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Abstract

The problem of kaleidoscopically tiling a surface by congruent triangles is equivalent to finding groups generated in certain ways. In order to admit a tiling, a group must have a specific set of generators as well as an involutary automorphism, θ, that acts to reverse the orientation of the tiles. The purpose of this paper is to explore group theoretic and computational methods for determining the existence of symmetry groups and tiling groups, as well as to classify the symmetry and tiling groups on hyperbolic Riemann surfaces of genus 6 and 7.

Contents

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1 Introduction

A surface is a topological space in which every point is contained in a neighborhood topologically equivalent to a plane. The genus of an compact, orientable surface is the number of "holes" in the surface. We will denote the genus by σ. A sphere has genus zero, a torus has genus one, and a surface consisting of n connected tori has genus n. See Figures 1 and 2 for pictures of a sphere and torus.

In these figures we have examples of tilings, i.e., non-overlapping coverings of the surface by polygons. The purpose of this paper is to classify kaleidoscopic, geodesic tilings (definition to follow) of surfaces of genus six and seven with congruent, hyperbolic triangles. We will assume that our surfaces have a metric of constant curvature, and that, with respect to this metric, the edges are geodesic segments, i.e., the shortest path between the endpoints. A closed geodesic will then be some closed loop consisting of smoothly connected geodesic segments. On a sphere, for example, the geodesics are great circles.

Definition 1 *A tiling is* kaleidoscopic *if each edge of the tiling is part of a geodesic on the surface such that there is a mirror reflection of the surface across the geodesic. The set of fixed points of the reflection is called the* mirror *of the reflection.*

The mirror of a reflection across an edge is a union of disjoint circles one of which contains the given edge. Furthermore, the triangles in the tiling are all mirror images of each other along an edge, and all the angles meeting at a vertex have the same measure.

Definition 2 *A tiling is* geodesic *if for each edge of the tiling, the mirror of the reflection is a union of edges of the tiling.*

The geodesic condition ensures that at any vertex, there is an even number of triangles. Otherwise, some geodesic would end at the vertex. These properties can be seen in the icosahedral tiling of the sphere shown in Figure 1.

Most triangular tilings on surfaces of genera less than six have already been classified. Broughton [2] classified all rotation groups in genus two and three, and Vinroot [7] classified most tilings by triangles on surfaces of genus four and five. Our main results are the tables in Section 9 where the tilings are classified by the tiling group. We have determined many, though not all, tiling groups for genus six and seven and found missing cases in [2] and [7]. We develop the material presenting these tables in the following sections. In Section 2 we introduce the notion of tiling groups and review the scheme for classifying tilings by tiling groups. A part of the classification problem requires finding an automorphism, θ, of a particular type of the automorphism group of a surface. Methods for finding θ are developed in Section 3. In the remaining sections an approach to the classification is described, along with necessary group theoretic material and some sample calculations. Some suggestions for further research are given in Section 8.

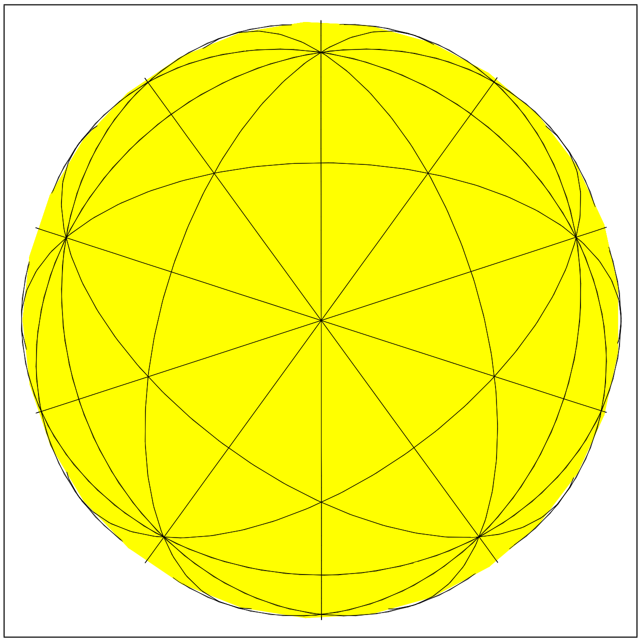

Figure 1. Icosahedral tiling - top view

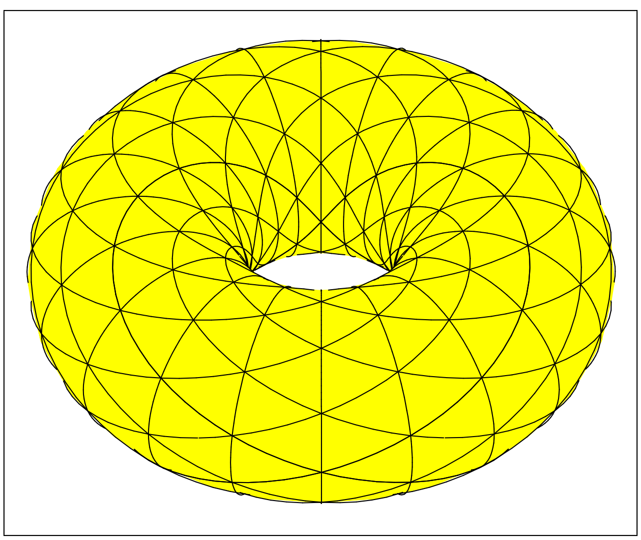

Figure 1. Icosahedral tiling - top view

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2 Rotation and tiling groups

In order to better understand the structure of a tiling, it is helpful to consider tilings from a group theoretic perspective. From the definition of kaleidoscopic tiling, the group G^* generated by reflections in the sides of tiles is a (tiling preserving) group of isometries of the surface, called the *tiling group.* We can view the tiling as arising from the group action of G^* in which one distinguished tile, called the master tile, is reflected across bounding geodesics to create the entire surface. Thus, the master tile can be carried to every other tile on the surface via the action of G^*, through repeated reflection. This is illustrated in the icosahedral tiling of the sphere shown in Figure 1. Now consider rotations about the vertices in the tiling. Through iterated rotations of the surface around the vertices of tiles, any triangle can by carried to any other triangle of the same orientation. These rotations comprise a subgroup of isometries G of the surface called the *orientation-preserving tiling group* (OP tiling group) or more simply a rotation group since it is generated by rotations. If we include a single reflection along with the rotations, allowing tiles with reversed orientation, we generate the entire tiling group.

We would like find specific generators of G^* and G. Let us consider the master tile, $\triangle RPQ$, as shown in Figure 3. Note that the triangle has curved edges to suggest it is hyperbolic. Let p, q, and r be reflections across their respective edges, and define $a = pq$, $b = qr$, and $c = rp$. Reflecting across two different lines is equivalent to rotating about the point of intersection of the lines of reflection. More specifically, it is a rotation by twice the angle of the intersection. Therefore, a, b, and c are simply rotations about the vertices of $\triangle RPQ$ by twice the angles of the triangles. If we define the angle measures of our triangle to be $\frac{\pi}{l}$, $\frac{\pi}{m}$, and $\frac{\pi}{n}$, then a, b, and c are counterclockwise rotations about R, P, and Q through angles of $\frac{2\pi}{l}$, $\frac{2\pi}{m}$, and $\frac{2\pi}{n}$ respectively. If we repeat the rotation a of our master tile l times, we will have rotated by 2π. Thus, we will be back at our master tile. Using similar arguments with b and c, it follows that

$$a^l = b^m = c^n = 1. \tag{2.1}$$

Also, since reflections have order two,

$$abc = pqqrrp = 1. \tag{2.2}$$

It is not to difficult to show that G^* is generated by reflections in the sides of the master tile, so that $G^* = \langle p, q, r \rangle$. The rotation group G, consists of the orientation-preserving elements of G^*. Therefore, each element is a product of an even number of reflections and hence a word in a, b, and c. It follows that $G = \langle a, b, c \rangle$ and G is a subgroup of index two, and is therefore normal in G^*. Since $abc = 1$, it immediately follows that $G = \langle a, b \rangle$.

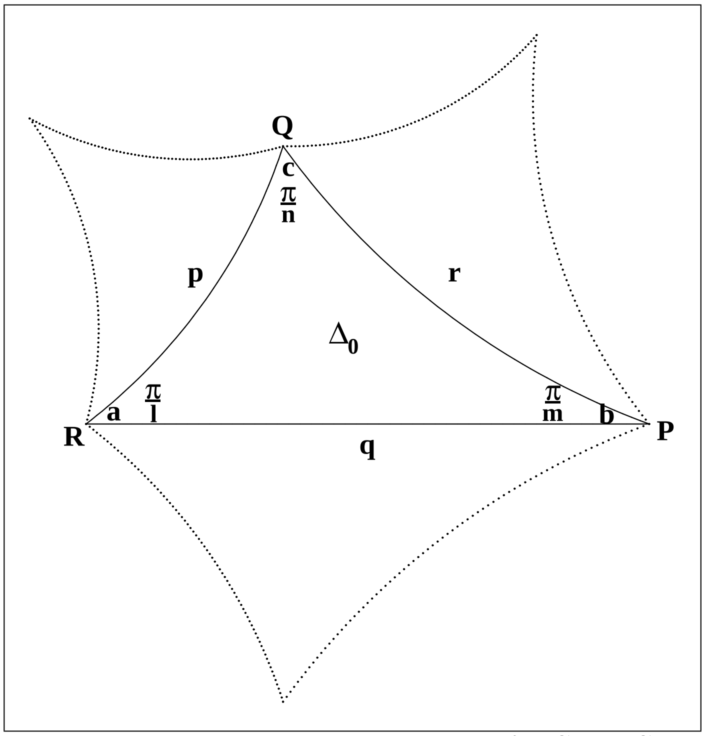

Figure 3. The master tile and generators for G and G^*

Remark 3 *It is possible to construct a group $G = \langle a, b, c \rangle$from rotations satisfying 2.1 and 2.2 with out necessarily coming from a triangular tiling. To allow for this possibility we will call G a rotation group of the surface if we do not want to assume that G is an OP-tiling group.*

When searching for tiling groups, it is often easier to find the rotation group G rather than directly looking for G^*. The following equation called the Riemann-Hurwitz equation,

$$\frac{2\sigma - 2}{|G|} = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \right), \tag{2.3}$$

as well as the relations in equations 2.1 and 2.2 aid in searching for rotation groups. The Riemann-Hurwitz equation, which can be proven using the Euler characteristic, relates the order of the rotation group, the genus, and the angles of the triangle (or orders of rotations). With this equation, we can create a list of all possible group orders and (l, m, n)-triples for a given genus. We call this list the branching data.

Using the Riemann-Hurwitz formula, we can see that if $\sigma \geq 2$, then the sum of the interior angles of the master tile must be less than π. Thus, we see that the underlying geometry must be hyperbolic. Since our geometry is hyperbolic, any two triangles with the same angle measures are congruent. We will call a triangle with angle measures $\frac{\pi}{l}$, $\frac{\pi}{m}$, and $\frac{\pi}{n}$ a (l, m, n)-triangle. If there is a triple (a, b, c) of elements from G which satisfy equations 2.1, 2.2, and 2.3, then this triple is called a *generating* (l, m, n)*-triple*.

Given a rotation group G, we wish to know if G is a subgroup of some tiling group on the surface or if it is just a group action on the surface that does not admit a tiling. We will see that the existence of the tiling group is intimately entwined with the existence of an automorphism of the orientation-preserving rotation group on the surface, arising from the reflections in the edges of tiles. We discuss this automorphism θ in the next section.

3 The elusive theta

Given a tiling group $G^* = \langle p, q, r \rangle$, we can define an automorphism θ of the corresponding rotation group G by conjugation by q

$$\begin{aligned} \theta(a) &= q^{-1}aq = qaq = qpqq = qp = a^{-1}, \\ \theta(b) &= q^{-1}bq = qbq = qqrq = rq = b^{-1}. \end{aligned} \tag{3.4}$$

Suppose we know that a rotation group G exists for a surface. How can we determine if a tiling group exists as well? The answer to this intriguing question lies with the automorphism θ. The existence of a θ implies the existence of a reflection q by the following theorem.

Theorem 4 *Let G have a generating (l, m, n)-triple and suppose that the quantity σ defined by 2.3 is an integer. Then there is always a surface of genus σ with an orientation preserving G-action. If, in addition, there is an involutary automorphism θ, i.e. $\theta^2 = id$, of G satisfying 3.4, then the surface has a tiling by (l, m, n)-triangles such that the orientation preserving tiling group as constructed above is the original G, and $G^* \cong \langle\theta\rangle \ltimes G$.*

Two Methods for Finding Theta One method for finding θ is shown in Ryan Vinroot's technical report [7]. This method involves embedding G in $S_{|G|}$ by finding the left regular representation of G using Cayley's Theorem [4]. Once this has been accomplished, we can use Magma to compute the normalizer of G in $S_{|G|}$. Any automorphism of G is equivalent to conjugation of G by some element in its normalizer. Therefore, we can search through the normalizer of G to find θ. If no θ satisfying equation 3.4 is found in the normalizer, then no tiling group exists. Unfortunately, as $|G|$ increases, $S_{|G|}$ increases factorially, making it extremely time-consuming to find the normalizer. For groups of order under

50, this method works well; however, for larger groups, this method becomes inefficient. In order to find θ for larger groups in a reasonable amount of time, a faster method is needed.

The second method uses a more direct construction for finding θ. We know from 3.4 that

$$\begin{aligned} \theta(a) &= a^{-1} \\ \theta(b) &= b^{-1}. \end{aligned} \tag{3.5}$$

Also, we know that a and b generate G. Accordingly, we can iteratively generate all elements g in G as a combination of a's and b's as follows. We start with a set containing 1, a and b. Then we multiply each element of the set on the left by a and b separately, adding the new elements to the set. For example the second set would be $\{1, a, b, a^2, ab, ba, b^2\}$. We repeat this process until we have created the entire group. To attempt to create θ we use a similar process to create all ordered pairs $(g, \theta(g))$, where g varies in G. This time start with the set $\{(1,1), (a, a^{-1}), (b, b^{-1})\} = \{(1, \theta(1), (a, \theta(a)), (b, \theta(b))\}$. Then at each iteration we select a pair (g, g') from the set and add $(ag, a^{-1}g')$ and $(bg, b^{-1}g')$. If θ exists then $g' = \theta(g)$ and so $\theta(ag) = \theta(a)\theta(g) = a^{-1}g'$. Thus we are certainly adding the right elements. We quit when we have $|G|$ pairs. Next, we check that this collection of ordered pairs defines a bijection from G to G by checking that the set of all the first elements in the ordered pairs equals G and the set of all the second elements equals G.

In order to show this bijection is an automorphism, we need to show that it satisfies the homomorphism condition. The simplest way to check this condition is to directly multiply each pair of elements in the group. With, $|G|^2$ calculations, we can verify that $\theta(ab) = \theta(a)\theta(b)$ for all a and b in G. Another way to show that the bijection is an automorphism is to construct θ in $S_{|G|}$. To construct θ, we need the ordering of the group G used in creating the left regular representation of G in $S_{|G|}$. Let $\#g$ be the index of g in G. We then construct the permutation element where $\#g$ is sent to $\#\theta(g)$ for all g in G. If this permutation normalizes G in $S_{|G|}$, then conjugation by this permutation element is our automorphism θ of G. The precise statement we need is the following.

Proposition 5 *Let G be a group and let $\theta : G \to G$ be a bijection satisfying $\theta(1) = 1$. For $g \in G$ let $L_g : G \to G$ be the left multiplication operator $L_g(x) = gx$. Then θ is an automorphism of G if and only if θ normalizes the subgroup $\{L_g : g \in G\}$ of $S_{|G|}$, the group of invertible mappings of G. In the more specific case that $G = \langle a, b\rangle$ and 3.5 holds then θ will be an automorphism if and only if*

$$\begin{aligned} \theta L_a \theta^{-1} &= L_{a^{-1}}, \\ \theta L_b \theta^{-1} &= L_{b^{-1}}. \end{aligned}$$

Proof. If $\theta \in \mathrm{Aut}(G)$, then for each $g, h \in G$

$$\theta L_g \theta^{-1}(h) = \theta(g\theta^{-1}(h)) = \theta(g)h = L_{\theta(g)}h.$$

Thus $\theta L_g \theta^{-1} = L_{\theta(g)}$, proving the necessity of the conclusion. Now lets prove sufficiency. The hypothesis shows that $\theta \in S_{|G|}$ normalizes the subgroup $G_L = \{L_g : g \in G\}$. Thus $Ad_\theta : L_g \to \theta L_g \theta^{-1}$ is an automorphism of G_L onto its image, which is also G_L. It follows that there is a bijective map $\psi : G \to G$ satisfying

$$\theta L_g \theta^{-1} = L_{\psi(g)}, \text{ or alternatively}$$
$$\theta(gx) = \psi(g)\theta(x) \text{ for all } g \text{ and } x.$$

However, $L : G \to G_L$ is an isomorphism, and by definition $\psi = L^{-1} \circ Ad_\theta \circ L$. Thus we have proven that ψ is a automorphism of G. We finish by noting that $\theta(g) = \theta(g \cdot 1) = \psi(g)\theta(1) = \psi(g)$, by hypothesis and previous calculation ■.

Example 6 *Consider the group S_3, of order 6, with θ as defined below.*

$\#g$	g	$\theta(g)$	$\#\theta(g)$
1	id	id	1
2	a	$a^{-1} = a^2$	4
3	b	$b^{-1} = b$	3
4	a^2	$a^{-2} = a$	2
5	ab	$a^{-1}b^{-1} = a^2b$	6
6	a^2b	$a^{-2}b^{-1} = ab$	5

Thus, $\theta(g)$ is conjugation of g by the following element

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 2 & 6 & 5 \end{bmatrix} = (2,4)(5,6).$$

Using this method, we can also create the automorphism group of G. Instead of mapping a to a^{-1} and b to b^{-1}, we can look at all automorphisms of G. Any automorphism must map (a, b, c) to another generating triple. Recall that a generating triple must satisfy equations 2.1 and 2.2. Once we decide where a and b map, we can use the above method to directly construct the automorphism. Thus, assuming we have a generating triple (a, b, c), we can construct every automorphism and create the entire automorphism group. This method can be generalized to any group with a finite number of generators; however, as the number of generators increases, the complexity and the computation time required increase significantly.

4 Finding tiling groups for genus 6 and 7

Building from the work of Broughton and Vinroot, we decided to classify rotation and tiling groups for surfaces of genus 6 and 7. Classifying these groups can be broken into two different parts. First, groups of low order are fairly well known, and it is, thus, possible to find databases of all such groups. For groups of larger order, additional group theoretic methods are needed to search for tilings.

Groups of Order Less Than 128 Magma [6] has a database of solvable, non-Abelian groups of orders less than or equal to one hundred. Furthermore, all solvable groups of order under 128, including the Abelian groups, can be found in polycyclic notation on the Internet [1]. We wrote a program that takes branching data from the Riemann-Hurwitz equation and determines all rotation groups and tiling groups. First it searches for a group of the given order that contains possible (a, b, c) triples for the given (l, m, n). Such a group is a rotation group. Then, the program proceeds to check for the existence of a θ, with the required properties, in the automorphism group of the rotation group. As mentioned in Section 2, if a theta exists, then a tiling group exists, and methods for finding θ were discussed in Section 3. The listing of all groups of order under 128 makes it easy to find all rotation groups of order less than 128. The tables in Section 9 show the triangular tiling groups for surfaces of genus three through seven. For genus six and seven, all of the orientation preserving rotation groups of order less than 128 were found using the method described above.

Groups of Order More Than 127 There are a variety of ways of analyzing groups of order greater than 127. In the next section we discuss the group theoretic background for these. There are very few non-solvable rotation groups G on surfaces of low genus. In fact the order of G must be $\leq 84(\sigma - 1)$. Thus for our search we may assume a maximal order of $84 \times (7 - 1) = 504$. The possible simple factors for G are A_5, A_6, $PSL_2(7)$, and $PSL_2(8)$ of orders 60, 360, 168 and 504 respectively. The possible orders for non-solvable G are 60, 120, 180, 240, 300, 360, 420, 480, 168, 336, and 504. Moreover some of these possibilities can be excluded on the basis of the branching data. Thus we exploit the fact that in a solvable group there is a normal subgroup $N \lhd G$ and try to construct G from our knowledge of the action of N on S and G/N on S/N. We are particularly fortunate in the cases where $N \hookrightarrow G \twoheadrightarrow G/N$ splits and that a semi-direct product may be constructed. With the aid of Magma, these semi-direct products can be created. Then, we can use the same steps as in the case of groups of order ≤ 127.

5 Group construction and representation

In this section we consider group representation and group constructions that are particularly applicable to solvable groups.

Polycyclic Notation In the previous subsection, we discussed how to use Magma to determine whether a group is a rotation group. To perform this calculation, we must first have the group represented in Magma. Polycyclic notation is one form of group representation that Magma can deal with efficiently.

Polycyclic notation is a convenient way to represent any finite solvable group. The representation for a group G in polycyclic form is a list of generators, $a_1, a_2, a_3, \ldots, a_n$, along with some defining relations. Each of the generators,

a_i, when raised to a certain prime p_i must equal either the identity or a word in the generators a_j where $j > i$. We also impose the condition that the subgroup generated by $a_{i+1}, \ldots, a_n$ is normal in the subgroup generated by $a_i, \ldots, a_n$. Not only can any element of the group be written in terms of these n generators, but the conditions imply that any element of the group can be written as a word with the generators in the proper order. I.e., there is a unique normal form for $g \in G$ of the form

$$g = a_1^{r_1} a_2^{r_2} a_3^{r_3} \ldots a_n^{r_n},$$

for some $0 \leq r_1 < p_i, \ldots, 0 \leq r_n < p_n$. It follows that $|G| = p_1 p_2 p_3 \cdots p_n$.

The defining relations tell us how to represent an arbitrary element of G in the proper form described above, in particular how to write products and inverses in normal form. These relations are

$$a_j^{p_i} = a_{i+1}^{s_{i+1}} \cdots a_n^{s_n}$$
$$a_j^{a_i} = a_{i+1}^{r_{i+1}} \cdots a_n^{r_n}$$

for all $j > i$ and the word $a_{i+1}^{r_{i+1}} \cdots a_n^{r_n}$ depends on both i and j.. The notation x^y means conjugation of x by y, $x^y = y^{-1}xy$. In Magma, trivial relations are often not listed, e.g., if a and b commute, then the relation $a^b = a$ is not shown with the defining relations. Any solvable group can be written in polycyclic notation. However, there is not necessarily a unique way to represent the group in polycyclic notation.

Semi-Direct Products Given branching data obtained from the Riemann-Hurwitz equation, we wish to check all groups of a certain order to see if any are orientation preserving rotation groups. As previously mentioned most groups are solvable and we can assume the existence of a normal subgroup. Using the normal subgroup we can often use semi-direct products to generate the possible rotation groups of the desired order.

Given a group G with a normal subgroup N and another subgroup H where $N \cap H = \{1\}$ and $|N||H| = |G|$, we can construct $G = NH = \{nh : n \in N, h \in H\}$. Note that NH is closed, because N is normal. See [3, p. 176] for more information.

We can generalize this notion to the semi-direct product. Here, we take two groups and construct a larger group in a way similar to that of direct products. Given groups H and K and a homomorphism $\phi : K \mapsto Aut(H)$, we construct

$$H \rtimes_\phi K = \{(h, k) : h \in H, k \in K\}$$

where multiplication is defined by

$$(h_1, k_1) \cdot (h_2, k_2) = (h_1\phi(k_1)(h_2), k_1k_2).$$

This will be a group of order $|H||K|$ with H isomorphic to a normal subgroup in $H \rtimes K$. Note that if ϕ is the homomorphism that maps every element of K to the identity element of $Aut(H)$, then $H \rtimes_\phi K = H \times K$.

The fact that a group G has a normal subgroup does not imply that G can be represented as a semi-direct product.

Example 7 *The quaternion group, Q_8 has a normal subgroup of order two. However, since every subgroup of order four contains the normal subgroup, Q_8 cannot be written as a semi-direct product. We say that Q_8 does not split.*

The question then arises how to tell if a group splits with a given normal subgroup N, i.e. whether or not it can be written as a semi-direct product. If we can show that there exists a subgroup H of the group G, where $H \cap N = \{1\}$ and $|H||N| = |G|$, then we know that our group splits and $G \cong N \rtimes H$. Such a subgroup H is called a complement of N. However, in trying to find tiling groups, this is not a very helpful method, as we do not have an easy way to see if H exists. In order to prove that a potential rotation group splits, we can sometimes use the following theorem from group cohomology theory.

Theorem 8 *Given a group G and a normal subgroup N, consider the standard homomorphism $\psi : G \mapsto G/N$. If for every prime p that divides the order of G/N there is a p-group P, such that $\psi|_P$ maps bijectively onto a Sylow p-subgroup of G/N, then there exists a ϕ such that $G \cong N \rtimes_\phi (G/N)$.*

The use of this theorem is described in [2]. The following example illustrates a basic application of this theorem.

Example 9 *Let $|G| = 160$ and $(l, m, n) = (2, 4, 5)$. Consider the case where G has a normal subgroup, $|N| = 16$, so that $|G/N| = 10$. We want to know if G can be written as a semi-direct product of N and G/N. The prime divisors of $|G/N|$ are 2 and 5, so we have two cases. First, we look at the case where $p = 5$. Since $|N|$ and 5 are relatively prime, we know that no order 5 element of G can be in the kernel of ψ. Thus, ψ maps every Sylow 5-subgroup of G onto a Sylow 5-subgroup of G/N. Next, we consider $p = 2$. Recall that (a, b, c) is a generating triple of G with $a^l = b^m = c^n = id$. Let $\psi(a) = \bar{a}$, $\psi(b) = \bar{b}$, $\psi(c) = \bar{c}$. Since a and b generate G, $\bar{a}$ and $\bar{b}$ generate G/N. If $|\bar{a}| = 1$, then $\bar{a}$ and $\bar{b}$ cannot generate G/N. Therefore, $a \notin N$. Since a is an element of some Sylow 2-subgroup of G and $a \notin N$, it follows that this Sylow 2-subgroup must map to $\langle \bar{a} \rangle$. It then follows from the above theorem that $G \cong N \rtimes G/N$. In fact, it turns out that for a particular ϕ, $G = (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\phi D_5$ is a rotation group for a surface of genus 5.*

If a group does not satisfy the conditions of the above theorem for a given N, then we have no guarantee that the semi-direct products we can create will be all of the possible rotation groups of order G. In that case, we need to use other methods to analyze the non-splitting groups.

6 Sample methods for classifying rotation groups

Our first method for classifying large rotation groups comes from Ryan Vinroot's technical report [7]. If we know that all groups of a given order contain a normal Sylow subgroup, we can often eliminate this branching data as illustrated in the following example.

Example 10 *Let* $|G| = 200 = 2^3 5^2$*,* $(l, m, n) = (2, 4, 5)$*, and* $\sigma = 6$*. Then, by Sylow's theorems,* G *has a normal Sylow 5-subgroup,* P_5*, of order 25. Furthermore, since* c *has order 5, it follows that* $c \in P_5$*. If we let* $H = \langle a \rangle$*, then* $P_5 H$ *is a subgroup of* G *such that* $\langle a, c \rangle \subseteq P_5 H$*. Since* $|P_5 H| = 25 \cdot 2 = 50 < 200$*, we see that* a *and* c *cannot generate all of* G*. Thus, we have arrived at a contradiction, and we can eliminate this branching data.*

To illustrate several more techniques for classifying rotation groups, consider the case where G is not simple. Then, G must have at least one normal subgroup N. Also, $|N| \Big| |G|$ so we must break our classification of rotation groups of a specific order into different cases based on the possible values of $|N|$.

Example 11 *Let* $|G| = 150$*,* $(l, m, n) = (2, 3, 10)$*, and* $\sigma = 6$*. Then,* $|N| \in \{2, 3, 5, 6, 10, 15, 25, 30, 50, 75\}$*.*

Next, we can look at the standard homomorphism $\psi : G \mapsto G/N$. Let $\psi(a) = \bar{a}$, $\psi(b) = \bar{b}$, and $\psi(c) = \bar{c}$. Since any two of (a, b, c) generate G, it follows that any two of $(\bar{a}, \bar{b}, \bar{c})$ generate G/N. Similarly, since $abc = 1$, $\bar{a}\bar{b}\bar{c} = 1$. Also, it follows from properties of homomorphisms that the orders of $\bar{a}$,$\bar{b}$, and $\bar{c}$ must divide l, m, and n respectively. These conditions force G/N to be a orientation preserving rotation group of a surface of lower genus. Thus, we need only see if such a rotation group exists by either looking at classification tables for lower genera or, if the order of G/N is less than 128, by using the programs described in section 4.

Example 12 *Continuing our example from above, suppose* $|N| = 15$*. Then,* $|G/N| = 10$*. Recall that* $|b| = 3$*. Thus,* $|\bar{b}| \Big| 3$ *and* $|\bar{b}| \Big| 10$*. This implies that* $|\bar{b}| = 1$*. Since* $|\bar{a}| = 1$ *or* 2*, we have that the group generated by* $\bar{a}$ *and* $\bar{b}$ *has order less than or equal to 2. However,* a *and* b *generate all of* G*, so* $\bar{a}$ *and* $\bar{b}$ *must generate all of* G/N*. Thus, we arrive at a contradiction, and we know that* $|N| \neq 15$*. Using similar reasoning, we can reduce the possible orders of* N*. Thus,* $|N| \in \{2, 3, 5, 25, 75\}$*.*

Example 13 *Again, continuing our above example, suppose* $|N| = 5$*. Then,* $|G/N| = 30$*. Also, the possible orders for* $\bar{a}, \bar{b}$*, and* $\bar{c}$ *are* $(2, 3, 5)$ *and* $(2, 3, 10)$ *respectively. Using previously described methods, we can show that there is no rotation group of order 30 with* $(l, m, n) = (2, 3, 5)$ *or* $(2, 3, 10)$*. Thus, we know* $|N| \neq 5$*.*

Suppose that the normal subgroup is a Sylow p-subgroup, P, and that one of our generating elements has order p. Then, we know that this generating element, which we will arbitrarily call a, is in the Sylow p-subgroup. Then, we can use the method in Example 10 to arrive at a contradiction, assuming $|P||\langle b \rangle| < |G|$, where $|b| \leq |c|$.

Example 14 *If* $|N| = 2$ *or* 3*, then* N *is a normal Sylow* p*-subgroup. Since* $|a| = 2$ *and* $|b| = 3$*, this Sylow* p*-subgroup must contain a generating element. Following the argument described above, we see that* $|N| \neq 2$*, and* $|N| \neq 3$*. Thus,* $|N| \in \{25, 75\}$*.*

If a normal subgroup in G can be shown to have a characteristic subgroup, then that characteristic subgroup is normal in G. This fact allows us to reduce some cases to other ones.

Example 15 *If $|N| = 75$, then Sylow's theorems tell us that N contains a characteristic subgroup of order 25. Thus, G contains a normal subgroup of order 25, and we need only consider the case where $|N| = 25$.*

Once we reduce the possible orders of $|N|$ through the methods described above, we must then attempt to create the group G with the information we have gained. If we are lucky, we can use the theorem described in section 5 to see that G splits, and then create all possible groups G using the semi-direct product.

Example 16 *If $|N| = 25$, then we can show that G splits, because 2 and 3 are relatively prime to 25. We find that $|\bar{a}| = 2$, $|\bar{b}| = 3$, and $|\bar{c}| = 2$, which forces G/N to be isomorphic to S_3. Since there are only two groups of order 25, we are left with two possible types of semi-direct products: $(\mathbb{Z}_5 \times \mathbb{Z}_5) \rtimes S_3$ and $\mathbb{Z}_{25} \rtimes S_3$. With the aid of a computer program, we were able to construct all such groups. After constructing these groups, we could use a different program to check if these groups were orientation preserving rotation groups. One of the semi-direct products for $(\mathbb{Z}_5 \times \mathbb{Z}_5) \rtimes S_3$ is indeed a rotation group, and there does exist an automorphism θ which implies the existence of a tiling group G^*.*

In the example above, it was helpful that there were only two possible groups of order 25. Sometimes, we are not as fortunate. However, the following argument allows us to restrict ourselves to the cases where N is Abelian, provided that G is solvable. First, we must present the following definition. [5]

Definition 17 *Let G be a group. The subgroup of G generated by the set $\{aba^{-1}b^{-1}\}$ is called the* commutator subgroup *of G and is denoted G'.*

Let $G^{(1)}$ be G', and let $G^{(n)}$ be the commutator subgroup of $G^{(n-1)}$. It is a fact that $G^{(n)}$ is characteristic in $G^{(n-1)}$ [4]. From this fact, we see that $G^{(n)}$ is normal in G for all n. Furthermore, if a group is solvable, then there exists an n such that $G^{(n)} = \{1\}$. It is trivial to show that if $G^{(n)} = \{1\}$, then $G^{(n-1)}$ must be a normal Abelian subgroup.

Given a solvable normal subgroup N in G, we can find an n such that $N^{(n)}$ is Abelian and normal in G. Thus, when using the methods described above to classify orientation preserving rotation groups, we can limit our search to normal subgroups that are Abelian. Furthermore, if the normal Abelian group is not of order p^m for some prime p and integer m, then it must contain a characteristic Sylow subgroup. This subgroup will be normal in G. Thus, we need only look at the cases where N is an Abelian group of order p^m. This greatly reduces the number of groups we need to search through.

What if G Does Not Split? If G does not split, we can use information about the normal subgroup and the quotient group to construct a polycyclic representation of G. Starting with the normal subgroup, we can add generators to the beginning of the list of polycyclic generators in such a way that the relationships described by the quotient group are preserved.

Example 18 *Consider $|G| = 192$ and $N = \mathbb{Z}_2$. In this case N lies in the center of G. The only possible quotient group that satisfies all the necessary conditions is*

$$G/N = \langle x, y, z, w, v, u : x^2, y^3, z^2 = v, w^2 = u, v^2, u^2, y^x = y^2, \\ z^x = zwv, z^y = wv, w^y = zw, v^x = vu, v^y = u, u^y = vu\rangle.$$

We would like to be able to add a generator t of order two to the end of the generator list. This t represents N, and thus, we would have to allow for the possibility that each relation may or may not be multiplied by t on the right hand side. This is true because when we mod out by $\mathbb{Z}_2$, all the t's would disappear, leaving the quotient group as shown above. Although this method would work, we would be forced to check 2^{21} possible groups, which would take far too long.

Luckily, there is another way for us to proceed. Instead of starting with the quotient group, we can start with the normal subgroup $\mathbb{Z}_2$ and add the generators one by one. The fact that t is normal in G tells us that $t^a = t$ for all a in the generator list. As we add generators, we only consider the possibilities that preserve the relations of the quotient group. As in the previous method, every relationship can also have a t multiplied on the right. After adding each generator, we can eliminate all representations that do not describe consistent groups. We can check consistency with a computer by attempting to construct the left regular representation of the generators of the group, and checking to see if they generate a group of order 192. The fact that we can eliminate groups at each step reduces the time required significantly. Once we have built our way up to groups of order 192, we can check for orientation preserving rotation groups. It is important to note that there can be many different polycyclic notations for the same group. Using this method, we found 32 different polycyclic representations for the single orientation preserving rotation group of order 192 on a surface of genus 5.

7 The classification

Using the methods described above, we classified most tilings on surfaces of genus six and seven, as well as a few additional tilings on surfaces of genus three, and five. There are a few group orders left unclassified for genus six and seven. These are listed as unclassified data in Tables 1.b, and 2.b. For instance, we do not know if there are any tiling groups for groups of order 180, 240, or 288. In this section, we will sketch how the classification of tiling groups of all other orders for surfaces of genus through seven is complete.

Consider $|G| = 120$. From our list of all solvable groups of order under 128, we can classify all solvable rotation groups of order 120. The only non-solvable groups of order 120 are $S_5, A_5 \times \mathbb{Z}_2$, and $SL_2(5)$. Thus, we can classify all rotation groups of order 120 using Magma.

Consider $|G| = 144$ with $(l, m, n) = (2, 3, 12), (2, 4, 6), (2, 3, 8)$, or $(2, 3, 9)$. Then, $|N| \in \{2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72\}$. Using the method illustrated in Examples 12 and 13 we can eliminate the possibilities of $|N|$ equaling 4, 9, 16, 18, and 36. For $|N| = 8$, the quotient group must be $\mathbb{Z}_3 \times D_3$ for $(l, m, n) = (2, 3, 12)$. We constructed the semi-direct products and found that none of them produced tiling groups. Since this case splits, we have eliminated this possibility. Similarly, for $|N| = 3$ and $(l, m, n) = (2, 3, 12)$, we must construct semi-direct products. This time, we find that one of the semi-direct products produces a tiling group. Again, because this case splits, we have completed this possibility. The remaining cases, excluding $|N| = 2$, reduce to previous cases because each must contain a characteristic subgroup of a previously discussed order. If $|N| = 2$, we are no longer guaranteed that G splits to produce a semi-direct product. Thus, we must use the method explained in example 18.

Similar arguments show that the cases $|G| = 160, 192$ and 216, as shown in the table are complete. The details of this will occur in a subsequent paper. We have now classified all triangular tiling groups on surfaces through genus 7, excluding groups of orders 180, 240, and 288.

8 Further directions

During this REU, we finished the classification of triangular tiling groups on surfaces of genus four and five that was started by Ryan Vinroot. We also classified most rotation groups on surfaces of genus six and seven. These results are listed in the tables in the next section. One obvious further direction is to complete the classification of all tiling groups on surfaces of genus four through ten. In addition, we have the following questions:

- What are necessary and sufficient conditions for the existence of θ given an orientation preserving rotation group?
- Are there cases where a rotation group admits a tiling group for some generating triples but not for others? So far, we have not found any examples of this phenomenon.
- Is there a more efficient way to deal with the cases that cannot be represented as semi-direct products?
- Are there more effective ways to use polycyclic notation to search for tiling groups?

9 Tables of rotation and tiling groups

In the tables the group order, branching data, and group presentation given unless the group has a particularly simple form. For each entry a generating triple was found and tested to see if an automorphism θ existed. Branching data which was not completely analyzed are put into tables of unclassified data.

Table 1.a Known genus 6 rotation and tiling groups

$\lvert G\rvert$	(l,m,n)	G	tiling group?
13	$(13,13,13)$	$\mathbb{Z}_{13}$	Yes
14	$(7,14,14)$	$\mathbb{Z}_{14}$	Yes
15	$(5,15,15)$	$\mathbb{Z}_{15}$	Yes
16	$(4,16,16)$	$\mathbb{Z}_{16}$	Yes
18	$(3,18,18)$	$\mathbb{Z}_{18}$	Yes
20	$(4,5,20)$	$\mathbb{Z}_{20}$	Yes
21	$(3,7,21)$	$\mathbb{Z}_{21}$	Yes
24	$(4,4,12)$	$\langle x,y,z,w : z^2=w^3=1,\ x^2=y^2=z,$ $y^x=yz,\ w^x=w^2\rangle$	Yes
24	$(4,6,6)$	$\mathbb{Z}_3 \oplus D_4$	Yes
24	$(4,6,6)$	$\langle x,y,z,w : x^3=w^4=1,\ y^2=z^2=w,$ $y^x=z,\ z^x=yzw,\ z^y=zw\rangle$	Yes
24	$(2,24,24)$	$\mathbb{Z}_{24}$	Yes
24	$(3,8,8)$	$\langle x,y,z,w : z^2=w^3=1,\ x^2=y,$ $y^2=z,\ w^x=w^2\rangle$	
26	$(2,13,26)$	$\mathbb{Z}_{26}$	Yes
28	$(2,14,14)$	$\mathbb{Z}_{14}\times\mathbb{Z}_2$	Yes
28	$(4,4,7)$	$\langle x,y,z : x^2=y^2=z^7=1,\ z^x=z^6\rangle$	Yes
30	$(2,10,15)$	$\mathbb{Z}_5\times D_3$	Yes
36	$(2,9,9)$	$\langle x,y,z,w : y^3=z^2=w^2=1,\ x^3=y,$ $z^x=zw,\ w^x=y\rangle$	Yes
39	$(3,3,13)$	$\langle x,y : x^3=y^{13}=1,\ y^x=y^3\rangle$	NO
48	$(2,4,24)$	$\langle x,y,z,w,v : x^2=w^2=v^3=1,\ y^2=z,\ z^2=w,$ $y^x=yz,\ z^x=zw,\ v^x=v^2\rangle$	Yes
48	$(2,6,8)$	$\langle x,y,z,w,v : x^2=w^2=v^3=1,\ y^2=zw,$ $z^3=w,\ y^x=yz,\ z^x=zw,\ v^y=v^2\rangle$	Yes
50	$(2,5,10)$	$\langle x,y,z : x^2=y^5=z^5=1,\ z^x=z^4\rangle$	Yes
56	$(2,4,14)$	$\langle x,y,z,w : x^2=y^2=z^2=w^7=1,\ y^x=yz\rangle$ $w^x=w^6\rangle$	Yes
72	$(2,4,9)$	$\langle x,y,z,w,v : x^2=z^3=w^2=v^2=1,\ y^3=z,$ $y^x=y^2z^2,\ z^x=z^2,\ w^y=wv,\ v^x=wv,\ v^2=w\rangle$	Yes
75	$(3,3,5)$	$\langle x,y,z : x^3=y^5=z^5=1,\ y^x=y^4z^4,$ $z^x=y\rangle$	Yes
120	$(2,4,6)$	S_5	Yes
150	$(2,3,10)$	$\langle x,y,z,w : x^2=y^3=z^5=w^5=1,\ y^x=y^2,$ $z^x=z^4,\ z^y=w^4,\ w^x=z^4w,\ w^y=zw^4\rangle$	Yes
504	$(2,3,7)$	$PSL_2(8)$	Yes

Table 1.b Unclassified branching data for genus 6

$\|G\|$	(l,m,n)	G	tiling group?
120	$(2,3,12)$	unknown	
120	$(2,4,6)$	unknown	
120	$(3,3,4)$	unknown	
180	$(2,3,9)$	unknown	
240	$(2,3,8)$	unknown	

Table 2.a: Known genus 7 rotation and tiling groups

$\|G\|$	(l,m,n)	G	tiling group?
15	$(15,15,15)$	$\mathbb{Z}_{15}$	Yes
16	$(8,16,16)$	$\mathbb{Z}_{16}$	Yes
18	$(6,9,18)$	$\mathbb{Z}_{18}$	Yes
20	$(4,10,20)$	$\mathbb{Z}_{20}$	Yes
21	$(3,21,21)$	$\mathbb{Z}_{21}$	Yes
24	$(3,8,24)$	$\mathbb{Z}_{24}$	Yes
24	$(4,6,12)$	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	Yes
24	$(6,6,6)$	$\langle x,y,z,w : x^3 = w^2 = 1,\ y^2 = z^2 = w,$	Yes
		$y^x = z,\ z^x = yzw,\ z^y = zw\rangle$	
27	$(3,9,9)$	$\mathbb{Z}_9 \times \mathbb{Z}_3$	Yes
27	$(3,9,9)$	$\langle x,y,z : x^3 = z,\ y^x = yz\rangle$	NO
28	$(2,28,28)$	$\mathbb{Z}_{28}$	Yes
28	$(4,4,14)$	D_{14}	Yes
30	$(2,15,30)$	$\mathbb{Z}_{30}$	Yes
32	$(2,16,16)$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_2$	Yes
32	$(2,16,16)$	$\langle x,y,z,w,v : x^2 = z,\ z^2 = w,\ w^2 = v,\ y^x = yv\rangle$	Yes
32	$(4,4,8)$	$\langle x,y,z,w,v : x^2 = w,\ y^2 = z^2 = v,\ y^x = yz,$	Yes
		$z^x = z^y = zv\rangle$	
32	$(4,4,8)$	$\langle x,y,z,w,v : x^2 = w,\ z^2 = w^2 = v,\ y^x = yz,$	Yes
		$z^x = z^y = zv\rangle$	Yes
32	$(4,4,8)$	$\langle x,y,z,w,v : x^2 = w,\ y^2 = z,\ z^2 = v,\ y^x = yz,$	Yes
		$z^x = zv\rangle$	
32	$(4,4,8)$	$\langle x,y,z,w,v : x^2 = w,\ y^2 = zv,\ z^2 = v,\ y^x = yz,$	Yes
		$z^x = zv\rangle$	Yes
36	$(3,4,12)$	$\langle x,y,z,w : y^3 = z^2 = w^3 = 1,\ x^2 = z,\ w^x = w^2\rangle$	Yes
42	$(2,6,21)$	$\langle x,y,z : x^2 = y^3 = z^7 = 1,\ z^x = z^6\rangle$	Yes
48	$(2,6,12)$	$\langle x,y,z,w,v : y^2 = v^2 = 1,\ x^2 = z^2 = w^2 = v,$	Yes
		$z^y = zw,\ w^y = z,\ w^z = wv\rangle$	
48	$(3,4,6)$	$\langle x,y,z,w,v : x^2 = y^3 = v^2 = 1,\ z^2 = w^2 = v,$	Yes
		$z^y = zw,\ w^y = z,\ w^z = wv\rangle$	
54	$(2,6,9)$	$\langle x,y,z,w : x^2 = y^3 = w^3 = 1,\ z^3 = w,$	Yes
		$z^x = z^2w^2,\ w^x = w^2\rangle$	
54	$(2,6,9)$	$\langle x,y,z,w : x^2 = y^3 = w^3 = 1,\ z^3 = w,$	NO
		$z^x = z^2w^2,\ z^y = zw^2,\ w^x = w^2\rangle$	

Table 2.a continued

56	$(2,4,28)$	$\langle x,y,z,w : y^2 = z^2 = w^7 = 1,\ x^2 = z,\ w^y = w^6\rangle$	Yes
56	$(2,7,7)$	$\langle x,y,z,w : x^7 = y^2 = z^2 = w^2 = 1,\ y^x = z,$ $z^x = yzw,\ w^x = zw\rangle$	NO
64	$(2,4,16)$	$\langle x,y,z,w,v,u : x^2 = w,\ z^2 = vu,\ v^2 = u,\ y^x = yz,$ $z^x = z^y = zv,\ v^x = v^y = vu\rangle$	Yes
64	$(2,4,16)$	$\langle x,y,z,w,v,u : x^2 = w,\ z^2 = v,\ v^2 = u,\ y^x = yz$ $z^x = zv,\ z^y = zvu,\ w^y = wu,\ v^x = v^y = vu\rangle$	Yes
72	$(3,3,6)$	$\langle x,y,z,w,v : x^3 = y^3 = v^2 = 1,\ z^2 = w^2 = v,$ $z^x = zw,\ w^x = z,\ w^z = wv\rangle$	Yes
144	$(2,3,12)$	$\langle x,y,z,w,v,u : y^3 = v^2 = 1,\ x^2 = z^2 = w^2 = v,$ $z^y = zw, w^y = z,\ w^z = wv,\ u^x = u^2$	Yes

Table 2.b Unclassified branching data for genus 7

$\|G\|$	(l,m,n)	G	tiling group?
108	$(2,3,18)$	unknown	
112	$(2,4,7)$	unknown	
120	$(2,3,15)$	unknown	
120	$(2,5,5)$	unknown	
126	$(2,3,14)$	unknown	
144	$(2,3,12)$	unknown	
144	$(2,4,6)$	unknown	
180	$(2,3,10)$	unknown	
216	$(2,3,9)$	unknown	
240	$(2,4,5)$	unknown	
288	$(2,3,8)$	unknown	

Table 3: Previously unclassified genus 3 tiling groups

$\|G\|$	(l,m,n)	G	tiling group?
48	$(2,3,12)$	$\langle x,y,z,w,v \mid y^3 = v^2 = 1, x^2 = z^2 = w^2 = v,$ $z^y = zw, w^y = z, w^z = wv\rangle$	Yes

Table 4: Previously unclassified genus 5 tiling groups

$\|G\|$	(l,m,n)	G	tiling group?
64	$(2,4,8)$	$\langle x,y,z,w,v,u : x^2 = w,\ z^2 = u,\ y^x = yz,\ z^x = zv$ $z^y = zu,\ w^y = wvu\rangle$	Yes
64	$(2,4,8)$	$\langle x,y,z,w,v,u : x^2 = w,\ y^x = yz,\ z^x = zv,\ w^y = wv$ $w^z = wu,\ v^x = vu\rangle$	Yes
160	$(2,4,5)$	$\langle x,y,z,w,v,u : x^2 = y^5 = z^2 = w^2 = v^2 = u^2 = 1,$ $y^x = y^4,\ z^x = zu,\ w^x = wu,\ v^x = zwvu,\ z^y = zwv,$ $w^y = zwu,\ v^y = zwvu,\ u^y = wv\rangle$	Yes
192	$(2,3,8)$	$\langle x,y,z,w,v,u,t : x^2 = y^3 = v^2 = u^2 = t^2 = 1,\ z^2 = u,$ $w^2 = t,\ y^x = y^2,\ z^x = zu,\ z^y = wvt,\ w^x = zwu,$ $w^y = zwvt,\ w^z = wv,\ u^y = t,\ t^x = t^y = vut\rangle$	Yes

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