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THE GRAPH THEORY OF BLACKWORK EMBROIDERY

JOSHUA HOLDEN

Abstract. Blackwork is an embroidery technique which, in English-speaking countries, is generally associated with Tudor England. In that period and place, it was generally done with black thread on light colored linen, hence the name. Other distinguishing features include a preponderance of straight lines and geometric shapes, which make it eminently suitable for a mathematical treatment.

1. A Brief History of Blackwork

The origins of blackwork, as with many crafts, can be best described with the cliché "lost in antiquity". Legend has it that the technique was brought to England from Spain around 1501 with the court of Catherine of Aragon, the intended bride of Prince Arthur, heir to the throne of England. When Arthur died a year later, Catherine was betrothed to the new heir, Arthur's younger brother Henry. Henry would later become Henry VIII, and Catherine would become the first of Henry's famous six wives.

This story, while romantic, is certainly false. While techniques which could be called blackwork can be found in Spanish embroidery before 1501, they also appear in England and many other countries before Catherine's time. However, the association with Catherine led to the use of the term "Spanish work" as another term for blackwork, and the association with the queen certainly gave a boost to the popularity of blackwork in England during this period.

Another reason for our association of blackwork with this period of English history lies in the paintings of Hans Holbein the Younger, court painter to Henry VIII. Holbein's paintings are so realistic that the stitching on the sleeves and collars of his subjects, which included Henry and at least three of his wives, are clearly identifiable as blackwork, and in fact can be reproduced by modern stitchers. Holbein's portrait of Jane Seymour ([7], also page 8 of [2]) is a good example of such a painting, and a chart of the pattern on the Queen's cuff can be found on page 105 of [2].

Blackwork dropped out of favor in England during the seventeenth century and other stitching techniques took its place. In the twentieth century blackwork has enjoyed a revival among embroiderers, historical reenactors, interior decorators, and others who enjoy its combination of art, technical challenge, and intellectual creativity. Modern stitchers can choose from a large number of patterns, both those reproduced from period examples (either from paintings or artifacts) and newly created modern designs. Modern patterns may follow the traditional colorways of black, or sometimes red, thread on a light-colored background. Or, they may use other colors including light thread on a black or dark background. Many examples

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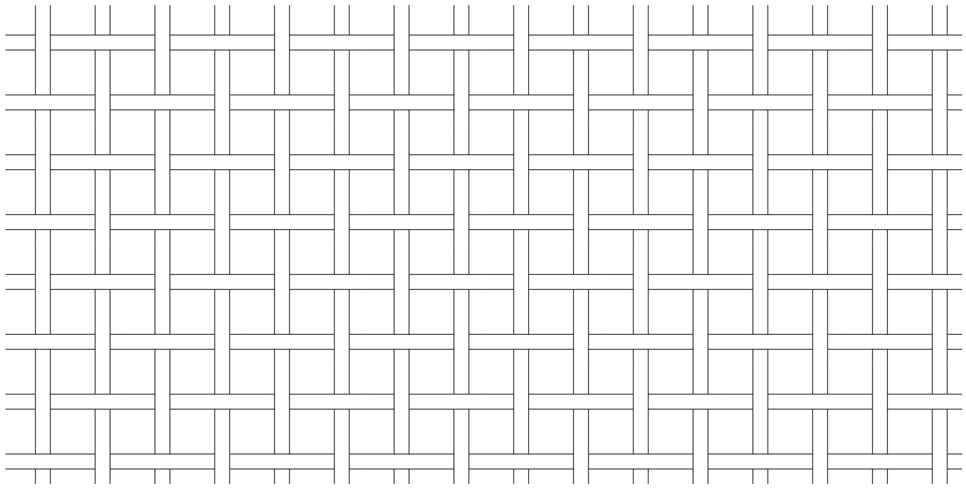

Figure 1. A typical piece of fabric used in blackwork

of each may be found in [1]. For more on the history and practice of blackwork, see [9] and the references in her bibliography.

2. How to Stitch Blackwork

In addition to the high-contrast color schemes and straight-line geometric patterns, there are specific methods of stitching which are characteristic of blackwork. Firstly, the linen or other cloth used (especially by modern stitchers) is selected for a regular grid of threads and holes, as can be seen in Figure 1. Unlike in more free-form methods of sewing and embroidery, the thread will only pass through these holes as it travels across the fabric.

Although any method of stitching which produces an unbroken straight line could be used for blackwork, the most traditional is the "double running stitch", also called "Holbein stitch" due to the association with Holbein's paintings. In double running stitch, the thread is brought up and down through the fabric while traveling in one direction ("running stitch"), as shown in Figure 2. (The numbers indicate in which order the thread should pass through the holes.) This produces a broken line on the front of the fabric and a complementary broken line on the back. Then the direction is reversed and the line is retraced in the other direction ("double running stitch"), producing identical continuous lines on the front and back of the fabric, as in Figure 3. Branching paths can also be made by taking "side trips", as shown in Figure 4.

If the patch of the stitching is exactly reversed, then the finished stitching will be "reversible", having the same appearance from either side of the fabric. (Technically the pattern will be a mirror image, of course, but we will ignore this for the moment.) This is one of the reasons blackwork was commonly used for sleeve cuffs and collars historically. It is also passible to use running stitch to produce pieces which have a different pattern on the front than on the back, or the same pattern but shifted by some number of threads. Or, if the back of the piece will never be seen, it may not be considered important to preserve any pattern at all on the back side. In this article, however, we will be interested primarily in "reversible" patterns stitched

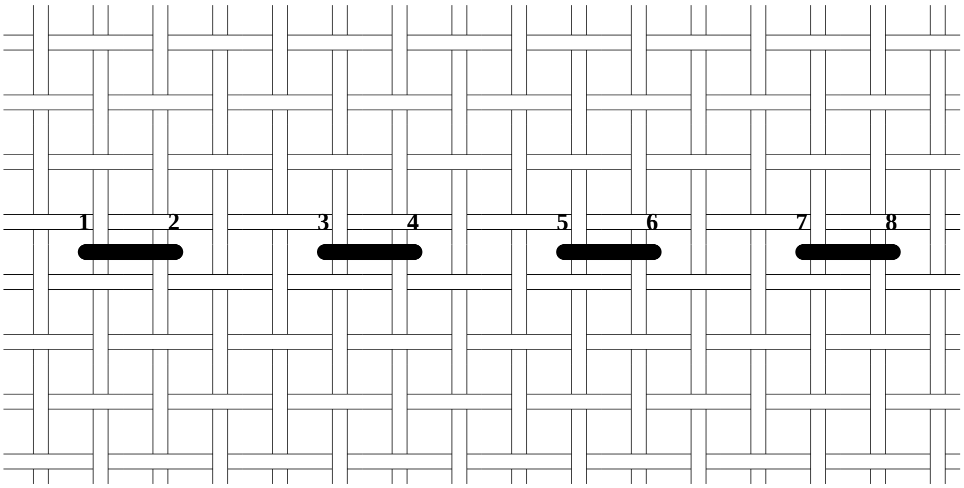

Figure 2. Running stitch

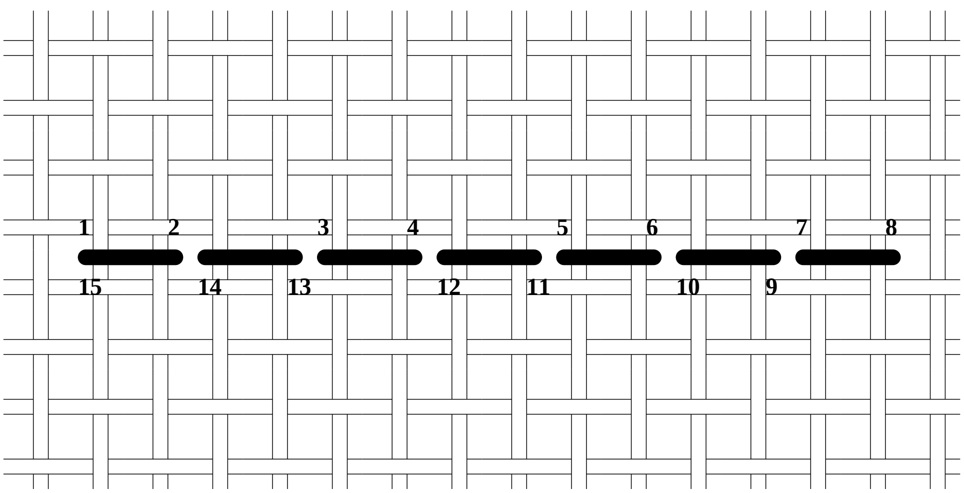

Figure 3. Double running stitch

using double running stitch. For more on the stitches used in blackwork, see, for example, [2].

3. Eulerian Graphs

With that introduction out of the way, we can embark on some mathematical analysis of blackwork embroidery. For us, the appearance of a blackwork pattern immediately brought to mind the mathematical concept of a graph. Unless noted otherwise, the definitions we need are more or less standard, but we present them as reminders and to fix the notation. (See for example, [4, Chap. 11] or [5] for more information.)

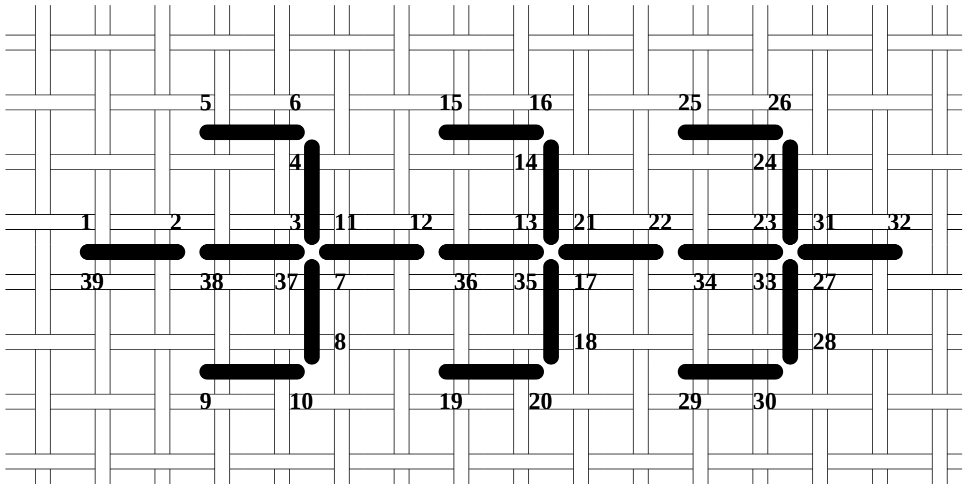

FIGURE 4. Double running stitch with "side trips"

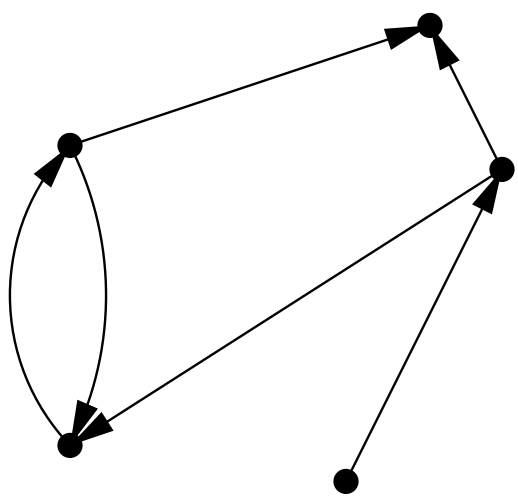

FIGURE 5. An example digraph

A *(loop-free) graph* is a set of vertices, V, and a set of edges, E, where each edge is an **unordered** pair of distinct vertices. A *(loop-free) directed graph*, or *digraph*, is a set of vertices, V, and a set of edges, E, where each edge is an **ordered** pair of distinct vertices. In a digraph, the order of vertices in each edge is often thought of as indicating a "direction" of travel, as in Figure 5. It should be clear that a graph may be associated with any digraph by simply forgetting about the ordering of the pairs. Of course, such a graph may come from any of several different digraphs. However, the standard way of associating a digraph to a graph is by including in the digraph **both** possible orders (or directions) of each edge, as indicated in Figure 6.

We will think of our blackwork pattern as a digraph, where the direction of the edge is the direction of stitching. Note that every other edge will end up on the back of the fabric, as was shown in Figure 3. Thus each pair of vertices must have two

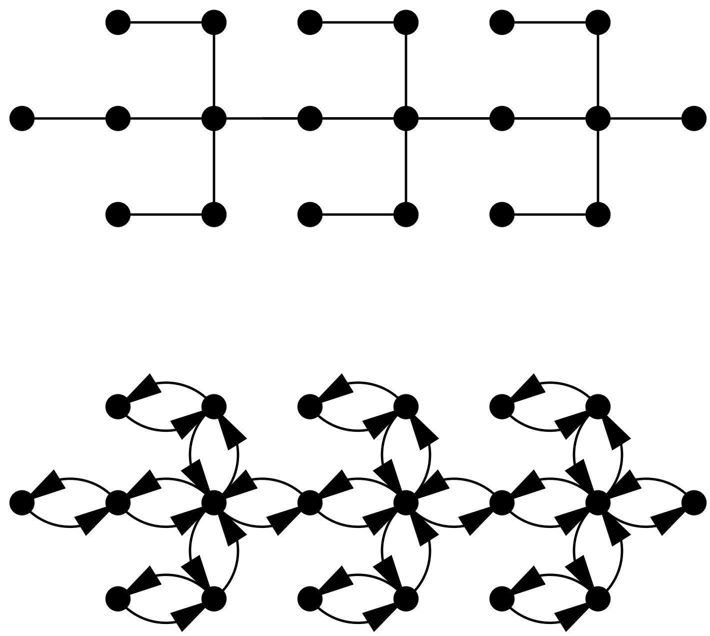

FIGURE 6. A graph and its associated digraph

edges between them, one in each direction. Clearly a reversible blackwork pattern is a digraph that comes from a graph — each edge is traveled in one direction on the front of the fabric and the other direction on the back. (In fact, the digraph in Figure 6 is the same as the pattern in Figure 4.)

Of course, this implies that the edges are traversed in some sort of order in the graph, as they are when one stitches. A *walk* on a graph is a finite alternating sequence of vertices and edges $x_0, e_1, x_1, \ldots, e_n, x_n$ where each $e_i = \{x_{i-1}, x_i\}$. A graph is *connected* if there is a walk between any two vertices. A *directed walk* on a digraph is a finite alternating sequence of vertices and edges $x_0, e_1, x_1, \ldots, e_n, x_n$ where each $e_i = (x_{i-1}, x_i)$. A digraph is *strongly connected* if there is a directed walk between any two vertices.

The proofs of the following two lemmas should then be clear:

Lemma 1. *If a digraph is strongly connected then the associated graph is connected.*

Lemma 2. *If a graph is connected then the associated digraph is strongly connected.*

If we can stitch a reversible blackwork pattern with one thread then the pattern, regarded as a digraph, must be strongly connected. Our interest will be to see if all patterns which are strongly connected digraphs can be stitched reversibly with one thread.

It is usually considered aesthetically undesirable (i.e. ugly) to have more than one thread passing between the same two holes on the same side of the fabric. So we need to keep track of whether edges are repeated in our directed walks. A *directed trail* is a directed walk where no edge is repeated. (Although vertices may be.) A *directed circuit* is a directed trail where $x_0 = x_n$. An *Eulerian circuit* of a digraph is a directed circuit which includes all of the edges of the digraph. A digraph is *Eulerian* if it has an Eulerian circuit.

Clearly if we can stitch a blackwork pattern reversibly (with one thread) then the digraph must be Eulerian. (Of course, being Eulerian implies being strongly connected.) Now, it is well known which digraphs are Eulerian. For each vertex v of a digraph, the *out degree* of v is the number of edges with v as the starting point. For each vertex v of a digraph, the *in degree* of v is the number of edges with v as the ending point.

Theorem 3. *A directed graph is Eulerian if and only if it is strongly connected and the in degree of each vertex is equal to the out degree.*

This is the digraph version of Euler and Hierholzer's solution to the Seven Bridges of Königsberg. A proof of the undirected graph version can be found in [4, Sec. 11.3]; a somewhat different proof is given in [6, Chap. 7]. Adapting these to the directed graph version is left as an exercise.

In our situation we can apply the following lemma:

Lemma 4. *The digraph associated to a graph has the in degree of each vertex equal to the out degree.*

We then get a nice corollary to the theorem:

Corollary 5. *The digraph associated to a graph is Eulerian if and only if the original graph is connected.*

So although we showed that a necessary condition for being able to stitch a blackwork pattern reversibly is that the digraph must be Eulerian, it turns out that this is not much of a condition.

4. Holbeinian Graphs

But this is not the end of the story! We have determined how to make sure each edge in the graph is traversed exactly twice, in two different directions. But we must make sure that the two different directions of travel lie on opposite sides of the fabric. For that we need some more definitions. The first two are a little less standard than the previous ones but still not particularly unusual. A digraph is *symmetric* for every edge (x, y) in the digraph, the edge (y, x) is also in the digraph. Equivalently, a digraph is symmetric if it is the digraph associated to some graph. If $x_0, e_1, x_1, \ldots, e_n, x_n$ is a directed trail on a digraph, we say that the *parity* of each edge e_i is the parity of i.

We have not been able to find the next two definitions in the literature, so we have taken the liberty of inventing names for them. The reader should note the analogy with the Eulerian case.

Definition 6. A *Holbeinian circuit* of a symmetric digraph is an Eulerian circuit where all edges (x, y) and (y, x) have opposite parities.

Definition 7. A symmetric digraph is *Holbeinian* if it has an Holbeinian circuit.

The parity of the edge corresponds to whether the thread is on the front side or the back side of the fabric, and a blackwork pattern can be stitched reversibly with one thread if and only if the corresponding digraph is Holbeinian. (Surprise!) So can we categorize Holbeinian digraphs?

Theorem 8. *A symmetric digraph is Holbeinian if and only if it is strongly connected.*

Corollary 9. *A symmetric digraph is Holbeinian if and only if the associated graph is connected.*

So it turns out that every connected blackwork pattern can be stitched reversibly with one thread, a fact which will come as no surprise whatsoever to any stitcher who has tried. Nevertheless, we think that the proof of the theorem is interesting and perhaps instructive. It is closely based on the same theorem for the Eulerian case (see, for example, [4, Thm. 11.3]) with the addition of the parity condition. (An earlier proof of this theorem was proposed by Lana Holden. That proof is somewhat more algorithmic in nature and we hope to present it in a future paper dealing with algorithms and blackwork.)

Proof of Theorem 8. Let G be the symmetric digraph. If G has a Holbeinian circuit, then it must pass through every vertex. Thus for any two vertices, there is a directed walk from one to the other, so G is strongly connected.

Now assume G is strongly connected, and let G^u be the associated graph. Let V the vertices of G^u, and E the edges. If E has only one edge $\{x, y\}$, then $x, (x, y), y, (y, x), x$ is a Holbeinian circuit of G.

By way of induction, suppose the result is true if E has less than n edges and that E has n edges. Let $\{x, y\}$ be any edge of E. Let H be the digraph obtained from G by removing the edges (x, y) and (y, x), and also either vertex x or y if there are no other edges going in or out. Now either H is still strongly connected, or it is divided into two strongly connected pieces, one of which contains x and the other y.

If H is still strongly connected, it has a Holbeinian circuit. A Holbeinian circuit of G is constructed by following the Holbeinian circuit of H until the vertex x is reached, then following $x, (x, y), y, (y, x), x$, then finishing the Holbeinian circuit of H. The detour of length 2 does not affect the parity of any of the edges in the circuit of H.

If H is no longer strongly connected, each piece has a Holbeinian circuit. Let the pieces be H_1 and H_2, with associated graphs H_1^u and H_2^u and associated undirected edge sets E_1 and E_2. A Holbeinian circuit of G is constructed by following the Holbeinian circuit of H_1 until the vertex x is reached, then following $x, (x, y), y$, then following the Holbeinian circuit of H_2 back to y, then following $y, (y, x), x$, and finally finishing the Holbeinian circuit of H. The detour of length $2 + 2|E_2|$ does not affect the parity of any of the edges in the Holbeinian circuit of H_1.

□

The reader might be interested to know that the idea of an Eulerian circuit of the symmetric digraph associated with a graph is well-studied; [3] refers to it as a *bidirectional double tracing*. Only the idea of parity is missing. The following theorem was apparently first proved by König, in [8, Thm. II.9]:

Theorem 10. *A graph has a bidirectional double tracing if and only if it is connected.*

Graphs exist where not all bidirectional double tracings are Holbeinian; the reader might enjoy attempting to construct one. There are also some classical algorithms for constructing bidirectional double tracings; Chapter III of [8] is a good reference. Some of these algorithms necessarily produce Holbeinian paths and some do not; we hope a future paper will explore this.

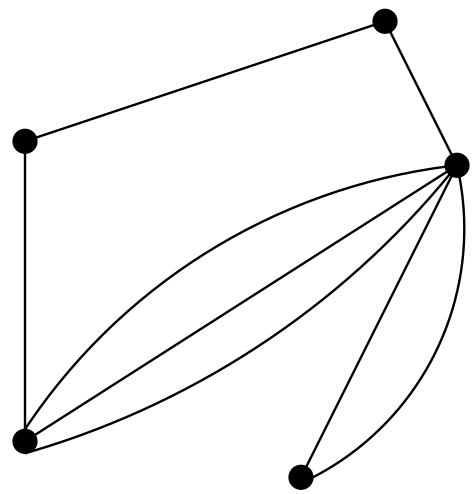

FIGURE 7. An example multigraph

5. Eulerian Multigraphs

Now, the reader may be wondering whether it's really necessary for the the two threads along each undirected edge to travel in opposite directions, as long as they lie on opposite sides of the fabric. After all, in most circumstances it's impossible to tell by looking which direction a thread was stitched in. Clearly relaxing this restriction doesn't gain us anything, because we can already stitch any pattern. But what if we require that the stitches *always* go in the same direction? We could call this "stitching unidirectionally". (We will assume that we are still requiring "reversibility" in the sense that the threads lying on the back of the fabric are in the same places as those in the front.)

In order to keep track of multiple edges going in the same direction, we need something a little different than a graph. A *(loop-free) multigraph* is a set of vertices, V, and a multiset of edges, E, where each edge is an unordered pair of distinct vertices. A pair of vertices has *multiplicity* n if there are exactly n edges between them. If n is a positive integer, a multigraph has *uniform multiplicity* n (or is n*-uniform*) if every pair of vertices which has an edge between them has exactly n edges.

The idea is that a multigraph is just like a graph except that, since a multiset have have more than one indistinguishable copy of each element, a pair of vertices in a multigraph may have more than one indistinguishable edge between them, as in Figure 7. We could now think of a blackwork pattern as a 2-uniform multigraph, letting us keep track of the number of times each edge has been stitched. (Hopefully, that will come out to be once for the front of the fabric and once for the back!)

A graph may be associated with any multigraph by simply forgetting about any edges except one between each pair of vertices. As in the situation with digraphs, a graph may come from any number of different multigraphs. However, we can uniquely associate a multigraph of uniform multiplicity n to a graph by including in the multigraph n edges between each pair of vertices which has an edge in the original graph, as indicated in Figure 7.

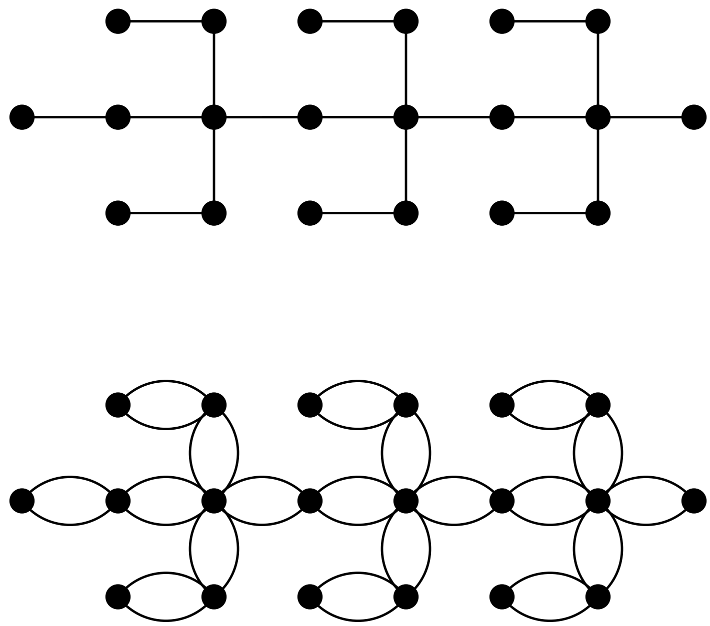

FIGURE 8. A graph and its associated multigraph of uniform multiplicity 2

The reader may have noticed that we have not assigned any directions to the edges in our multigraph; it turns out that the proof is easier that way. This does, however, require a few modifications to some of our earlier definitions. (These are all still quite standard.) A *walk* on a multigraph is a finite alternating sequence of vertices and edges $x_0, e_1, x_1, \ldots, e_n, x_n$ where each $e_i = \{x_{i-1}, x_i\}$. (This is not really different from a walk on a graph.) A multigraph is *connected* if there is a walk between any two vertices.

And of course, we have:

Lemma 11. *If a multigraph is connected then the associated graph is connected.*

Lemma 12. *If a graph is connected then the associated n-uniform multigraph is connected.*

We still need to make sure that we don't stitch an edge too many times, so we define a *trail* in a multigraph to be a walk where no edge is repeated more times than the multiplicity of that edge. A *circuit* in a multigraph is a trail where $x_0 = x_n$. Finally, an *Eulerian circuit* of a multigraph is a circuit which includes all of the edges of the multigraph exactly as many times as the multiplicity of each, and a multigraph is *Eulerian* if it has an Eulerian circuit.

If we can stitch a blackwork pattern unidirectionally (with one thread) then the multigraph must be Eulerian. (Again in this case, being Eulerian implies being connected.) It is again well-known which multigraphs are Eulerian — this actually is the original Seven Bridges of Königsberg Theorem. (Only the "only if" part was rigorously proved by Euler, although he gave a heuristic reason why the "if" part should be true. A valid proof of the "if" part had to wait for Hierholzer over a hundred years later! See [11] for more details and references.) For each vertex v of a multigraph, the *degree* of v is the number of edges with v as one endpoint.

Theorem 13 (Euler (1736); Hierholzer (1873))**.** *A multigraph is Eulerian if and only if it is connected and the degree of each vertex is even.*

6. Aragonian Graphs

So now we just need to keep track of the direction of the stitching and whether the threads lie on the correct side of the fabric. If $x_0, e_1, x_1, \ldots, e_n, x_n$ is a trail on a multigraph, we say that the *parity* of each edge e_i is the parity of i. (Technically, this definition is an abuse of notation since it may require us to distinguish between the multiple copies of each edge, which are supposed to be indistinguishable. But it shouldn't present a problem.)

The next two definitions are again analogous with the Eulerian and Holbeinian case, and we have continued to name them somewhat whimsically.

Definition 14. An *Aragonian circuit* of a multigraph of uniform multiplicity 2 is an Eulerian circuit where the two edges $\{x, y\}$ are both traversed in the order $x, \{x, y\}, y$ and are traversed with opposite parities.

Definition 15. A multigraph of multiplicity 2 is *Aragonian* if it has an Aragonian circuit.

A blackwork pattern can be stitched reversibly with one thread with stitches on the front and back going in the same direction if and only if it is Aragonian. We can characterize Aragonian multigraphs almost as easily as Holbeinian ones. (The reader might try to think about how to do this before reading on. When the authors first tried, they initially came up with conditions that were considerably stronger than necessary!)

Theorem 16. *A multigraph of uniform multiplicity* 2 *is Aragonian if and only if it is connected, it has a circuit of odd length, and every vertex has degree divisible by* 4.

Corollary 17. *A multigraph of uniform multiplicity* 2 *is Aragonian if and only if the associated graph is connected, has a circuit of odd length, and every vertex has even degree.*

Corollary 18. *A multigraph of uniform multiplicity* 2 *is Aragonian if and only if the associated graph is Eulerian and has a circuit of odd length.*

This time, the proof is virtually the same as that for Eulerian circuits on undirected graphs (again, see [4, Sec. 11.3] or [6, Chap. 7]), with the addition of parity.

Proof of Theorem 16. Let G be the multigraph. If G has an Aragonian circuit, then it must pass through every vertex. Thus for any two vertices, there is a walk from one to the other, so G is connected.

For any vertex v of G other than the starting vertex, each time the circuit comes to v then it departs from v. Thus the circuit has traversed two (new) edges with v as an endpoint. Since it must do this twice for each edge, the degree of v must be a multiple of 4. (For the starting vertex, consider the first edge of the circuit and the last edge.)

To show that G has a circuit of odd length, consider the first edge $\{x, y\}$ to appear twice in the Aragonian circuit. Since it must appear with opposite parities each time, the part of the circuit which includes the first occurrence of $\{x, y\}$ and stops just short of the second occurrence of $\{x, y\}$ will be a circuit of odd length.

Now assume G is connected, has a circuit of odd length, and every vertex has degree divisible by 4. Let V the vertices of G, and E the edges. If E has only three distinct edges $\{x, y\}$, $\{y, z\}$, $\{z, x\}$ then

$$x, \{x, y\}, y, \{y, z\}, z, \{z, x\}, x, \{x, y\}, y, \{y, z\}, z, \{z, x\}, x$$

is an Aragonian circuit of G.

By way of induction, suppose the result is true if E has less than n edges and that E has n edges. We know that G has a circuit of odd length; call if C. If the circuit contains a copy of every edge in E, then we are done; simply follow the circuit C twice. If not, let K be the sub-multigraph of G obtained by removing the edges in C, and also any vertices which are left without edges going in or out. K has all vertices of degree a multiple of 4, but it may not still be connected. However, each component of K is connected and its associated graph has an Eulerian circuit.

We construct an Aragonian circuit on K by following the circuit C until we arrive at a vertex x_1 which is on the Euler circuit of a component of K. If K has an odd number of edges, we follow this Eulerian circuit twice. Otherwise we follow it once. In either case, the detours do not change the parities of the edges of C.

When we arrive back at the first edge of C, we have switched parities, since C has an odd number of edges. We follow C again, this time only traversing the even components of K. Once again, the detours do not change the parities.

□

Once again, the idea of an Eulerian circuit a multigraph of uniform multiplicity 2 has been studied; [3] refers to it as a *double tracing*. Reference is also made to double tracings where the two edges $\{x, y\}$ are traversed in the same direction, although no special terminology is given, possibly because conditions on the existence of such tracings are fairly trivial; if you can trace a graph once, you can trace it any number of times.

Lemma 19. *A multigraph of uniform multiplicity* 2 *has a double tracing which traverses the two edges* $\{x, y\}$ *in the same direction if and only if the associated graph is Eulerian.*

Again, this is the same as an Aragonian graph except for the idea of parity. Note, however, that there are graphs which can be traversed in the fashion described in the lemma but which are not Aragonian!

On the other hand, Vestergaard proved the following in [10] (the terminology is that of [3]):

Theorem 20. *Let G be a connected multigraph and E_0 be a subset of E, the edges of G. G has a double tracing which traverses the edges of E_0 in each direction and the edges of $E \setminus E_0$ in the same direction if and only if the degree of each vertex of $G \setminus E_0$ is even.*

It would be interesting to know when a graph can be traversed such that the edges are always traversed twice with opposite parities, but some edges are specified as "Holbeinian" and others as "Aragonian".

7. Generalizations

One obvious way to generalize the theorems in this paper would be to replace the two "sides of the fabric" with a larger number. This does not make a lot of sense from the point of view of stitching, but in terms of abstract graph theory it seems perfectly reasonable.

Theorem 16 is fairly easy to generalize in this way. Let k be a positive integer which represents the number of "sides". If $x_0, e_1, x_1, \ldots, e_n, x_n$ is a trail on a multigraph, we say that the *residue class* of each edge e_i is the residue class of i modulo k.

Definition 21. An *k-Aragonian circuit* of a multigraph of uniform multiplicity k is an Eulerian circuit where the k edges $\{x, y\}$ are all traversed in the order $x, \{x, y\}, y$ and are traversed with distinct residue classes.

Definition 22. A multigraph of multiplicity k is *k-Aragonian* if it has a k-Aragonian circuit.

Then the proofs of the following are essentially the same as in the $k = 2$ case.

Theorem 23. *A multigraph of uniform multiplicity k is k-Aragonian if and only if it is connected, it has a circuit of length relatively prime to k, and every vertex has degree divisible by $2k$.*

Corollary 24. *A multigraph of uniform multiplicity k is k-Aragonian if and only if the associated graph is connected, has a circuit of length relatively prime to k, and every vertex has even degree.*

Corollary 25. *A multigraph of uniform multiplicity k is k-Aragonian if and only if the associated graph is Eulerian and has a circuit of length relatively prime to k.*

Generalizing Theorem 8 in this way is more problematic, mainly because the concept of a directed graph doesn't generalize obviously to more "directions". The problem is not so much in the proofs of the theorems, but in formulating definitions that are both intuitively reasonable and useful. One would certainly want to start with the idea of a "hypergraph", where an "edge" may have more than two vertices. We will leave the details for the advanced reader; once the correct definitions are formulated the proofs should be easy.

Happy stitching!

Acknowledgements

The author would like to especially thank Lana Holden, not only for much advice on stitching but also for help with the original mathematical formulation of the Holbeinian graph problem and for the original proof of Theorem 8.

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