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Abstract

We examine a model for the propagation of heat through a one-dimensional object with an interior "flaw". The flaw is modeled as a nonlinear relationship between the flux and temperature jump at an interior point of the object. Under realistic hypotheses, the resulting nonlinear initial boundary value problem is shown to have a unique and suitably smooth solution.

Key words: Parabolic differential equations, nonlinear boundary condition, jump condition.
AMS(MOS) subject classifications: 35K60

In this paper we study a parabolic initial-boundary value problem which arises in the study of the noninvasive use of thermal energy to detect the presence of cracks and other "defects" in the interior of an object. More specifically, consider an object Ω in $\mathbb{R}^n$, $1 \leq n \leq 3$, with surface $\partial\Omega$. The object may contain a "crack" or other defect in its interior. We expect the presence of the defect to affect the dynamics of heat flow in Ω. To identify the flaw one could introduce a specified heat flux on some portion of $\partial\Omega$ and then monitor the temperature on that portion of $\partial\Omega$ for a certain time interval. Our ultimate goal is to determine if this data is sufficient to detect the presence of a defect, determine the shape and location of the defect, and understand the precise effect the defect has on the dynamics of heat in the object.

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This is similar to the situation considered in the authors' works ([1]-[3]) on corrosion detection. In the present case, however, we are interested in the possibility of determining the shape and location of an interior defect using a more general and possibly more realistic model for the effective boundary conditions at the defect, and even of identifying the effective boundary conditions themselves from data. An essential part of this study is to establish the well-posedness of the relevant forward problems under reasonable boundary conditions.

As a first step we develop a mathematical model for the flow of heat through a one-dimensional object with an interior flaw, which could represent (in the analogous two or three dimensional case) a crack or disbond. The resulting model is an initial boundary value problem (IBVP) for the heat equation. We model the defect as a point in the interior of the object, with a flux condition across the defect that depends non-linearly on the temperature jump across the defect. Our goal in this paper is to establish existence and uniqueness of classical solutions for this IBVP. These results will provide the basis for future work on the inverse problems mentioned above.

The IBVP is presented in section 1. In sections 2 and 3, the IBVP is reformulated as a system of nonlinear integral equations. In section 4, we prove that this system has a unique continuous solution. In section 5, we establish that this solution is classical, i.e., is sufficiently smooth and satisfies the stated boundary conditions.

1 Introduction

Consider the flow of heat through a one-dimensional object described by the interval $\Omega = (0,1)$ in $\mathbb{R}^1$, with unit thermal diffusivity and conductivity. We assume that the interior of the object contains a point "flaw" at position $x = \sigma$, $0 < \sigma < 1$, which impedes the flow of heat through the bar. We model the presence of the flaw as a "contact resistance" between the regions on either side of the flaw, by assuming that the flaw causes a jump in the temperature at $x = \sigma$, in which the jump is a function of the heat flux across this point. This can be quantified by letting $u(t,x)$ denote the temperature in the bar and taking $u(t,x)$ to satisfy

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u(t,x) &= 0 \text{ in } (0,\sigma)\cup(\sigma,1) && (1)\\
-\frac{\partial u}{\partial x}(t,0) &= g_0(t) && (2)\\
\frac{\partial u}{\partial x}(t,1) &= g_1(t) && (3)\\
\frac{\partial u}{\partial x}(t,\sigma) &= F(u^+(t) - u^-(t)) && (4)\\
u(0,x) &= \phi(x) && (5)
\end{aligned}
$$

where u^+ and u^- are the limiting values of u from above and below $x = \sigma$, respectively, and we assume that $\frac{\partial u}{\partial x}$ is continuous across the flaw at $x = \sigma$. The functions $g_0(t)$ and $g_1(t)$ define the input heat flux at each end of the interval, while ϕ denotes the initial temperature. The function F governs the flow of heat across the defect, modeled as a nonlinear contact resistance. On physical grounds we take F to be non-decreasing with $F(0) = 0$.

The goal here is to establish unique solvability of the initial-boundary value problem (IBVP) (1)-(5). We seek a solution $u(t,x)$ which is smooth in t and C^1 on the closed intervals $[0,\sigma^-]$ and $[\sigma^+,1]$. To this end, we make the following assumptions:

A1. The functions $g_0(t)$ and $g_1(t)$ from $\mathbb{R}^+$ to $\mathbb{R}$ are continuously differentiable on $0 \leq t \leq T$ for each $T > 0$ and of exponential order.

A2. F is uniformly Lipschitz continuous on any bounded interval, i.e., for any $a > 0$ and $x, y \in (-a, a)$ we have $|F(x) - F(y)| \leq C(a)|x - y|$ for some constant $C(a)$.

A3. For simplicity, we take $\phi \equiv 0$.

For consistency, we assume that $g_0(0) = g_1(0) = 0$.

Our main result is:

Theorem 1.1 *Under assumptions A1-A3, the initial-boundary value problem (1)-(5) has a unique solution for $t > 0$. The solution is C^1 on $[0, \sigma^-] \cup [\sigma^+, 1]$ and smooth in t.*

We shall prove this theorem by first reformulating the IBVP as a system of integral equations. We show that this system has a unique continuous solution, which will provide a formal solution to the IBVP. We then demonstrate that this formal solution is in fact a classical solution.

2 Preliminaries

Let $G(t, x) = e^{-x^2/(4t)}/\sqrt{4\pi t}$, the fundamental solution for the heat equation in one spatial dimension. For $x \in [0, 1]$, define the linear integral operators

$$
\begin{aligned}
A_x(h) &\equiv 2\int_0^T G(T-t, x)h(t)\,dt = \int_0^T \frac{e^{-\frac{x^2}{4(T-t)}}}{\sqrt{\pi(T-t)}}h(t)\,dt \\
K_x(h) &\equiv -2\int_0^T \frac{\partial G}{\partial x}(T-t, x)h(t)\,dt = \int_0^T \frac{xe^{-\frac{x^2}{4(T-t)}}}{2\sqrt{\pi(T-t)^3}}h(t)\,dt\,.
\end{aligned}
$$

Each of these is a bounded linear operator on $C[0, T]$ (with the usual supremum norm) for any $T > 0$. For $x > 0$, each one is a smoothing operator (mapping continuous functions to C^∞ functions), and both $A_x(h)(T)$ and $K_x(h)(T)$ satisfy the heat equation for $T > 0$.

It is useful to note that $||K_x|| < 1$ in the operator norm on $C[0, T]$. This is easily verified by integrating the kernel for K_x directly.

Let $v(t, x)$ denote a $C^1((0, \infty); C^1[a, b] \cap C^2(a, b))$ solution to the heat equation on some interval (a, b) with $v(0, x) = 0$. The following is a rather standard result, easily proved by using integration by parts in x on the product $\frac{\partial^2 v}{\partial x^2}(t, x)G(T-t+\epsilon, x)$, noting that $G(T-t+\epsilon, x)$ satisfies the backward heat equation, letting ϵ approach zero from above, and using the fact that $\int_{-\infty}^{\infty} G(t, x)\,dx = 1$.

Lemma 2.1 *Let $v(t, x)$ denote a $C^1((0, \infty); C^1([a, b]))$ solution to the heat equation on some interval (a, b) with $v(0, x) = 0$. Then for $T > 0$ we have*

$$
\begin{aligned}
v(T, a) &= A_0(g_a)(T) + A_{b-a}(g_b)(T) + K_{b-a}(v(\cdot, b))(T) \\
v(T, b) &= A_0(g_b)(T) + A_{b-a}(g_a)(T) + K_{b-a}(v(\cdot, a))(T) \\
v(T, x) &= \frac{1}{2}A_{a-x}(g_a)(T) + \frac{1}{2}A_{b-x}(g_b)(T) - \frac{1}{2}K_{a-x}(v(\cdot, a)) + \frac{1}{2}K_{b-x}(v(\cdot, b)), \quad x \in (a, b)
\end{aligned}
$$

where $g_a(t) = -\frac{\partial v}{\partial x}(t, a)$ and $g_b(t) = \frac{\partial v}{\partial x}(t, b)$.

3 Integral Equation Formulation

Let u satisfy the IBVP (1)-(5). We will use the notation $u_0(t) = u(t,0)$ and $u_1(t) = u(t,1)$, and use $u^-(t)$ and $u^+(t)$ to denote the limiting values of $u(t,x)$ as x approaches σ from the left or right, respectively. Application of Lemma 2.1 to u on each subinterval now yields

$$\begin{aligned} u_0(T) &= A_\sigma\left(F\left(u^+ - u^-\right)\right) + K_\sigma(u^-) + A_0(g_0) && (6)\\ u^-(T) &= A_0\left(F\left(u^+ - u^-\right)\right) + K_\sigma(u_0) + A_\sigma(g_0) && (7)\\ u^+(T) &= -A_0\left(F\left(u^+ - u^-\right)\right) + K_{1-\sigma}(u_1) + A_{1-\sigma}(g_1) && (8)\\ u_1(T) &= -A_{1-\sigma}\left(F\left(u^+ - u^-\right)\right) + K_{1-\sigma}(u^+) + A_0(g_1) && (9)\end{aligned}$$

where we have used the jump condition (4) to replace $\partial u/\partial x$ at $x = \sigma$. This system of integral equations must be satisfied by a sufficiently smooth solution to the initial-boundary value problem.

We can in fact distill the system (6)-(9) down into a single integral equation. First, by substituting (6) into (7) and (9) into (8), we can reduce to a pair of equations:

$$\begin{aligned} u^-(T) &= A_0\left(F\left(u^+ - u^-\right)\right) + K_\sigma A_\sigma\left(F\left(u^+ - u^-\right)\right) + K_\sigma^2(u^-) + K_\sigma A_0(g_0) + A_\sigma(g_0)\\ u^+(T) &= -A_0\left(F\left(u^+ - u^-\right)\right) - K_{1-\sigma}A_{1-\sigma}\left(F\left(u^+ - u^-\right)\right) + K_{1-\sigma}^2(u^+) + K_{1-\sigma}A_0(g_1) + A_{1-\sigma}(g_1)\end{aligned}$$

for functions u^- and u^+. Let us then rewrite the above equations as

$$\begin{aligned} (I - K_\sigma^2)(u^-) &= A_0\left(F\left(u^+ - u^-\right)\right) + K_\sigma A_\sigma\left(F\left(u^+ - u^-\right)\right) + K_\sigma A_0(g_0) + A_\sigma(g_0) && (10)\\ (I - K_{1-\sigma}^2)(u^+) &= -A_0\left(F\left(u^+ - u^-\right)\right) - K_{1-\sigma}A_{1-\sigma}\left(F\left(u^+ - u^-\right)\right) + K_{1-\sigma}A_0(g_1) + A_{1-\sigma}(g_1) && (11)\end{aligned}$$

where I is the identity operator. Since $||K_w|| < 1$ for $w > 0$ the operator $(I - K_w^2)$ can be inverted with a Neumann series, and

$$(I - K_w^2)^{-1} = I + \sum_{j=1}^{\infty} K_w^{2j} \equiv H_w \, .$$

Setting $\tilde{H}_w \equiv H_w - I$, we see that the kernel of $\tilde{H}_w$, denoted by $\tilde{h}_w(x,y)$, is continuous. Furthermore, by expressing $\tilde{h}_w = \sum_{j=1}^{\infty} k_w^{2j}(x,y)$, where $k_w^{2j}(x,y) = \int_y^x k_w(x,z)k_w^{2j-1}(z,y)\,dz$ (with $k_w^0 = k_w$) and k_w is the kernel of K_w, it becomes clear that $\tilde{h}_w$ is a positive function (since k_w is positive). Also, the operator H_w is convolutional, and so $\tilde{h}_w(x,y) = \tilde{h}_w(x-y)$. Apply H_σ to both sides of equation (10), $H_{1-\sigma}$ to equation (11), and subtract to obtain

$$v = L(F(v)) + g \tag{12}$$

where $v = u^+ - u^-$, $L = -(H_{1-\sigma}A_0 + H_{1-\sigma}K_{1-\sigma}A_{1-\sigma} + H_\sigma A_0 + H_\sigma K_\sigma A_\sigma)$ is convolutional, and $g = H_{1-\sigma}(K_{1-\sigma}A_0 + A_{1-\sigma})(g_1) - H_\sigma(K_\sigma A_0 + A_\sigma)(g_0)$. Note that L is a bounded linear operator on $C[0,T]$ and g is smooth.

We will prove

Lemma 3.1 *Equation (12) has a unique continuous solution on* $[0,\infty)$.

From this we can recover the functions u_0, u_1, u^-, and u^+, and so construct (using a representation formula with the fundamental solution G) a solution to the IBVP (1)-(5).

4 Proof of Lemma 3.1

First note that equation (12) can be cast into the form

$$v(t) = -\int_0^t \frac{2}{\sqrt{\pi(t-s)}} F(v(s))\,ds - A(F(v))(t) + g(t) \tag{13}$$

where A is the convolutional operator given by

$$A = \tilde{H}_{1-\sigma}A_0 + \tilde{H}_{1-\sigma}K_{1-\sigma}A_{1-\sigma} + K_{1-\sigma}A_{1-\sigma} + \tilde{H}_\sigma A_0 + \tilde{H}_\sigma K_\sigma A_\sigma + K_\sigma A_\sigma. \tag{14}$$

Let $a(t)$ denote the kernel of A, so $A(\phi)(t) = \int_0^t a(t-s)\phi(s)\,ds$. It is clear that a is smooth and positive (since the kernels of the various operators defining A also possess these properties.) One can easily establish that $|a(t)| \leq Ct^\beta$ for any $\beta > 0$, so that a decays rapidly near zero, a consequence of the rapid decay of the kernels of $K_\sigma, K_{1-\sigma}, A_\sigma$, and $A_{1-\sigma}$ near zero. Note also that the function g is C^∞.

Equation (13) is similar to integral equations studied by Roberts and Mann, Padmavally, J.J. Levin, and N. Levinson, all in connection with parabolic PDE's with nonlinear boundary conditions. In those cases, however, the additional $A(F(v))$ term was not present. But we can in fact use a slight modification of the argument in [4] to establish unique solvability of equation (13) in the class of continuous functions on $[0, \infty)$.

Before proving Lemma 3.1 we first establish several lemmas.

Lemma 4.1 *The kernel a of the operator A in equation (13) is non-decreasing.*

Proof: Since all of the operators involved in the definition of A in equation (14) are convolutional, we can compute a by convolving the kernels of these operators, and this is easily accomplished by taking the products of the Laplace transforms of the kernels of A_x, A_0, and K_x. If we denote the kernels in lower case and use $\mathcal{L}(\phi)$ to denote the Laplace transform of ϕ then, for $x > 0$,

$$\mathcal{L}(a_x)(s) = \frac{e^{-x\sqrt{s}}}{\sqrt{s}}, \quad \mathcal{L}(a_0)(s) = \frac{1}{\sqrt{s}}, \quad \mathcal{L}(k_x)(s) = e^{-x\sqrt{s}}. \tag{15}$$

We can compute the Laplace transform of H_x as $\mathcal{L}(H_x) = 1/(1-\mathcal{L}(K_x)^2)$ and of $\tilde{H}_x = K_x^2(I-K_x^2)^{-1}$ as $\mathcal{L}(\tilde{H}_x) = \mathcal{L}(K_x)^2/(1-\mathcal{L}(K_x)^2)$.

Equation (14) and the elementary properties of the Laplace transform show, after a bit of algebra, that the Laplace transform of the function a is given by

$$\mathcal{L}(a)(s) = \frac{2}{\sqrt{s}}\left(\frac{1}{e^{2\sigma\sqrt{s}}-1} + \frac{1}{e^{2(1-\sigma)\sqrt{s}}-1}\right).$$

Since the function a is smooth we can prove that a is non-decreasing by showing that $a' \geq 0$ at all points. Since $a(0) = 0$, the Laplace transform of a' is given by

$$\mathcal{L}(a')(s) = 2\sqrt{s}\left(\frac{1}{e^{2\sigma\sqrt{s}}-1} + \frac{1}{e^{2(1-\sigma)\sqrt{s}}-1}\right). \tag{16}$$

We will not explicitly invert the transform to find a', but rather will show that $a' \geq 0$ by using the Post-Widder Inversion Formula [5] for the Laplace transform, which states that if $F = \mathcal{L}(f)$ for a continuous function f of exponential order then

$$f(t) = \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right)$$

where $F^{(k)}$ denotes the kth derivative of F. An immediate consequence of the inversion formula (and the definition of the Laplace transform) is that a function $f(t)$ is non-negative for all $t \geq 0$ if and only if $(-1)^k F^{(k)}(s) \geq 0$ for all $k \geq 0$ and all $s > 0$. In short, the Laplace transform of a non-negative function must alternate sign. We will establish this for the transform of a'.

In fact, it suffices to establish this property for $\frac{\sqrt{s}}{e^{2\sigma\sqrt{s}}-1}$, since the second term on the right in equation (16) is of exactly the same form (note both σ and $1-\sigma$ are positive.) Also, if functions ϕ_1 and ϕ_2 satisfy $(-1)^k\phi_1^{(k)} \geq 0$ and $(-1)^k\phi_2^{(k)} \geq 0$, respectively, then an easy induction argument shows that the product $\phi = \phi_1\phi_2$ also satisfies $(-1)^k\phi^{(k)} \geq 0$. Since $\sqrt{s}$ obviously satisfies this alternating sign property, we need only to establish it for $\frac{1}{e^{2\sigma\sqrt{s}}-1}$, or equivalently, for $\frac{1}{e^{2\sqrt{s}}-1}$.

Let $b = e^{2\sqrt{s}} - 1$. Clearly, $\frac{1}{b} \geq 0$ for $s > 0$. The first derivative of $1/b$ is

$$\frac{db}{ds} = -\frac{1}{\sqrt{s}}\frac{1}{b} - \frac{1}{\sqrt{s}}\frac{1}{b^2}.$$

Note that both terms are of the form $s^{-\beta}b^{-k}$ for $\beta, k > 0$, and more generally the derivative of an expression of the form $s^{-\beta}b^{-k}$ consists of terms of the same form but with opposite sign, for

$$\frac{d}{ds}(s^{-\beta}b^{-k}) = -\beta s^{-\beta-1}b^{-k} - ks^{-\beta-1/2}b^{-k} - ks^{-\beta-1/2}b^{-k-1}.$$

This makes it apparent that the second derivative of $1/b$ will be non-positive (all terms will change sign), and similarly the third derivative will be non-negative, etc. An easy induction argument shows that $(-1)^k\frac{d^kb}{ds^k} \geq 0$ for $s > 0$. Thus the Laplace transform of a' has the alternating sign property, and hence $a' \geq 0$, so a is non-decreasing. This completes the proof of Lemma 4.1. □

We use $C^\beta(I)$, $0 < \beta < 1$ to denote the uniformly Hölder continuous functions with exponent β on an interval I, i.e., those ϕ for which

$$\sup_{t_1,t_2\in I, t_1\neq t_2} \frac{|\phi(t_1) - \phi(t_2)|}{|t_1 - t_2|^\beta} < \infty.$$

For $\beta = 0$ we take $C^0(I)$ as the bounded continuous functions on I.

Lemma 4.2 *Let H be the operator defined by*

$$H(u)(t) \equiv \int_{t_0}^t \frac{h(u(s))}{\sqrt{t-s}}\, ds$$

where h is uniformly Lipschitz continuous on any bounded interval. Then for $0 \leq \beta < 1$, H maps functions in $C^\beta[t_0, t)$ into $C^{(\beta+1)/2}[t_0, t)$.

Proof: We may assume that $t_0 = 0$. It is easy to verify that for any $0 < \alpha \leq 1$, $t > 0$, and δ sufficiently small we have

$$H(u)(t) - H(u)(t-\delta) = I_1 + I_2 + I_3 + I_4 + I_5$$

where

$$\begin{aligned}
I_1 &= \int_{t-\delta^\alpha}^{t} \frac{h(u(s)) - h(u(s-\delta))}{\sqrt{t-s}}\, ds \\
I_2 &= \int_0^\delta \frac{h(u(s))}{\sqrt{t-s}}\, ds \\
I_3 &= \int_\delta^{t-\delta-\delta^\alpha} h(u(s)) \left(\frac{1}{\sqrt{t-s}} - \frac{1}{\sqrt{t-s-\delta}} \right) ds \\
I_4 &= \int_{t-\delta-\delta^\alpha}^{t-\delta^\alpha} \frac{h(u(s))}{\sqrt{t-s}}\, ds \\
I_5 &= \int_0^\delta \frac{h(u(s))}{\sqrt{t-s-\delta}}\, ds.
\end{aligned}$$

If $u \in C^\beta$ then it's easy to bound

$$|I_1| \leq C_1 \delta^\beta \int_{t-\delta^\alpha}^{t} \frac{1}{\sqrt{t-s}}\, ds = C_1 \delta^{\beta+\alpha/2}.$$

Trivially $|I_2| \leq C_2\delta$, while

$$\begin{aligned}
|I_3| &\leq C_3 \int_\delta^{t-\delta-\delta^\alpha} \left(\frac{1}{\sqrt{t-s}} - \frac{1}{\sqrt{t-s-\delta}} \right) ds \\
&= 2C_3(-\sqrt{\delta+\delta^\alpha} + \delta^{\alpha/2} + \sqrt{t-\delta} - \sqrt{t-2\delta}) \\
&\leq C_3'(\delta^{1-\alpha/2} + O(\delta)) \\
&\leq C_3''\delta^{1-\alpha/2}
\end{aligned}$$

where we have used the fact that $\delta^{\alpha/2} - \sqrt{\delta+\delta^\alpha} = \delta^{\alpha/2}(1 - \sqrt{1+\delta^{1-\alpha}}) = O(\delta^{\alpha/2}\delta^{1-\alpha}) = O(\delta^{1-\alpha/2})$ for $0 < \alpha < 1$. A similar estimate gives

$$|I_4| \leq C_4(\sqrt{\delta+\delta^\alpha} - \sqrt{\delta^\alpha}) \leq C_4'\delta^{1-\alpha/2}$$

and trivially $|I_5| \leq C_5\delta$. All in all we obtain

$$|H(u)(t) - H(u)(t-\delta)| \leq C_1\delta^{\beta+\alpha/2} + C_2'\delta^{1-\alpha/2}$$

where we've absorbed $O(\delta)$ terms into the $C_2'\delta^{1-\alpha/2}$ term (which is of order less than $O(\delta)$.) The choice $\alpha = 1-\beta$ gives the desired (and best) conclusion. □

Remark

Lemma 4.2 immediately shows that if v is a bounded continuous solution to equation (13) on $[0,t)$ then v must be in the class $C^\beta([0,t))$ for all $\beta < 1$, for v is explicitly the sum of a C^β function and a smooth function.

We will need the following extension lemma.

Lemma 4.3 *Let $v(t)$ be a bounded continuous solution to equation (13) on the interval $[0, t_1)$. Then v can be extended as a solution to equation (13) on an interval $[0, t_1 + \delta)$, where $\delta > 0$.*

Proof: Define a function $k(t) = \frac{2}{\sqrt{\pi t}} + a(t)$ where $a(t)$ is the kernel of A defined in equation (14). We can write equation (13) as

$$v(t) = -\int_0^t k(t-s)F(v(s))\,ds + g(t).$$

Define a modification $\tilde{F}$ of the function F as

$$\tilde{F}(x) = \begin{cases} F(x), & |F(x)| \leq M \\ F(M), & F(x) > M \\ F(-M), & F(x) < -M \end{cases}$$

and M is any positive constant, so that $\tilde{F}$ is just F "chopped off" at $\pm M$. It's easy to check that $\tilde{F}$ is uniformly Lipschitz continuous with $|\tilde{F}(x) - \tilde{F}(y)| \leq K_1|x-y|$ for all real x, y and some constant $K_1 = K_1(M, F)$ independent of x, y.

Consider the integral equation

$$v(t) = -\int_{t_0}^t k(t-s)\tilde{F}(v(s))\,ds + g(t) \equiv L(\tilde{F}(v)) + g\,, \tag{17}$$

where g is continuous on $[0, \infty)$ and $t_0 \geq 0$ (note that we are taking lower limit t_0, not 0.)

Claim 1: For any $t_0 \geq 0$ the integral equation (17) has a unique continuous solution on $[t_0, t_0 + h]$ if h is sufficiently small. Moreover, h depends only on M, F, and the function k.

Proof: This is a rather standard fixed-point/contraction mapping argument. Let v_1, v_2 be continuous functions on $[t_0, t_0 + h]$ for some $h > 0$. Using the uniform Lipschitz continuity of $\tilde{F}$ we can estimate that if $t_0 \leq t \leq t_0 + h$

$$\begin{aligned} |L(\tilde{F}(v_1))(t) - L(\tilde{F}(v_2))(t)| &= \left|\int_{t_0}^t k(t-s)(\tilde{F}(v_2(s)) - \tilde{F}(v_1(s)))\,ds\right| \\ &\leq K_1\|v_1 - v_2\|_{L^\infty[t_0,t_0+h]}\int_{t_0}^t |k(t-s)|\,ds. \end{aligned}$$

By choosing h sufficiently small the integral on the right can be made arbitrarily small (say $< 1/(2K_1)$) and we obtain $\|L(\tilde{F}(v_1)) - L(\tilde{F}(v_2))\|_{L^\infty[t_0,t_0+h]} \leq C\|v_1 - v_2\|_{L^\infty[t_0,t_0+h]}$ for some constant $C < 1$. Thus $\phi \to L(\tilde{F}(\phi))$ is a contraction mapping on the complete space $C^0[t_0, t_0 + h]$, and hence has a unique fixed point $C^0[t_0, t_0 + h]$, the solution to $v = L(\tilde{F}(v)) + g$. Note that the choice of h does not depend on the function g, but only on M, F, and the function k. This proves Claim 1.

Claim 2: Suppose that equation (17) has a continuous solution on some interval $[t_0, t_1]$. Then this solution extends to a continuous solution on $[t_0, t_1 + h)$ for some $h > 0$, where h depends only on M, F, and k.

Proof: Let $v_0(t)$ be a continuous solution to $v = L(\tilde{F}(v)) + g$ on some interval $[t_0, t_1]$. Let $h > 0$. For any continuous function v on $[t_0, t_1 + h]$ and $t \in [t_1, t_1 + h]$, we have

$$
\begin{aligned}
& - \int_{t_0}^{t} k(t-s)\tilde{F}(v(s))\,ds + g(t) \\
= & - \int_{t_1}^{t} k(t-s)\tilde{F}(v(s))\,ds - \int_{t_0}^{t_1} k(t-s)\tilde{F}(v(s))\,ds + g(t) \\
= & - \int_{t_1}^{t} k(t-s)\tilde{F}(v(s))\,ds - \int_{t_0}^{t_1} k(t_1-s)\tilde{F}(v(s))\,ds - \int_{t_0}^{t_1} k(t-s)\tilde{F}(v(s))\,ds \\
+ & \int_{t_0}^{t_1} k(t_1-s)\tilde{F}(v(s))\,ds + g(t) \\
= & - \int_{t_1}^{t} k(t-s)\tilde{F}(v(s))\,ds + v_0(t_1) - \int_{t_0}^{t_1} (k(t-s) - k(t_1-s))\tilde{F}(v_0(s))\,ds + g(t) - g(t_1) \quad (18) \\
= & - \int_{t_1}^{t} k(t-s)\tilde{F}(v(s))\,ds + \tilde{g}(t) \quad (19)
\end{aligned}
$$

where $\tilde{g}(t) = v_0(t_1) - \psi(t) + g(t) - g(t_1)$ and $\psi(t)$ denotes the second integral on the right in equation (18). The function $\tilde{g}(t)$ is continuous on the interval $[t_0, t_1 + h)$. By Claim 1 (with t_1 replacing t_0 and $\tilde{g}$ replacing g) we know that the equation

$$
v(t) = - \int_{t_1}^{t} k(t-s)\tilde{F}(v(s))\,ds + \tilde{g}(t) \tag{20}
$$

is uniquely solvable on the interval $[t_1, t_1 + h]$ for some $h > 0$, where h depends only on M, F, and k. Let the solution on this interval be denoted by $v_1(t)$. Since $v_1(t_1) = v_0(t_1)$, the function $u(t)$ defined by

$$
v(t) = \begin{cases} v_0(t), & t_0 \le t \le t_1 \\ v_1(t), & t_1 < t \le t_1 + h \end{cases}
$$

is continuous and by reversing the equations leading up to equation (19) we see that $v(t)$ is the (unique) solution to $v = L(\tilde{F}(v)) + g$ on $[t_0, t_1 + h]$ with h dependent on M, F, and k. This concludes the proof of Claim 2.

We can now finish the proof of Lemma 4.3. Let $v(t)$ be a bounded continuous solution to $v = L(F(v)) + g$ on some interval $[0, t_1)$, with $A = \sup_{0 \le t < t_1} |v|$. Choose $M > f(A)$ in the definition of $\tilde{F}$, so that v also satisfies $v = L(\tilde{F}(v)) + g$ on $[t_0, t_1)$ (for if $|v| \le A$ then $f(v) \le f(A) < M$ and so $F(v) = \tilde{F}(v)$. Here we make use of the fact that F is non-decreasing.) Now v is a solution to $v = L(\tilde{F}(v)) + g$ on any interval $[t_0, t_1 - \epsilon]$ with $\epsilon > 0$, and by Claim 2 we can extend v as a solution to $v = L(\tilde{F}(v)) + g$ to any interval $[t_0, t_1 - \epsilon + h]$ where h is independent of ϵ, so that v extends as a solution to $v = L(\tilde{F}(v)) + g$ on $[t_0, t_1 + \delta)$ for some $\delta > 0$. But since v is continuous it is clear that v must also extend for some distance past $t = t_1$ as a solution of $v = L(F(v)) + g$ (until $|f(v(t))| > M$). This completes the proof of Lemma 4.3. □

From Lemma 4.3 it follows that if a solution to equation (13) fails to exist beyond some point $t = T$ then the solution must be unbounded on $[0, T)$. Using the positivity of the function a and the condition $xF(x) \ge 0$ one can verify that we cannot simply have $\lim_{t \to T-} v = \infty$ or $\lim_{t \to T-} v = -\infty$; the solution must thus exhibit some kind of unbounded oscillatory behavior if it fails to exist beyond $t = T$. However, we can rule this out (and thereby prove that solutions exist on $[0, \infty)$).

Lemma 4.4 *A continuous solution $v(t)$ to equation (13) on an interval $[0,T)$ must be bounded on $[0,T)$.*

Proof: Consider the quantity $M(t) = \sup_{0\le s\le t} v(s)$. Let us suppose that for some real number A we have $M(t_1) = A$ for some $t_1 < T$. We will show that $F(A) \le \frac{G}{2}\sqrt{\pi T}$ (where G is the Lipschitz constant for g), which will immediately imply that v is bounded. Let t_A denote any point in $[0, t_1]$ for which $v(t_A) = A$ (so the supremum on $[0, t_1]$ is actually attained at t_A.) We have $v(t) \le v(t_A)$ for all $t < t_A$.

Let $\epsilon > 0$ and let $t_2 = \sup\{t : t < t_A, v(t) = A - \epsilon\}$. (See Figure 1 below.)

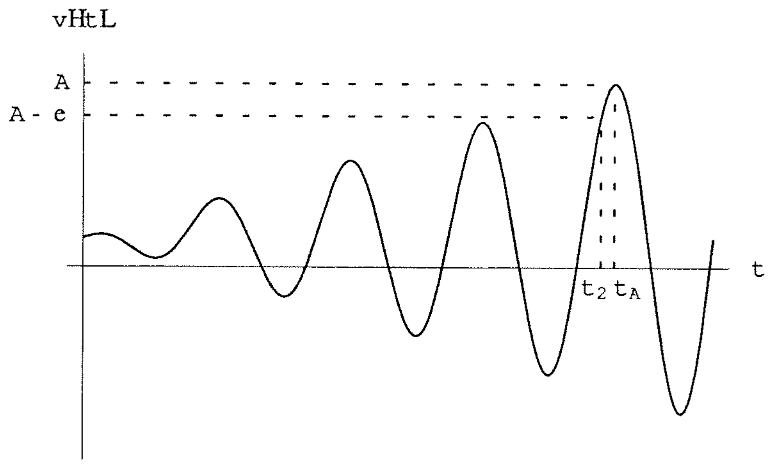

Figure 1.

Continuity of v implies $v(t_2) = A - \epsilon$, so t_2 is the largest root of $v(t) = A - \epsilon$ with $t < t_A$. It is clear from continuity of v that $A - \epsilon < v(t) \le A$ for $t \in (t_2, t_A)$ and that as $\epsilon \to 0$ we have $t_2 \to t_A$. Set $\delta = t_A - t_2$. Now, from equation (19) with $t = t_A$ (and using $\tilde{F} = F$ with t_2 in place of t_1—the same identity holds in this case) we have

$$v(t_A) = -\int_{t_2}^{t_A} k(t_A - s)F(v(s))\,ds + v(t_2) - \psi(t_A) + g(t_A) - g(t_2) \tag{21}$$

(recall $\psi(t)$ is defined as the second integral on the right in equation (18).) We can bound the first integral on the right above as

$$\begin{aligned} -\int_{t_2}^{t_A} k(t_A - s)F(v(s))\,ds &= -\int_{t_2}^{t_A} F(v(s)) \left(\frac{2}{\sqrt{\pi(t_A - s)}} + a(t_A - s) \right) ds \\ &\le -F(v(t_2)) \int_{t_2}^{t_A} \left(\frac{2}{\sqrt{\pi(t_A - s)}} + a(t_A - s) \right) ds \\ &\le -F(v(t_2)) \frac{4}{\sqrt{\pi}} \sqrt{\delta} \end{aligned} \tag{22}$$

using the fact that $v(t_2) < v(t) \le v(t_A)$ for $t \in (t_2, t_A)$, F non-decreasing, and a positive (so its contribution to the integral can be dropped.) A similar estimate gives

$$\begin{aligned}
-\psi(t_A) &= \int_0^{t_2} (k(t_2 - s) - k(t_A - s))F(v(s))\, ds \\
&= \int_0^{t_2} \left(\frac{2}{\sqrt{\pi(t_2 - s)}} - \frac{2}{\sqrt{\pi(t_A - s)}} + a(t_2 - s) - a(t_A - s) \right) F(v(s))\, ds \\
&\le \int_0^{t_2} \left(\frac{2}{\sqrt{\pi(t_2 - s)}} - \frac{2}{\sqrt{\pi(t_A - s)}} \right) F(v(s))\, ds \\
&\le F(v(t_A)) \int_0^{t_2} \left(\frac{2}{\sqrt{\pi(t_2 - s)}} - \frac{2}{\sqrt{\pi(t_A - s)}} \right) ds \\
&= \frac{4}{\sqrt{\pi}} F(v(t_A))(\sqrt{\delta} + \sqrt{t_2} - \sqrt{t_A}) \qquad (23)
\end{aligned}$$

where we have used the fact (Lemma 4.1) that a is non-decreasing, so that $a(t_2 - s) - a(t_A - s) \le 0$.

Inserting the estimates (22) and (23) into equation (21) and rearranging yields

$$v(t_A) - v(t_2) \le \frac{4}{\sqrt{\pi}} \sqrt{\delta}(F(v(t_A)) - F(v(t_2))) + \frac{4}{\sqrt{\pi}} F(v(t_A))(\sqrt{t_2} - \sqrt{t_A}) + g(t_A) - g(t_2).$$

We can use the inequality $\sqrt{y} - \sqrt{x} \le \frac{y-x}{2\sqrt{x}}$ (valid for $0 \le x, y$, easily derived from $\sqrt{xy} \le (x+y)/2$) with $y = t_2$ and $x = t_A$ to obtain

$$v(t_A) - v(t_2) \le \frac{4}{\sqrt{\pi}} \sqrt{\delta}(F(v(t_A)) - F(v(t_2))) - \frac{2}{\sqrt{\pi}} \frac{F(v(t_A))\delta}{\sqrt{t_A}} + g(t_A) - g(t_2).$$

Now since $v(t_A) - v(t_2) \ge 0$ by construction, we must have

$$0 \le \frac{4}{\sqrt{\pi}} \sqrt{\delta}(F(v(t_A)) - F(v(t_2))) - \frac{2}{\sqrt{\pi}} \frac{F(v(t_A))\delta}{\sqrt{t_A}} + g(t_A) - g(t_2). \qquad (24)$$

By Lemma 4.2 and the remarks which follow that lemma we have $v \in C^\beta$ for all $\beta < 1$ and so $v(t_A) - v(t_2) < C(A, \beta)(t_A - t_2)^\beta$ for all $\beta < 1$. Also, because F is uniformly Lipschitz continuous on any bounded interval we have $F(v(t_A)) - F(v(t_2)) \le C_2(A)(v(t_A) - v(t_2))$, where we've noted that C_2 will depend on A. Inserting these facts into inequality (24) we find that

$$0 \le C_3 \delta^{\beta + 1/2} - \frac{2}{\sqrt{\pi}} \frac{F(v(t_A))\delta}{\sqrt{t_A}} + G\delta$$

for all $\beta < 1$, where G is the Lipschitz constant for g at $t = t_A$. Dividing through by δ and considering the limit as δ approaches zero shows that

$$0 \le -\frac{2}{\sqrt{\pi}} \frac{F(v(t_A))}{\sqrt{t_A}} + G$$

or

$$F(v(t_A)) \le \frac{G\sqrt{\pi}}{2} \sqrt{t_A} \le \frac{G\sqrt{\pi}}{2} \sqrt{T}. \qquad (25)$$

Thus $F(v(t))$ is bounded above on $[0,T)$. But inserting this into the original integral equation (13) immediately shows that v is bounded below on $[0,T)$, for

$$v(t) = -\int_0^t F(v(s))k(t-s)\,ds + g(t) \geq -\frac{G\sqrt{\pi T}}{2}\int_0^t k(t-s)\,ds - GT. \tag{26}$$

We can insert this lower bound for v back into the integral equation and perform a similar estimate to bound v above. Thus v is bounded on $[0,T)$. This completes the proof of Lemma 4.4. □

Proof of Lemma 3.1: First, equation (12) has a solution on some non-empty interval $[0,t_1)$. To see this, choose $M > g(0)$ in the definition of $\tilde{F}$ in the proof of Lemma 4.3. Claim 1 in the proof of Lemma 4.3 makes it clear that $v = L(\tilde{F}(v)) + g$ has a unique continuous solution on some interval $[0,t_1')$. Since $\tilde{F}(v) = F(v)$ for $|v| \leq M$ and v is continuous with $v(0) = g(0)$ the function also satisfies $v = L(F(v)) + g$ on some interval $[0,t_1)$. Thus $v = L(F(v)) + g$ has a unique continuous solution on some interval $[0,t_1)$.

We prove global existence by contradiction: suppose that equation (13) does NOT have a solution on $[0,\infty)$. Let $[0,T)$ the largest interval on which the equation does possess a continuous solution (from the above paragraph such an interval exists.) From Lemma 4.4 we know that v is bounded on $[0,T)$. But Lemma 4.3 then shows that v can be extended to some interval $[0,T+\delta)$, a contradiction. Thus the solution must exist on $[0,\infty)$. Uniqueness follows analogously from Lemma 4.3. □.

Since equation (13) has a unique solution, we can now insert $v = u^+ - u^-$ into the right side of equations (10) and (11) and apply the operators $H_\sigma = (I - K_\sigma^2)^{-1}$ and $H_{1-\sigma} = (I - K_{1-\sigma}^2)^{-1}$, respectively, to solve for the continuous functions u^- and u^+ which satisfy the pair of equations (10) and (11). We can then insert u^- and u^+ into equations (6) and (9) to obtain continuous functions u_0 and u_1; the functions u_0, u^-, u^+, u_1 are the unique continuous solutions to the system (6)-(9).

5 Solution of the IBVP

For the IBVP (1)-(5), we can use the last formula in Lemma 2.1 to define a formal solution to IBVP, given for x in $(0,\sigma) \cup (\sigma,1)$ and $T > 0$, by

$$u(T,x) = \begin{cases} \frac{1}{2}A_x(g_0) + \frac{1}{2}A_{\sigma-x}(F(u^+ - u^-)) + \frac{1}{2}K_x(u_0) + \frac{1}{2}K_{\sigma-x}(u^-), & 0 < x < \sigma; \\ \frac{1}{2}A_{1-x}(g_1) - \frac{1}{2}A_{\sigma-x}(F(u^+ - u^-)) + \frac{1}{2}K_{1-x}(u_1) - \frac{1}{2}K_{\sigma-x}(u^+), & \sigma < x < 1; \end{cases} \tag{27}$$

where u_0, u_1, u^+, u^- are the unique continuous solutions to equations (6)-(9). It is easy to see that this function is a smooth solution of the heat equation in $(0,\sigma) \cup (\sigma,1)$ for $T > 0$, and that $u(0,x) = 0$ for all $x \in [0,1]$.

To prove Theorem 1.1 it remains to show that this formal solution in fact satisfies the boundary conditions (2)-(4). This is the focus of the following two lemmas.

Lemma 5.1 *For each $T > 0$, $u(T,\cdot)$, as defined in (27), is in $C[0,\sigma] \cup C[\sigma,1]$ with $\lim_{x \searrow 0} u(T,x) = u_0(T)$, $\lim_{x \nearrow \sigma} u(T,x) = u^-(T)$, $\lim_{x \searrow \sigma} u(T,x) = u^+(T)$, and $\lim_{x \nearrow 1} u(T,x) = u_1(T)$.*

Proof. Since $u(T,x)$ is smooth for $x \in (0,\sigma)\cup(\sigma,1)$, we need only check continuity at the endpoints. Let us first consider the behavior of u defined by equation (27) as $x \to 0+$; it is clear that the first integral on the right approaches $\frac{1}{2}A_0(g_0)$. The second and fourth integrals are easily seen to approach $\frac{1}{2}A_\sigma(F(u^+ - u^-))$ and $\frac{1}{2}K_\sigma(u^-)$, respectively. Now to evaluate the third integral one can directly verify by integration that

$$\lim_{x\to 0+}\int_0^\epsilon \frac{xe^{-\frac{x^2}{4t}}}{4\sqrt{\pi t^3}}\,ds = \lim_{x\to 0+}\frac{1}{2}(1-\mathrm{erf}(\frac{x}{2\sqrt{\epsilon}})) = \frac{1}{2},\quad \lim_{x\to 0+}\int_\epsilon^\infty \frac{xe^{-\frac{x^2}{4t}}}{4\sqrt{\pi t^3}}\,ds = \lim_{x\to 0+}\frac{1}{2}\mathrm{erf}(\frac{x}{2\sqrt{\epsilon}}) = 0 \quad (28)$$

for any ϵ, from which we can deduce (using continuity of u_0) that $\frac{1}{2}K_x(u_0)(T)$ approaches $\frac{1}{2}u_0(T)$. All in all we find that $u(t,x)$ defined by equation (27) has a well-defined limit as $x \to 0+$ and so u extends continuously to $[0,\sigma)$ as

$$u(t,0) = \frac{1}{2}u_0(t) + \frac{1}{2}A_2(g_0) + \frac{1}{2}A_1(F(u^+ - u^-)) + \frac{1}{2}K_1(u^-).$$

In light of (6) however, we have simply $u(t,0) = u_0(t)$. One can use (7)-(9) to similarly verify that $u(t,\sigma^-) = u^-(t), u(t,\sigma^+) = u^+(t)$, and $u(t,1) = u_1(t)$ extend u continuously to $[0,\sigma]\cup[\sigma,1]$. □

Lemma 5.2 *For each $T > 0$, $u(T,\cdot)$ is in $C^1[0,\sigma]\cup C^1[\sigma,1]$.*

Proof We compute $\frac{\partial u}{\partial x}$ using equation (27) and take the limit as x approaches 0 from the right. We have for $x \in (0,1)$

$$2\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}A_1(x)(g_0) + \frac{\partial}{\partial x}A_1(\sigma - x)(F(u^+ - u^-)) + \frac{\partial}{\partial x}K_1(x)(u_0) + \frac{\partial}{\partial x}K_1(\sigma - x)(u^-). \quad (29)$$

Denote the right side above as $-I_1 - I_2 - I_3 + I_4$, where the I_j denote the integrals in the order they appear. Since $x \in (0,\sigma)$ we can move $\frac{\partial}{\partial x}$ inside the integrals.

As x approaches zero from the right one can use (28) to see that I_1 approaches $\frac{1}{2}g_0(t)$. The behavior of I_2 presents no difficulty and I_2 is seen to approach $\frac{1}{2}K_1(F(u^+ - u^-))(t)$. One can see that I_4 approaches the well-defined limit

$$I_4 = \frac{1}{8\sqrt{\pi}}\int_0^t \frac{(\sigma^2 - 2(t-s))e^{-\frac{\sigma^2}{4(t-s)}}}{(t-s)^{5/2}}u^-(s)\,ds.$$

Only the existence of a limiting value for I_3 is non-trivial. However, since g_0 is C^1 can check that $A_2(g_0)$ is also C^1, and equation (6) makes it clear that u_0 must also be C^1 (since A_1 and K_1 are smoothing.) We can thus integrate by parts in I_3 to find that for $x > 0$

$$I_3 = -A_x(u_0').$$

As $x \to 0^+$ this integral approaches a well-defined limit. We conclude that $\frac{\partial u}{\partial x}$ extends continuously to $x = 0$, and similar arguments show that $\frac{\partial u}{\partial x}$ extends continuously to $[0,\sigma]\cup[\sigma,1]$.

□

Proof of Theorem 1.1: In light of Lemma 5.2, it remains only to show that the Neumann boundary conditions are satisfied. We confine our attention to the interval $[0, \sigma]$. We now know that our formal solution $u(t, x)$ defined by (27) is a classical $C^1[0, \sigma]$ solution to the heat equation, satisfies $u(0, x) = 0$ as well as $u(T, 0) = u_0(T)$ and $u(T, \sigma^-) = u^-(T)$. From Lemma 2.1 we can derive

$$
\begin{aligned}
u(T,0) &= u_0(T) = A_0(\tilde{g}_0) + A_\sigma(\tilde{g}_\sigma) + K_\sigma(u^-) && (30)\\
u(T,\sigma^-) &= u^-(T) = A_0(\tilde{g}_\sigma) + A_\sigma(\tilde{g}_0) + K_\sigma(u_0)\,; && (31)
\end{aligned}
$$

where

$$\tilde{g}_0 = -\lim_{x \searrow 0} \frac{\partial v}{\partial x}(T, x) \text{ and } \tilde{g}_\sigma = \lim_{x \nearrow \sigma} \frac{\partial v}{\partial x}(T, x)\,,$$

both of which exist, by virtue of Lemma 5.2. Since we have shown that u_0, u^-, u^+, u_1 satisfy equations (6)-(9), comparison of (30) with (6) and (31) with (7) shows that

$$
\begin{aligned}
A_0(g_0) + A_\sigma(F(u^+ - u^-)) &= A_0(\tilde{g}_0) + A_\sigma(\tilde{g}_\sigma)\\
A_0(F(u^+ - u^-)) + A_\sigma(g_0) &= A_0(\tilde{g}_\sigma) + A_\sigma(\tilde{g}_0)\,.
\end{aligned}
$$

These can be rearranged to yield

$$
\begin{aligned}
A_0(f_1) + A_\sigma(f_2) &= 0 && (32)\\
A_0(f_2) + A_\sigma(f_1) &= 0\,, && (33)
\end{aligned}
$$

where $f_1 \equiv g_0 - \tilde{g}_0$ and $f_2 \equiv F(u^+ - u^-) - \tilde{g}_\sigma$.

The inequalities (25) and (26) can be used to show that both $v(t) = u^+(t) - u^-(t)$ and $F(v(t))$ are of exponential order in t (i.e., bounded by Ae^{ct} for some constants A and c for $t \geq t_0$.) Moreover, one can easily show that if $h(t)$ is a continuous function of exponential order and x is positive then $A_0(h)$, $A_x(h)$, and $K_x(h)$ are all of exponential order. Then, assumptions A1-A2, along with the representation (27), allow us to conclude that f_1 and f_2 are of exponential order. Consequently, we can take the Laplace transform of both sides of (32) and (33). So doing, we obtain, for transform variable $s > 0$,

$$
\begin{aligned}
\hat{f}_1(s) &= -e^{-\sigma\sqrt{s}}\hat{f}_2(s) && (34)\\
\hat{f}_2(s) &= -e^{-\sigma\sqrt{s}}\hat{f}_1(s)\,, && (35)
\end{aligned}
$$

where the hat denotes the Laplace transform. Here, we have used the facts that

$$\hat{G}(t, 0) = \frac{1}{\sqrt{s}} \text{ and } \hat{G}(t, x) = \frac{e^{-x\sqrt{s}}}{\sqrt{s}}$$

for $x > 0$.

Combining (34) and (35) yields

$$\hat{f}_1(s) = e^{-2\sigma\sqrt{s}}\hat{f}_1(s)\,,$$

which can hold for $s > 0$ only if $\hat{f}_1(s)$ is the zero function. Consequently, $f_1(t) \equiv 0$, so that $\tilde{g}_0 = g_0$, i.e. v satisfies the Neumann condition at $x = 0$. A similar argument shows that $\tilde{g}_\sigma = F(u^+ - u^-)$, and that the corresponding boundary conditions on $[\sigma, 1]$ also hold, completing the proof. □

6 Conclusion

As mentioned in the introduction, our real interest is the inverse problem of recovering the defect location and dynamics from boundary measurements. In the present one-dimensional case we would like to investigate whether one can, using data at $x = 0$ and/or $x = 1$, determine the crack location σ (considering the function F known), or determine the function F knowing the crack location σ, or possibly determine both from the data, as well as develop stability estimates. Knowledge of the function F might yield useful information about the nature and severity of the defect at $x = \sigma$.

We would also of course like to extend these existence and uniqueness results, as well as analysis of the inverse problem, to two and three dimensional bodies containing, respectively, one and two dimensional cracks.

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