_c(\mathbb{Z}_{2q})$ but $\Gamma(\mathbb{Z}_{2p}) \ncong \Gamma(\mathbb{Z}_{2q})$.

Example 3.2. *Figure 3 displays the zero-divisor graphs and compressed zero-divisor graphs of $\mathbb{Z}_{10}$ and $\mathbb{Z}_{14}$.*

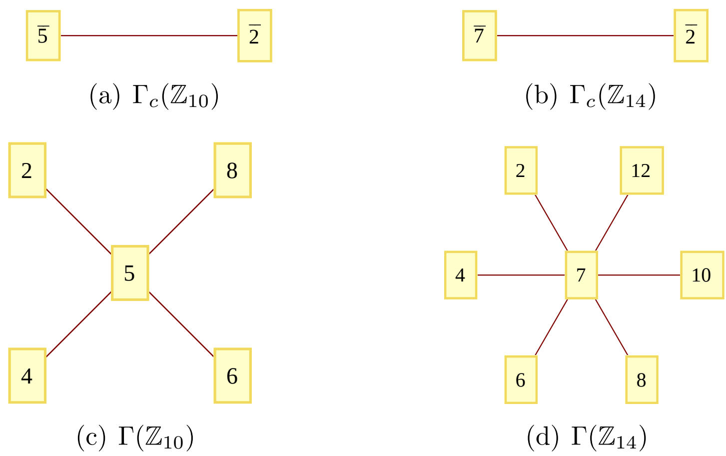


Figure 3: The zero-divisor graphs and compressed zero-divisor graphs of $\mathbb{Z}_{10}$ and $\mathbb{Z}_{14}$

The next result has been proven in more generality in [5]. We provide an alternate proof for the finite case.

Theorem 3.3. *Let R be a finite commutative ring such that $\Gamma(R)$ is not complete r-partite. A set A is a cut-set in $\Gamma(R)$ if and only if $\bar{A}$ is a cut-set in $\Gamma_c(R)$.*

Proof. For ease of notation, let the set of vertices $A = \{a_1, a_2, \ldots, a_n\}$ and let $\bar{A} = \{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n\}$, where $\bar{A}$ is the of equivalence classes for all elements in A.

($\Rightarrow$) Let $\Gamma(R)$ be a zero-divisor graph that is not complete r-partite and let A be a cut-set in $\Gamma(R)$. Since $\Gamma(R)$ is connected [3, Theorem 2.3], we have $A \neq \emptyset$. Then there exists distinct $c, d \in V(\Gamma(R))\backslash A$ such that every path from c to d involves at least one vertex in A.

Case 1: Assume for all $c, d \in V(\Gamma(R))\backslash A$, we have $c \equiv d$. Since $\Gamma(R)$ is not complete r-partite, then there must exist $a_i, a_j \in A$ such that $a_i a_j \neq 0$. Notice that $diam(\Gamma(R)) \leq 3$ by [3, Theorem 2.3] and that $diam(\Gamma(R)) \neq 3$ since all c, d are connected to every element in the cut-set A. So by Theorem 4.5 in [6], $\Gamma(R)$ is star-shaped reducible. By Theorem 2.3 in [6], $Z(R)$ forms an ideal. Consider $c + a_i$. If $c + a_i \in A$, then $(c + a_i)d = 0$. However,

$(c + a_i)d = cd \neq 0$, since c and d are separated by A. So, $c + a_i \in V(\Gamma(R))\backslash A$ which means that $(c+a_i)A = 0$, however, $(c+a_i)a_j = a_i a_j \neq 0$. Hence, $c+a_i \notin V(\Gamma(R))\backslash A$ and therefore $c + a_i \notin Z(R)$. Thus, this case is not realizable as a zero-divisor graph.

Case 2: There exists $c, d \in V(\Gamma(R))\backslash A$ such that $c \not\equiv d$. Then $\bar{c}, \bar{d}$ also exists as distinct vertices in $\Gamma_c(R)$. If $a_i \equiv a_j$ for some $i \neq j$ and $1 \leq i, j \leq n$, then let $\bar{a}_i$ represent the equivalence class of a_i in the graph. If the path from c to d involved an element in $\{\bar{a}_i\}$ in $\Gamma(R)$, then the path goes through $\bar{a}_i$ in $\Gamma_c(R)$. So, $\bar{A} = \{a_1, \dots, \bar{a}_i, \dots, a_n\}$ separates $\Gamma_c(R)$. If $\bar{A}$ was not minimal, then there would exist a $\bar{f}, \bar{g} \in V(\Gamma_c(R))\backslash\bar{A}$ such that $\bar{A}\backslash\{\bar{a}_i\}$ would separate $\bar{f}$ and $\bar{g}$. However, this would mean that f and g would be separated by $A\backslash\{\bar{a}_i\}$, which is a contradiction on the minimality of A. Thus, $\bar{A}$ is a cut-set in $\Gamma_c(R)$.

If $a_i \not\equiv a_j$ for all $i \neq j$ and $1 \leq i, j \leq n$, then any path between $\bar{c}$ and $\bar{d}$ still involves at least one vertex in $\bar{A}$. So, $\bar{A}$ separates the graph in $\Gamma_c(R)$. If $\bar{A}$ is not minimal in $\Gamma_c(R)$, then there exists an $\bar{a}_i \in \bar{A}$ and $\bar{c}, \bar{d} \in V(\Gamma_c(R))\backslash\bar{A}$ such that $\bar{c}$ and $\bar{d}$ are separated by $\bar{A}\backslash\{\bar{a}_i\}$. This would mean that c and d are separated by $A\backslash\{\bar{a}_i\}$ in $\Gamma(R)$, which is a contradiction on the minimality of A. Thus, $\bar{A}$ is minimal and is therefore a cut-set in $\Gamma_c(R)$.

($\Leftarrow$) Let $\bar{A}$ be a cut-set in $\Gamma_c(R)$. Then there exist distinct $\bar{c}, \bar{d} \in V(\Gamma_c(R))\backslash\bar{A}$ such that every path involves at least one element of $\bar{A}$. Notice that A separates $\Gamma(R)$ because if it did not, then there would exist $\bar{c}, \bar{d}$ that does not involve $\bar{A}$.

Case 1: Let $\bar{A} = \{\bar{a}_i\}$ for some $a_i \in A$. Every path from c to d passes through A and for all $a_j \in \{\bar{a}_i\}$, there exists a path from c to d that involves a_j. Since every $a_i \equiv a_j$ for all $a_i, a_j \in A$, if there is an edge $x — a_i$ for some $x \in V(\Gamma(R))$, then x is connected to a_i for all i. Thus, A is minimal in $\Gamma(R)$ and is therefore a cut-set.

Case 2: There exist distinct $a_i, a_j \in A$ such that $a_i \not\equiv a_j$. If A is not a cut-set, then there exists an $a_k \in A$ such that $A\backslash\{a_k\}$ separates $\Gamma(R)$. Thus, for all $c, d \in V(\Gamma(R))\backslash A$ there exists a path that does not involve a_k. Therefore, there exists a path from $\bar{c}$ to $\bar{d}$ in $\Gamma_c(R)$ that does not involve $\bar{a}_k$. Thus, $\bar{A}\backslash\{\bar{a}_k\}$ separates $\Gamma_c(R)$, which is a contradiction on the minimality of $\bar{A}$. Hence, A is minimal and therefore is a cut-set in $\Gamma(R)$. □

Recall that a complete r-partite zero-divisor graph is the disjoint union of r nonempty vertex sets and two distinct vertices are adjacent if and only if they are in distinct vertex sets. By Theorem 3.1 in [1], if $\Gamma(R)$ is complete r-partite then $|V_i| > 1$ for at most one $1 \leq i \leq r$, and if $V_j = \{x\}$ then $x^2 = 0$. This means that for all x, x_k in the vertex sets of order 1, $\text{ann}(x) = \text{ann}(x_k)$, which means that they are in the same equivalence class and will appear as a single vertex in the compressed zero-divisor graph. Also, for all vertices $b, b_m \in V_i$ such that $|V_i| > 1$, $\text{ann}(b) = \text{ann}(b_m)$, which means that they are all in the same equivalence class. Thus, every complete r-partite graph compresses into a graph with two vertices that are connected to each other. Since there are only two vertices, there can be no cut-set.

4 Connections between $\Gamma_c(R)$ and $\Lambda(R)$

In [9], cut-sets were classified for all finite, nonlocal rings as annihilator ideals. Notice that in the local ring $\mathbb{Z}_8[x]/(x^2+2x)$, shown in Figure 4, a cut-set is $\{\overline{2x}, \overline{4x}, \overline{2x+4}\}$ in the compressed zero-divisor graph. By Theorem 3.3, we know that this corresponds to a cut-set in the zero-divisor graph, which is $\{2x, 4x, 6x, 2x+4, 6x+4\}$. Notice that this cut-set (union with $\{0\}$) is not an ideal in the ring. Also, in the ring $\mathbb{Z}_4[x]/(x^2+2x)$, the vertex $2x$ is a cut vertex in the zero-divisor graph, but $\{0, 2x\}$ cannot be realized as the annihilator of a single element. However, we can identify the cut-set of $2x$ as the root in $\Lambda(\mathbb{Z}_4[x]/(x^2+2x))$. Because of both of these examples, we hope that studying zero-divisor lattices will help us understand more about the structure and properties of cut-sets.

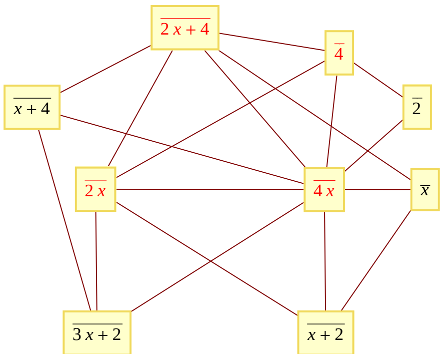


Figure 4: $\Gamma_c(\mathbb{Z}_8[x]/(x^2+2x))$

We begin by proving two theorems relating the structure of $\Gamma(R)$, $\Gamma_c(R)$, and $\Lambda(R)$.

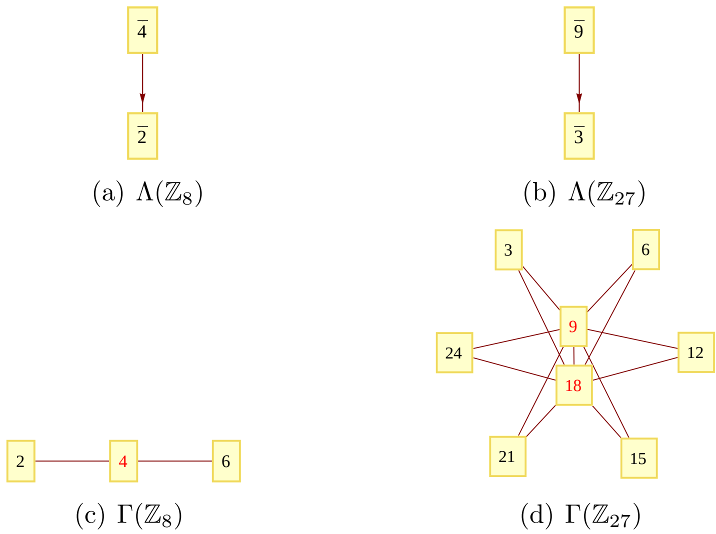
Theorem 4.1. *Let R and S be finite commutative rings. If $\Gamma(R) \cong \Gamma(S)$, then $\Lambda(R) \cong \Lambda(S)$.*

Proof. Suppose $V(\Gamma(R)) = \{r_1, r_2, \dots, r_n\}$ and $V(\Gamma(S)) = \{s_1, s_2, \dots, s_n\}$ such that the isomorphism $\phi : \Gamma(R) \to \Gamma(S)$ satisfies $\phi(r_i) = s_i$ for each $i \in \{1, 2, \dots, n\}$. By Lemma 2.1, $\phi(\text{ann}(r_i)) = \text{ann}(s_i)$ for each i, and if $\text{ann}(r_i) = \text{ann}(r_j)$ for any $1 \le i, j \le n$, then $\text{ann}(s_i) = \text{ann}(s_j)$, and if $\text{ann}(r_i) \subseteq \text{ann}(r_j)$ for any $i \neq j$, then $\text{ann}(s_i) \subseteq \text{ann}(s_j)$. Thus, the mapping of edges $\phi : E(\Lambda(R)) \to E(\Lambda(S))$ which sends the edge $\bar{r}_i \longrightarrow \bar{r}_j$ in $\Lambda(R)$ to the edge $\bar{s}_i \longrightarrow \bar{s}_j$ in $\Lambda(S)$ is a well-defined bijection. Thus, $\Lambda(R) \cong \Lambda(S)$. □

The converse of this theorem is false for the same reason that the converse for Theorem 3.1 is false. An example is given in Figure 5.

Theorem 4.2. *Let R and S be finite commutative rings. If $\Gamma_c(R) \cong \Gamma_c(S)$, then $\Lambda(R) \cong \Lambda(S)$.*

Proof. Suppose $V(\Gamma_c(R)) = \{r_1, r_2, \dots, r_n\}$ and $V(\Gamma_c(S)) = \{s_1, s_2, \dots, s_n\}$ such that the isomorphism $\phi : \Gamma_c(R) \to \Gamma_c(S)$ satisfies $\phi(r_i) = s_i$ for each $i \in \{1, 2, \dots, n\}$. By Lemma 2.1, $\phi(\text{ann}(r_i)) = \text{ann}(s_i)$ for each i, and if $\text{ann}(r_i) = \text{ann}(r_j)$ for any $1 \le i, j \le n$, then



(a) $\Lambda(\mathbb{Z}_8)$ (b) $\Lambda(\mathbb{Z}_{27})$ (c) $\Gamma(\mathbb{Z}_8)$ (d) $\Gamma(\mathbb{Z}_{27})$

Figure 5: The zero-divisor graphs and zero-divisor lattices of $\mathbb{Z}_8$ and $\mathbb{Z}_{27}$

$i = j$, and if $\text{ann}(r_i) \subseteq \text{ann}(r_j)$ for any $i \neq j$, then $\text{ann}(s_i) \subseteq \text{ann}(s_j)$. Thus, the mapping of edges $\phi : E(\Lambda(R)) \to E(\Lambda(S))$ which sends the edge $\bar{r}_i \longrightarrow \bar{r}_j$ in $\Lambda(R)$ to the edge $\bar{s}_i \longrightarrow \bar{s}_j$ in $\Lambda(S)$ is a well-defined bijection. Thus, $\Lambda(R) \cong \Lambda(S)$. $\square$

We believe the converse to this theorem is true, because $V(\Gamma_c(R)) = V(\Lambda(R))$, which remedies the reason why the converse for both Theorems 3.1 and 4.2 are false. However, this remains an open question.

5 Connectivity of $\Lambda(R)$

By Theorem 3(4) on page 752 in [10], any finite commutative ring R with identity can be written as $R \cong L_1 \times L_2 \times \cdots \times L_n \times F_1 \times F_2 \times \cdots \times F_m$, where each L_i is local and each F_j is a field. We will use this fact in the upcoming results.

Theorem 5.1. *Let $M_1, M_2, \ldots, M_n$ be ideals of a commutative ring R that is not a field. Then $M_1, M_2, \ldots, M_n$ are the maximal ideals in R if and only if $\Lambda(R)$ has n roots.*

Proof. ($\Rightarrow$) Let $M_1, M_2, \ldots, M_n$ be the maximal ideals of R. By Corollary 2.6 and Lemma 2.8, $M_i = \text{ann}(m_i)$ for $1 \leq i \leq n$. Since for any $1 \leq i \leq n$, we have $\text{ann}(m_i) \not\subseteq \text{ann}(m_k)$ for all $1 \leq k \leq n$, then m_i is a root in $\Lambda(R)$. Thus, $\Lambda(R)$ has n roots. If there exists another root m_r where $r \notin \{1, 2, \ldots, n\}$, then $\text{ann}(m_r) = M_r$ would be a maximal ideal in R. However, $M_1, M_2, \ldots, M_n$ are the only maximal ideals in R.

($\Leftarrow$) Let $\Lambda(R)$ have n roots, namely $m_1, m_2, \ldots, m_n$. By Corollary 2.6, $\text{ann}(x)$ is a maximal ideal for some $x \in R$. Obviously, $|\text{ann}(x)| \geq 2$ since R is not a field, so $x \in V(\Lambda(R))$. Also, $\text{ann}(x)$ is not properly contained in $\text{ann}(r)$ for all $r \in V(\Lambda(R))$. So, $\text{ann}(x)$ is a root of

$\Lambda(R)$. Therefore, $\text{ann}(x) = \text{ann}(m_i)$ for some $1 \leq i \leq n$. Hence, R has at most n maximal ideals. By definition of a root, $\text{ann}(m_i) \nsubseteq \text{ann}(r)$ for any $1 \leq i \leq n$ and every $r \in V(\Lambda(R))$. Thus $\text{ann}(m_i) = M_i$ is a maximal ideal of R for every $1 \leq i \leq n$. Since all maximal ideals are annihilator ideals, we must have each $\text{ann}(m_i)$ is maximal. Thus, $M_1, M_2, \ldots, M_n$ are maximal in R. □

The next four lemmas will help us prove the connectedness of the zero-divisor lattice. For each of them, recall from Corollary 2.6 that the maximal ideal of a finite ring can be written as the annihilator of a single element.

Remark 5.2. *Since all rings considered here are finite, we cannot have an infinitely ascending chain of ideals. Hence, if* $\text{ann}(x) \subsetneq \text{ann}(y)$, *it is either the case that* $y \to x$, *or there exist* $z_1, z_2, \ldots, z_n$ *such that* $y \to z_1 \to z_2 \to \cdots \to z_n \to x$. *Thus, to show* x *and* y *are connected in a lattice (treated as an undirected graph), it suffices to show, without loss of generality, that* $\text{ann}(x) \subsetneq \text{ann}(y)$. *This fact will be used in the following results.*

Lemma 5.3. *If* R *is a finite local ring, then* $\Lambda(R)$ *is connected.*

Proof. Let M be the maximal ideal of R. By Corollary 2.6 we know that $M = \text{ann}(x)$, where $x \in R$, so we know that $\bar{x}$ is a vertex of $\Lambda(R)$. Also, for any other vertex $\bar{y}$ in the lattice, $\text{ann}(y) \subset \text{ann}(x)$ since $\text{ann}(x) = M$. Thus, $\bar{y} < \bar{x}$. □

Lemma 5.4. *If* $R \cong L_1 \times L_2$, *where* L_1, L_2 *are finite local rings, then* $\Lambda(R)$ *is connected.*

Proof. By Lemma 2.7 and 2.8, we can write the maximal ideals of R as $M_1 \times L_2 = \text{ann}((m_1, 0))$ and $L_1 \times M_2 = \text{ann}((0, m_2))$, where M_i is the maximal ideal of L_i. Since these are the only maximal ideals in R, all other ideals (and therefore all other annihilator sets) are subsets of either $M_1 \times L_2$ or $L_1 \times M_2$. This means that the vertices $(m_1, 0)$ and $(0, m_2)$ are roots of $\Lambda(R)$. To show that $\Lambda(R)$ is connected, we need to show that there exists a vertex x with $x < (m_1, 0)$ and $x < (0, m_2)$. Notice $\text{ann}((m_1, 0)) = \{(l_1, y) \mid m_1 l_1 = 0 \textit{ and } y \in L_2\}$ and $\text{ann}((0, m_2)) = \{(x, l_2) \mid m_2 l_2 = 0 \textit{ and } x \in L_1\}$. Further notice that $\text{ann}((m_1, 1)) = \{(l_1, 0) \mid m_1 l_1 = 0\}$, that $\text{ann}((m_1, 1)) \subsetneq \text{ann}((m_1, 0))$, and that $\text{ann}((m_1, 1)) \subsetneq \text{ann}((0, m_2))$. Thus, $(m_1, 1) < (m_1, 0)$ and $(m_1, 1) < (0, m_2)$, and $\Lambda(R)$ is connected. □

Lemma 5.5. *If* $R \cong L \times F$, *where* L *is a finite local ring and* F *is a finite field, then* $\Lambda(R)$ *is connected.*

Proof. By Lemma 2.7 and 2.8, we can write the maximal ideals of R as $M_1 \times F = \text{ann}((l, 0))$, where $M_1 = \text{ann}(l)$ is the unique maximal ideal of L, and $L \times \{0\} = \text{ann}((0, 1))$. Since these are the only maximal ideals in R, all other ideals (and therefore all other annihilator sets) are subsets of either $M_1 \times F$ or $L \times 0$. Thus, $(l, 0)$ and $(0, 1)$ are the roots of $\Lambda(R)$. Notice that $\text{ann}((l, 0)) = \{(k, y) \mid kl = 0 \text{ and } y \in F\}$ and $\text{ann}((0, 1)) = \{(x, 0) \mid x \in L\}$. In order to show that $\Lambda(R)$ is connected, we need to show the annihilator of some element is a proper subset of the annihilator sets of $(l, 0)$ and $(0, 1)$. Consider $\text{ann}((l, 1)) = \{(k, 0) \mid kl = 0\}$. Obviously, $\text{ann}((l, 1)) \subsetneq \text{ann}((l, 0))$ and $\text{ann}((l, 1)) \subsetneq \text{ann}((0, 1))$; therefore, $(l, 1) < (l, 0)$ and $(l, 1) < (0, 1)$, and $\Lambda(R)$ is connected. □

Lemma 5.6. *If $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are finite commutative rings, then $\Lambda(R)$ is connected.*

Proof. By Lemma 2.7 and 2.8, we can write the maximal ideals of R as $M_1 \times R_2 \times R_3 = \text{ann}((m_1, 0, 0))$, $R_1 \times M_2 \times R_3 = \text{ann}((0, m_2, 0))$, and $R_1 \times R_2 \times M_3 = \text{ann}((0, 0, m_3))$ where $M_i = \text{ann}(m_i)$ for $i = 1, 2, 3$. Notice that $\text{ann}((m_1, 0, 0)) = \{(l_1, y, z) \mid m_1 l_1 = 0,\ y \in R_2,\ z \in R_3\}$, $\text{ann}((0, m_2, 0)) = \{(x, l_2, z) \mid m_2 l_2 = 0,\ x \in R_1,\ z \in R_3\}$, and $\text{ann}((0, 0, m_3)) = \{(x, y, l_3) \mid m_3 l_3 = 0,\ x \in R_1,\ y \in R_2\}$. To show that $\Lambda(R)$ is connected, we will find a vertex whose annihilator set is a subset of each of the maximal ideals. Consider $\text{ann}((m_1, 1, 1)) = \{(l_1, 0, 0) \mid m_1 l_1 = 0\}$. Notice that $\text{ann}((m_1, 1, 1)) \subsetneq \text{ann}((m_1, 0, 0))$, $\text{ann}((m_1, 1, 1)) \subsetneq \text{ann}((0, m_2, 0))$, and $\text{ann}((m_1, 1, 1)) \subsetneq \text{ann}((0, 0, m_3))$. Thus, $(m_1, 1, 1) < (m_1, 0, 0), (0, m_2, 0), (0, 0, m_3)$, and $\Lambda(R)$ is connected.

If R_1 has more than one maximal ideal. We can simply consider the vertex $(1, m_2, 1)$ since $\text{ann}((1, m_2, 1)) \subsetneq \text{ann}((m_1, 0, 0))$ and $\text{ann}((1, m_2, 1)) \subsetneq \text{ann}((m_1', 0, 0))$. Thus, $\Lambda(R)$ is connected. □

Now we prove that the zero-divisor lattice is connected in all but one specific case.

Theorem 5.7. *Let R be a finite commutative ring. Then $\Lambda(R)$ is connected if and only if $R \ncong F_1 \times F_2$ for fields F_1 and F_2.*

Proof. ($\Leftarrow$) Lemmas 5.3, 5.4, 5.5, and 5.6 show that if $R \ncong F_1 \times F_2$, then $\Lambda(R)$ is connected.

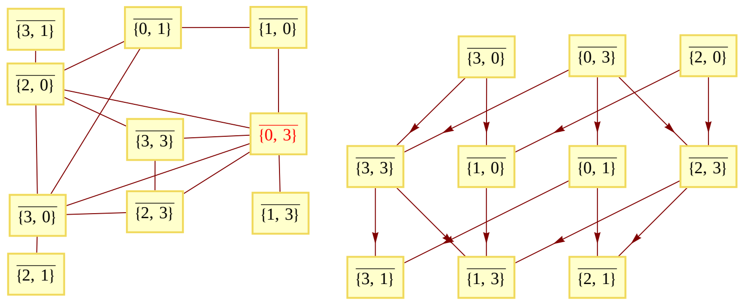
($\Rightarrow$) It suffices to show that if $R \cong F_1 \times F_2$, then $\Lambda(R)$ is disconnected. Let $R \cong F_1 \times F_2$. Then there exists two maximal ideals, namely $F_1 \times 0$ and $0 \times F_2$. Again, each is the annihilator of a single element, call them $(0, 1)$ and $(1, 0)$, respectively. Since these are the only maximal ideals in R, all other ideals (and therefore all other annihilator sets) are subsets of either $F_1 \times 0$ or $0 \times F_2$. To show that $\Lambda(R)$ is not connected, we need to show that there is no vertex whose annihilator set is a subset of both maximal ideals. Notice $\text{ann}((0, 1)) = \{(f, 0) \mid f \in F_1\}$ and $\text{ann}((1, 0)) = \{(0, f) \mid f \in F_2\}$. Further notice that $\text{ann}((0, 1)) \cap \text{ann}((1, 0)) = \{(0, 0)\}$, and the only elements in the ring whose annihilator set is the zero element are units. Since we do not allow units in the zero-divisor lattice, there is no vertex whose annihilator set is a subset of both maximal ideals; thus, $\Lambda(R)$ is disconnected. □

Corollary 5.8. *Let $\mathbb{Z}_{pq}$ be a ring with integers p and q. Then $\Lambda(\mathbb{Z}_{pq})$ is disconnected if and only if p and q are distinct primes.*

Example 5.9. *In $\mathbb{Z}_6 \times \mathbb{Z}_9$, the cut-sets are $\overline{(2, 0)}$, $\overline{(3, 0)}$, and $\overline{(0, 3)}$ in $\Gamma_c(\mathbb{Z}_6 \times \mathbb{Z}_9)$. Notice that these vertices are the roots of $\Lambda(\mathbb{Z}_6 \times \mathbb{Z}_9)$. So, the cut-sets of the graph are easily identifiable by looking at the lattice. Figure 6 shows these graphs.*

6 Conclusion

The hope of studying zero-divisor lattices is that it will be another way to identify the cut-sets of the zero-divisor graph. For example, in some cases of direct products of rings, we can



(a) $\Gamma_c(\mathbb{Z}_6 \times \mathbb{Z}_9)$ (b) $\Lambda(\mathbb{Z}_6 \times \mathbb{Z}_9)$

Figure 6: The compressed zero-divisor graph and zero-divisor lattice of $\mathbb{Z}_6 \times \mathbb{Z}_9$

quickly identify the cut-sets of the compressed zero-divisor graph by looking at the roots of the zero-divisor lattice, as is the case in Example 5.9. For future research we will attempt to create an algorithm that will identify the cut-sets of the zero-divisor graph by using the zero-divisor lattice. The motiviation behind the zero-divisor lattice arose from trying to solve the problem where in a finite local ring, a cut-set union with 0, does not always form an ideal. Notice that in the compressed zero-divisor graph of $\mathbb{Z}_{40}$, Figure 7, there are two cut-sets, $\{\overline{20}\}$ and $\{\overline{8}\}$. Notice further that these are the roots of the zero-divisor lattice of $\mathbb{Z}_{40}$, shown in Figure 8. This also occurs with $\mathbb{Z}_{30}$, as you can check in Figures 1(b) and 2. Because the cut-sets seem to be easily identifiable in the zero-divisor lattice, further research into zero-divisor lattices may shed light upon the problem with the cut-sets in finite local rings.

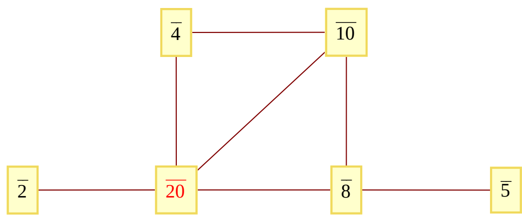


Figure 7: $\Gamma_c(\mathbb{Z}_{40})$

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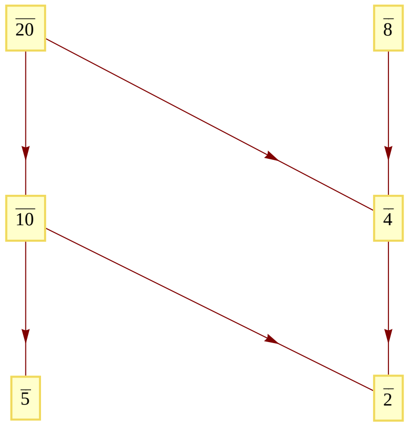

Figure 8: $\Lambda(\mathbb{Z}_{40})$

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