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Abstract. A numerical semigroup S is a set of nonnegative integers such that S contains 0, S is closed under addition, and the complement of S is finite. This paper considers pairs (S, I) of a given numerical semigroup S and corresponding relative ideal I such that $\mu(I)\mu(S-I) = \mu(I+(S-I))$, where μ denotes the size of the minimal generating set and $S-I$ is the dual of I in S. We will present recent results in the research of such pairs (perfect bricks) with $\mu(I) > 2$ and $\mu(S-I) > 2$. We will also show the existence of an infinite family of perfect bricks.

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1 Introduction

There are well-known problems in recreational mathematics that have a direct correlation to the content of this paper. We usually refer to the most famous of these as the Chicken McNugget problem, the postage stamp problem, or the coin problem.

For example, consider the postage stamp problem: Let $\{a_1, a_2, \ldots, a_n\}$ be a set of n distinct stamp denominations, with $a_1 < a_2 < \ldots < a_n$ and $\gcd(a_1, \ldots, a_n) = 1$. Find the largest postage that cannot be formed by using an unlimited number of each of these stamp denominations.

Example 1.1. Suppose we have infinitely many 7-cent, 8-cent, and 11-cent stamps. We need to consider all linear combinations of 7, 8, and 11 over the nonnegative integers. That is,

$$\begin{aligned}\{\text{all possible postages}\} &= \{7k_1 + 8k_2 + 11k_3 \mid k_1, k_2, k_3 \in \mathbb{N} \cup \{0\}\} \\ &= \{0, 7, 8, 11, 14, 15, 16, 18, 19, 21, 22, 23, 24, 25, 26, 27, \ldots\}.\end{aligned}$$

Since we have found seven consecutive integers, we know that we can obtain every integer by adding the smallest stamp denomination. Thus the largest postage one cannot make with these stamp denominations is 20 cents.

This largest "unformable" stamp value only exists when the greatest common divisor of the given stamp denominations is 1. This largest integer not contained in the set is usually referred to as the Frobenius number.

Example 1.2. Let 4 and 7 be the stamp denominations (or generators) of our set.

$$\begin{aligned}\{\text{all possible postages}\} &= \{4k_1 + 7k_2 \mid k_1, k_2 \in \mathbb{N} \cup \{0\}\} \\ &= \{0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, 23, \ldots\}\end{aligned}$$

The largest integer not contained in this set is 17. Notice that $17 = 4 \cdot 7 - (4 + 7)$.

There is a known closed form for finding the Frobenius number, denoted $F(S)$, for two generators a_1, a_2:

$$F(S) = a_1a_2 - (a_1 + a_2)$$

where $a_1, a_2 \in \mathbb{Z}^+$ are relatively prime [7]. Finding a closed form associated with the Frobenius number for 3 or more generators is an open problem in mathematics, but there are results for specific cases of three generated numerical semigroups. For more information, see [1].

In this paper, we are not concerned with the Frobenius number of a given set of generators but with the entire set of values that is formed by the generators, namely numerical semigroups. In particular, we will focus on special pairs (S, I) of numerical semigroups and related sets of integers called relative ideals that correspond to analogous structures in ring theory.

The next section of this paper will define and show through example numerical semigroups and their properties. In Section 3, we discuss a bridge between numerical semigroups and commutative ring theory. This bridge, a natural valuation map, allows algebraists to translate problems from commutative ring theory into a simpler context (numerical semigroups). This connection has been explored since the early 1990's.

In Section 4, we discuss the aforementioned pairs (S, I) of numerical semigroups and relative ideals such that $\mu(I) = 2$, $\mu(S - I) = 2$, and $\mu(I)\mu(S - I) = \mu(I + (S - I))$ where $S - I$ is the dual of I in S and $\mu(\cdot)$ denotes the size of the minimal generating set. We call such pairs 2×2 perfect bricks. We conclude this paper with a discussion of pairs (S, I) with $\mu(I) > 2$, $\mu(S - I) > 2$, and $\mu(I)\mu(S - I) = \mu(I + (S - I))$, that is, perfect bricks with dimension higher than 2×2.

2 Numerical Semigroups

Before discussing perfect bricks and the aforementioned connection to ring theory, we need to define a numerical semigroup and its corresponding structures.

Definition 2.1. A *numerical semigroup* is a set S of nonnegative integers such that

(i) S is closed under addition

(ii) S contains 0

(iii) the complement $\mathbb{N} \setminus S$ is finite.

We say $\{a_1, a_2, \ldots, a_n\}$ is a generating set for S provided

$$S = \{k_1 a_1 + \ldots + k_n a_n \mid k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}\}$$

and $\{a_1, \ldots, a_n\}$ is the *minimal generating set* for S if no proper subset forms a generating set. We write $S = \langle a_1, \ldots, a_n \rangle$ where $0 < a_1 < \ldots < a_n$, and we let $\mu(S)$ denote the size of the minimal generating set of S.

Example 2.1. Consider the numerical semigroup S generated by 4 and 7. Then

$$\begin{aligned} S = \langle 4, 7 \rangle &= \{4k_1 + 7k_2 \mid k_1, k_2 \in \mathbb{N} \cup \{0\}\} \\ &= \{0, 4, 7, 8, 11, 12, 15, 16, 18, 19, 20, 21, 22, 23, \ldots\} \\ &= \{0, 4, 7, 8, 11, 12, 15, 16, 18, \rightarrow\}. \end{aligned}$$

Clearly every integer after 18 can be formed; we denote this with an arrow.

The next structure we need in order to discuss perfect bricks is a relative ideal I of a numerical semigroup S.

Definition 2.2. Let S be a numerical semigroup. A nonempty set of integers I is called a *relative ideal* provided

(i) I has a smallest element

(ii) for all $i \in I$ and $s \in S$, $i + s \in I$.

It follows from the definition that a relative ideal I of S can be expressed as a finite union of cosets $z + S$ where $z \in \mathbb{Z}$. The set $\{z_1, \ldots, z_k\}$ is the *minimal generating set* for the relative ideal if I cannot be written as a union of cosets using any proper subset of $\{z_1, \ldots, z_k\}$. We write

$$I = (z_1, \ldots, z_k) = (z_1 + S) \cup \ldots \cup (z_k + S)$$

where $0 < z_1 < \ldots < z_k$, and we let $\mu(I)$ denote the size of the minimal generating set of I.

Example 2.2. Let $S = \langle 4, 7 \rangle = \{0, 4, 7, 8, 11, 12, 15, 16, 18, \rightarrow\}$. Then consider the relative ideal generated by 0 and 1:

$$\begin{aligned} I_1 &= (0,1) = (0 + S) \cup (1 + S) \\ &= \{0, 4, 7, 8, 11, 12, 15, 16, 18, \rightarrow\} \cup \{1, 5, 8, 9, 12, 13, 16, 17, 19, \rightarrow\} \\ &= \{0, 1, 4, 5, 7, 8, 9, 11, 12, 13, 15, \rightarrow\}. \end{aligned}$$

Now consider the relative ideal of $S = \langle 4, 7 \rangle$ generated by -1 and 0. We have

$$I_2 = (-1, 0) = \{-1, 0, 3, 4, 6, 7, 8, 10, 11, 12, 14, \rightarrow\}.$$

Note that the second relative ideal is just a linear translation of the first and that the two relative ideals are isomorphic.

Remark 2.1. For the remainder of this paper, we will assume, without loss of generality, that the smallest generator of a relative ideal is 0.

Lemma 2.1. *Let I and J be relative ideals of a numerical semigroup S. Then the ideal sum $I + J = \{i + j \mid i \in I \text{ and } j \in J\}$ is also a relative ideal of S.*

Proof. Recall that a set K of integers is a relative ideal of a numerical semigroup S provided that K contains a smallest element and if for all $k \in K$ and for all $s \in S$, $k + s \in K$. Consider the set of integers $I + J = \{i + j \mid i \in I, j \in J\}$, where I and J are relative ideals of S.

We first need to show that $I + J$ has a smallest element. By definition, I has a smallest element, say i_m. Similarly, J has a smallest element, say j_m. Then $i_m + j_m \in I + J$ and must be the minimum element in $I + J$.

Next we need to show that $I + J$ satisfies the closure property of relative ideals. Let $i \in I$, $j \in J$, and $s \in S$. Then $j + s = j' \in J$ since J is a relative ideal, and we have

$$\begin{aligned} i + j + s &= i + (j + s) \\ &= i + j' \\ &\in I + J. \end{aligned}$$

Thus $I + J$ is a relative ideal of S. □

Definition 2.3. Let I be a relative ideal of a numerical semigroup S. We define the *dual of I in S* to be

$$S - I = \{z \in \mathbb{Z} \mid z + I \subseteq S\}.$$

Informally, the dual of I in S is the set of integers which "knocks" I into S by addition. Again we let $\mu(S - I)$ denote the size of the minimal generating set of $S - I$.

Lemma 2.2. *If I is a relative ideal of a numerical semigroup S, then the dual of I in S is also a relative ideal of S.*

Proof. Let S be a numerical semigroup and I be a relative ideal of S. Recall that the dual of I in S is defined as $S - I = \{z \in \mathbb{Z} \mid z + I \subseteq S\}$. In order to conclude that $S - I$ is a relative ideal of S, we need to show that it has a smallest element and that for all $z \in S - I$ and for all $s \in S$, $z + s \in S - I$.

By Remark 2.1 and since S and I are both bounded below, we know that the elements of $S - I$ are nonnegative integers. Therefore $S - I$ must have a smallest element.

Let $z \in S - I$ and $s \in S$. Since $z + I \subseteq S$ and S is closed under addition, $(z + s) + I \subseteq S$. This implies that $z + s \in S - I$. Thus $S - I$ is a relative ideal of the numerical semigroup S. □

Via Lemmas 2.1 and 2.2, the ideal sum of the dual of I in S and a relative ideal I, that is $I + (S - I)$, is itself a relative ideal of S.

Example 2.3. Consider $S = \langle 5, 6, 13\rangle = \{0, 5, 6, 10, 11, 12, 13, 15, \rightarrow\}$ and $I = (0, 1) = \{0, 1, 5, 6, 7, 10, \rightarrow\}$. Then, to find the dual of I in S, we consider elements of $\mathbb{N} \cup \{0\}$, starting with 0:

$$0 \in S - I \iff 0 + I \subseteq S \iff \{0, 1, 5, 6, \ldots\} \subseteq S \implies 0 \notin S - I$$

$$1 \in S - I \iff 1 + I \subseteq S \iff \{1, 2, 6, 7, \ldots\} \subseteq S \implies 1 \notin S - I$$

It is clear that 2, 3, and 4 will not be elements of $S - I$ so we can just skip to 5:

$$5 \in S - I \iff 5 + I \subseteq S \iff \{5, 6, 10, 11, \ldots\} \subseteq S \implies 5 \in S - I$$

Continuing this process, we find that $S - I = \{5, 10, 11, 12, 15, \rightarrow\}$.

To determine the generators of $S - I$, we use the coset-based construction of the minimal generating set of a relative ideal. The smallest element of $S - I$, namely 5, must be a generator. Consider the coset $5 + S$:

$$5 + S = \{5 + s \mid s \in S\} = \{5, 10, 11, 15, 16, 17, 18, 20, \rightarrow\}.$$

The smallest element of $S - I$ which is not in this coset is 12. Thus 12 is also a generator, so we consider the coset $12 + S$:

$$12 + S = \{12 + s \mid s \in S\} = \{12, 17, 18, 22, 23, 24, 25, 27, \rightarrow\}.$$

Since every element of $S - I$ is contained in the union of $5 + S$ and $12 + S$, the only generators are 5 and 12. Therefore $S - I = (5, 12)$.

3 Motivation

There exists a connection between the properties and constructs of numerical semigroups and those in certain types of commutative rings.

We consider power series rings of the form $R = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_n}]]$, where k is a field and each a_i is a positive integer. Let $\bar{R}$ be the integral closure of R. An element of $\bar{R}$ is of the form $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$ where $a_i \in k$. There exists a well-defined function ν from the integral closure into the nonnegative integers that maps a power series element to its 'smallest exponent'. That is, $\nu : \bar{R} \rightarrow \mathbb{Z}^+$ is defined as

$$\nu(p(t)) = m$$

where a_m is the first nonzero coefficient in $p(t)$.

The natural valuation map ν creates a bridge between this type of power series ring and its corresponding numerical semigroup. Thus there is a one-to-one correspondence between power series ring monomials and numerical semigroup elements.

Example 3.1. Consider the power series ring $R = \mathbb{Q}[[t^5, t^8, t^9]]$. Via the natural valuation map $\nu(R)$, the ring $\mathbb{Q}[[t^5, t^8, t^9]]$ corresponds to the numerical semigroup generated by 5, 8, and 9 – that is, $S = \langle 5, 8, 9 \rangle = \{5, 8, 9, 10, 13, \rightarrow\}$.

Another powerful connection numerical semigroups have to ring theory is through Gorenstein rings.

Definition 3.1. Define $n(S) = |\{x \in S \mid x < F(S)\}|$. Then S is *symmetric* provided

$$n(S) = \frac{F(S) + 1}{2}$$

and $F(S)$ is odd.

Theorem 3.1 (Kunz [6])**.** *A ring $R = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_n}]]$ is Gorenstein if and only if $S = \nu(R)$ is a symmetric numerical semigroup.*

It can be shown that $\mu(I)\mu(S - I) \geq \mu(I + (S - I))$ holds for every pair (S, I). For the sake of determining perfect bricks, we are interested in the equality

$$\mu(I)\mu(S - I) = \mu(I + (S - I)) \tag{1}$$

when $\mu(I) \geq 2$. It is only when the above equality holds that nonzero torsion might exist in the corresponding ring context.

This fact about the generating sets of numerical semigroups and relative ideals is a consequence of the corresponding concept in commutative ring theory. Namely, if $(R, \mathbf{m})$ is a one-dimensional Noetherian local domain, I is a nonprincipal fractional ideal of R, and I^{-1} is the inverse of the ideal, then $\mu_R(I)\mu_R(I^{-1}) \geq \mu_R(II^{-1})$ always holds.

In this context, the strict inequality $\mu_R(I)\mu_R(I^{-1}) > \mu_R(II^{-1})$ implies the existence of nonzero torsion in $I \otimes_R I^{-1}$, the tensor product of I and its inverse. The concept of nonzero torsion in ring theory is somewhat analogous to the existence of zero divisors in group theory. The details surrounding this consequence are beyond the scope of this paper. For more information, see [3], [4].

For further reading about the correspondence between numerical semigroups, power series rings, fractional ideals, duals, and torsion in tensor products, see [3], [4], [6].

4 Perfect Bricks

Definition 4.1. A set of four nonnegative integers $A = \{a_1, a_2, a_3, a_4\}$ is said to be *balanced* provided

(i) $a_1 < a_2 < a_3 < a_4$

(ii) $\gcd(a_1, a_2, a_3, a_4) = 1$

(iii) each a_i cannot be written as a linear combination of elements of $A \setminus \{a_i\}$

(iv) $\mathrm{CS}(A) = a_1 + a_4 = a_2 + a_3$.

If the minimal generating set for a numerical semigroup S is a balanced set, then S is said to be *balanced*.

Definition 4.2. Let $A = \{a_1, a_2, a_3, a_4\}$ be a balanced set. If

$$\mathrm{CQ}(A) = \frac{\mathrm{CS}(A)}{\gcd(a_1, a_4) \cdot \gcd(a_2, a_3)} = 1$$

then A is a *unitary* set.

Similarly, if the minimal generating set for a numerical semigroup S is a unitary set, then S is said to be *unitary*. We call $\mathrm{CS}(A)$ the common sum and $\mathrm{CQ}(A)$ the common quotient.

Definition 4.3. Let S be a numerical semigroup, I be a relative ideal of S, and $S - I$ be its dual. Then the pair (S, I) is called a $k \times m$ *perfect brick* provided

(i) $\mu(I) = k$

(ii) $\mu(S - I) = m$

(iii) $\mu(I + (S - I)) = km$

(iv) $I + (S - I) = S \setminus \{0\}$.

From the definition it is clear that pairs (S, I) must satisfy our motivating equality (1) in order to possibly correspond to a perfect brick.

Theorem 4.1 (Herzinger et al. [5], [2])**.** *Let $S = \langle a_1, a_2, a_3, a_4 \rangle$ and $n = a_2 - a_1 = a_4 - a_3$. Then the pair*

$$(S = \langle a_1, a_2, a_3, a_4 \rangle, I = (0, n))$$

is a 2×2 perfect brick if and only if S is unitary.

Before this result was published, there was only one known 2×2 perfect brick, namely

$$(S = \langle 14, 15, 20, 21 \rangle, I = (0, 1)).$$

Now, as a consequence of the theorem, all of the infinitely many 2×2 perfect bricks are known.

Example 4.1. Let $S = \langle 15, 22, 33, 40\rangle$. We can clearly see that the requirements for balanced hold and that the common sum is $\mathrm{CS}(S) = 15 + 40 = 22 + 33 = 55$. Then we have

$$\mathrm{CQ}(S) = \frac{55}{\gcd(15, 40) \cdot \gcd(22, 33)} = \frac{55}{5 \cdot 11} = 1.$$

Therefore S is unitary. Note that the common difference between generators is $n = 22-15 = 40-33 = 7$. By Theorem 4.1,

$$(S = \langle 15, 22, 33, 40\rangle, I = (0, 7))$$

is a 2×2 perfect brick.

5 Higher Dimension Perfect Bricks

In order to begin the discussion of perfect bricks (S, I) with dimension higher than 2×2 (that is, with $\mu(I) > 2$ and $\mu(S - I) > 2$), we need more machinery to precisely define the minimal generating sets.

Definition 5.1. A *δ-cluster* $C = \{a_1, \ldots, a_k\}$ is a set of at least two integers such that

(i) $a_1 < a_2 < \ldots < a_k$

(ii) $\gcd(a_1, a_2, \ldots, a_k) = 1$

(iii) $\delta = a_2 - a_1 = \ldots = a_k - a_{k-1}$.

Example 5.1. Let $C_1 = \{40, 41, 42\}$, $C_2 = \{55, 56, 57\}$, and $C_3 = \{70, 71, 72\}$. Then we have

$$S = \langle 40, 41, 42, 55, 56, 57, 70, 71, 72\rangle.$$

Note that in this example, $\delta = 1$. We call these clusters 1-clusters.

If the minimal generating set for a numerical semigroup S is a union of δ-clusters, then in order to hope to generalize the concepts of balanced and unitary to $k \times m$ perfect bricks, the following condtions must be met:

- the minimal generating set for I is 0 and multiples of δ up to $(k - 1)\delta$
- the minimal generating set for $S - I$ is the first element of each cluster.

That is,

$$\begin{aligned} S &= \langle C_1 \cup C_2 \cup \ldots \cup C_k \rangle \\ I &= (0, \delta, 2\delta, \ldots, (k-1)\delta) \\ S - I &= (a_{11}, a_{21}, \ldots, a_{k1}). \end{aligned}$$

Definition 5.2. Let $A = C_1 \cup C_2 \cup \ldots \cup C_k$ where $C_i = \{a_{i1}, \ldots, a_{ik}\}$ is a δ-cluster for all i and δ is constant. Assume there exists some $\gamma \in \mathbb{Z}$ such that $C_i = C_{i-1} + \gamma$ for $i = 1, \ldots, k$. Then the set A is said to be *uniformly balanced* provided

(i) $a_{11} < \ldots < a_{kk}$

(ii) $\gcd(a_{11}, \ldots, a_{1k}, a_{21}, \ldots, a_{2k}, \ldots, a_{k1}, \ldots, a_{kk}) = 1$

(iii) each a_{ij} cannot be written as a linear combination of elements of $A \setminus \{a_{ij}\}$

(iv) $\mathrm{CS}(A) = a_{11} + a_{kk} = a_{1k} + a_{k1}$.

If the minimal generating set for S is a uniformly balanced set, then S is said to be *uniformly balanced.*

Example 5.2. Let $S = \langle 40, 41, 42, 55, 56, 57, 70, 71, 72 \rangle$. By observation we can see that requirements (i)-(iv) for uniformly balanced hold, and the common sum is $\mathrm{CS}(S) = 40+72 = 42+70 = 112$.

Then we have three 1-clusters, namely

$$\begin{aligned} C_1 &= \{40, 41, 42\} \\ C_2 &= \{55, 56, 57\} = C_1 + 15, \text{ and} \\ C_3 &= \{70, 71, 72\} = C_2 + 15. \end{aligned}$$

Thus $S = \langle C_1 \cup C_2 \cup C_3 \rangle = \langle 40, 41, 42, 55, 56, 57, 70, 71, 72 \rangle$ is uniformly balanced. Also we have $I = (0, \delta, 2\delta) = (0, 1, 2)$ and $S - I = (a_{11}, a_{21}, a_{31}) = (40, 55, 70)$. In fact, $(\langle 40, 41, 42, 55, 56, 57, 70, 71, 72 \rangle, (0, 1, 2))$ is a 3×3 perfect brick.

Definition 5.3. Let $A = C_1 \cup C_2 \cup \ldots \cup C_k$ be a uniformly balanced set. If

$$\mathrm{CQ}(A) = \frac{\mathrm{CS}(A)}{\gcd(a_{11}, a_{kk}) \cdot \gcd(a_{1k}, a_{k1})} = 1$$

then A is a *unitary* set.

If the minimal generating set for S is a unitary set, then S is said to be *unitary.*

Example 5.3. Let $S = \langle 77, 78, 79, 80, 98, 99, 100, 101, 119, 120, 121, 122, 140, 141, 142, 143\rangle$. By observation, S is uniformly balanced, and the common sum is $\mathrm{CS}(S) = 220$. Then

$$\mathrm{CQ}(S) = \frac{220}{\gcd(77, 143) \cdot \gcd(80, 140)} = \frac{220}{11 \cdot 20} = 1.$$

Therefore S is unitary. In fact, the pair $(S, I = (0, 1, 2, 3))$ is a 4×4 perfect brick.

Example 5.4. Let $S = \langle 27, 28, 29, 35, 36, 37, 43, 44, 45\rangle$. By observation, S is uniformly balanced, and the common sum is $\mathrm{CS}(S) = 72$. Then

$$\mathrm{CQ}(S) = \frac{72}{\gcd(27, 45) \cdot \gcd(29, 43)} = \frac{72}{9 \cdot 1} \neq 1.$$

Therefore S is not unitary. However, the pair $(S, I = (0, 1, 2))$ is actually 3×3 perfect brick.

Since we have both unitary and not unitary $k \times k$ perfect bricks, we cannot generalize Theorem 4.1 to $k \times k$ perfect bricks as a biconditional. However, there is clearly some opportunity for additional investigation into these notions.

5.1 An Infinite Family of Perfect Bricks

The following array (Rushall, 2010) represents an infinite family of $k \times m$ perfect bricks with $k \geq 2, m \geq 3$.

$$\begin{array}{cccccccc}
\star & \star & \star & \star & \star & \star & \star & \ldots \\
\star & \star & 17 & 22 & 27 & 32 & 37 & \cdots \\
\star & \star & 27 & 35 & 43 & 51 & 59 & \cdots \\
\star & \star & 37 & 48 & 59 & 70 & 81 & \cdots \\
\star & \star & 47 & 61 & 75 & 89 & 103 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The stars represent nonentries in the array since we do not have $k = 1$ or $m = 1, 2$. Notice that every row and column consists of consecutive elements from arithmetic progressions. Consequently, we can easily compute perfect bricks whose pairs (S, I) are constructed from an arbitrarily large number of generators. The array has been proven to yield perfect bricks (to appear, Herzinger). To build the unique $k \times m$ perfect brick in this family, let

$$A = \{0, 1, \ldots, k - 1\} \text{ and } B = \{(k, m), (k, m + 1), \ldots, (k, 2m - 1)\}$$

where (k, m) is the entry in the k^{th} row, m^{th} column. Then the pair

$$(S = \langle A + B\rangle, I = (A))$$

is a $k \times m$ perfect brick.

Since A contains exactly k elements and B contains exactly m elements, the size of the minimal generating set of S is $k \cdot m$ which preserves the equality for which we are searching – that is, $\mu(I)\mu(S - I) = \mu(I + (S - I))$.

Example 5.5. Let $A = \{0, 1, 2, 3\}$ and $B = \{37, 48, 59\}$. Then $I = (0, 1, 2, 3)$, $S - I = (37, 48, 59)$, and

$$S = \langle A + B \rangle = \langle 37, 38, 39, 40, 48, 49, 50, 51, 59, 60, 61, 62 \rangle.$$

Note that the minimal generating set for S contains 12 elements; thus the equality (1) holds. It is left to the reader to verify that $I + (S - I) = S \setminus \{0\}$.

The numerical semigroups that correspond to the $k \times k$ perfect bricks in the Rushall family are neither unitary nor symmetric. As previously noted, we have both unitary and not unitary $k \times k$ perfect bricks, so we clearly cannot generalize Theorem 4.1 to another biconditional. We must consider other conditions. For instance, is a symmetric and unitary numerical semigroup sufficient to imply the pair (S, I) is a $k \times k$ perfect brick?

6 Open Questions

We are currently investigating perfect bricks with the hope of answering the following questions:

(i) Can we extend the theorem for 2×2 unitary perfect bricks to hold for $k \times k$ unitary perfect bricks?

(ii) Is a symmetric numerical semigroup sufficient for a $k \times k$ perfect brick? Does symmetric imply unitary in $k \times k$ perfect bricks?

(iii) Are there conditions similar to uniformly balanced and unitary for $k \times m$ perfect bricks with $k \neq m$?

(iv) Is the Rushall family actually a 3D array with infinitely many layers?

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