10-16-2002

A Review of Selected Works on Crack Indentification

Kurt M. Bryan
Rose-Hulman Institute of Technology, bryan@rose-hulman.edu

A Review of Selected Works on Crack Identification

Kurt Bryan*and Michael S. Vogelius

MS TR 02-11

October 16, 2002

Department of Mathematics
Rose-Hulman Institute of Technology
http://www.rose-hulman.edu/Class/ma/HTML/index.html

FAX: (812) 877-8883

PHONE: (812) 877-8391

A review of selected works on crack identification.

Kurt Bryan *and Michael S. Vogelius †

October 16, 2002

Abstract

We give a short survey of some of the results obtained within the last 10 years or so concerning crack identification using impedance imaging techniques. We touch upon uniqueness results, continuous dependence results, and computational algorithms.

1 The Forward Problem

Consider first the two-dimensional forward problem, which has certain special features not found in higher dimensions. Let Ω be a bounded, simply connected domain in $\mathbb{R}^2$ with smooth boundary and $\gamma : \bar{\Omega} \to \mathbb{R}$ a bounded function with $\inf_\Omega \gamma \geq \delta > 0$; for the moment we assume that γ is real-analytic, although this restriction will later be relaxed. The domain Ω represents the object in which we wish to detect cracks and γ is the *reference* or *background* conductivity, considered known a priori.

Unless otherwise noted, we define a *crack* in Ω as a curve σ contained in Ω which can be parameterized by a twice continuously differentiable map from $[0, 1] \to \Omega$ with non-vanishing derivative; we also require that σ does not self-intersect, although this condition will later be relaxed. We use $\Sigma = \cup_{k=1}^{n}\{\sigma_k\}$

*Rose-Hulman Institute of Technology, Terre Haute, IN

†Rutgers University, New Brunswick, N.J. 08903, USA. This work was partially supported by the National Science Foundation under grant DMS-00-72556.

to denote a collection of n cracks. We assume that the cracks are pairwise disjoint. Note that Σ may be empty.

Let ϕ denote an applied electrical potential on $\partial\Omega$. If we assume that the collection of cracks Σ is perfectly insulating (completely blocking the flow of electrical current) then the electrical potential v inside Ω satisfies

$$\begin{aligned} \nabla \cdot (\gamma \nabla v) &= 0 \quad \text{in } \Omega \setminus \Sigma, \\ \gamma \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \Sigma \\ v &= \phi \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where ν is a unit normal vector field on Σ, consistently oriented on each crack (e.g., if a crack σ is parameterized by $c : [0,1] \to \mathbb{R}^2$, then take $\nu(c(t)) = (c'(t))^{\perp}/|c'(t)|$ where $\perp$ denotes a counter-clockwise rotation through an angle $\pi/2$.) The inverse problem of interest is to determine Σ from one or more pairs of boundary-voltage and current data, $(\phi, \gamma\frac{\partial v}{\partial \nu})$.

The boundary value problem (1) with insulating cracks is equivalent to a problem involving perfectly conducting cracks, and this framework is slightly preferable for the analysis that follows. By a *perfectly conducting* crack we mean a crack which maintains constant electrical potential along its length. The boundary value problem (1) can be transformed into an equivalent problem for perfectly conducting cracks by considering the "γ-harmonic conjugate" of v. Specifically, let u be a function related to v by $(\nabla u)^{\perp} = \gamma \nabla v$. It is not difficult to verify the existence of u, given that v satisfies (1) with $\gamma\frac{\partial v}{\partial \nu} = 0$ on Σ. The function u is determined only up to an additive constant.

If v is a solution to (1) then the function u satisfies the boundary value problem

$$\begin{aligned} \nabla \cdot (\gamma^{-1} \nabla u) &= 0 \quad \text{in } \Omega \setminus \Sigma, \\ u &= c_k \quad \text{on } \sigma_k \\ \gamma^{-1} \frac{\partial u}{\partial \nu} &= \psi \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

where $\psi = \frac{\partial \phi}{\partial s}$, ν is an outward unit normal vector field on $\partial\Omega$, and $s = (\nu)^{\perp}$. The constants c_k are determined by

$$c_k = -\int_p^{\sigma_k} \gamma \frac{\partial v}{\partial \nu}\, ds + u(p)$$

where $\nu = -s^{\perp}$, p is a fixed point on $\partial\Omega$ (or in Ω) and the integration is carried out along any path connecting p to a point in σ_k.

Note that knowledge of pairs $(\phi, \gamma\frac{\partial v}{\partial \nu}|_{\partial\Omega})$ for the perfectly insulating problem is entirely equivalent to knowledge of pairs $(u|_{\partial\Omega}, \psi)$ for the perfectly conducting problem.

The boundary value problem (2) can also be cast in an energy minimization form, specifically, u is the minimizer of

$$\begin{aligned} Q(w) &= \frac{1}{2}\int_{\Omega} \gamma^{-1}|\nabla w|^2\,dx - \int_{\partial\Omega} \psi w\,ds \qquad \text{over} \\ K &= H^1(\Omega) \cap \{w = \text{constant on each } \sigma_k\} \enspace . \end{aligned} \tag{3}$$

This minimizer, as well as the solution to (2), is uniquely determined modulo an additive constant.

Note that the constant values assumed by u on the cracks are determined as part of the minimization process. We also notice that:

Remark 1.1 *Let Γ be a simple closed curve in $\Omega \setminus \Sigma$. Then*

$$\int_{\Gamma} \gamma^{-1}\frac{\partial u}{\partial \nu}\,ds = \int_{\Gamma} \frac{\partial v}{\partial s}\,ds = 0. \tag{4}$$

In particular, we may select Γ so that it encloses a single crack. Therefore, for the energy minimizing solutions no crack acts as a source or sink of current. This criterion also serves to uniquely characterize the constants $\{c_k\}_{k=1}^{n}$ See [26] for a more detailed discussion.

However, one could also consider the non-physical boundary value and inverse problem obtained by allowing the constants c_k in (1) to assume arbitrary specified values (which don't necessarily minimize the above energy.) In this case individual cracks will act as net sources or sinks for current. This has important implications for the inverse problem.

In three or more dimensions the forward problem is again governed by equation (1) in the case of perfectly insulating cracks, or equation (2) for conducting cracks. A crack in a three-dimensional conductor is defined to be a suitably smooth (e.g., C^2) surface which does not self-intersect. The main difference between the two and higher dimensional cases is that in the latter there is no duality between the perfectly conducting and the perfectly insulating problems.

2 The Inverse Problem

2.1 Uniqueness Results in The Two Dimensional Case

The first uniqueness result concerning the determination of cracks inside a conductor was proved in [35]. It was demonstrated that with the Dirichlet data corresponding to two input current fluxes of a specified form one could uniquely determine the precise shape and location of a single conductive crack inside a conductor with real-analytic background conductivity; an analogous result was also proved for insulating cracks. In [35] it was also shown that in general two sets of measurements are required to determine a single crack. These results were generalized in [25] to show that $n + 1$ input fluxes of a specified form and the resulting Dirichlet data uniquely determine a collection of n conductive cracks. Although the authors remarked that this result holds in the case in which the crack constants are arbitrarily specified, the proof in fact requires that the constants assumed by the potential on the cracks be the "energy-minimizing" constants in the variational form of the problem (3).

The results of [25] were improved, simultaneously and independently, in [9] and [36]. In [36] it is shown that with real-analytic background conductivity one can determine a collection of any number of conducting cracks using only two inputs fluxes of a specified form and the corresponding Dirichlet data. The same result is proven in [9] with much weaker assumptions about the background conductivity γ and the smoothness of the cracks. The authors also provide analogous results for insulating cracks.

In all of the identifiability proofs an essential ingredient is the fact that the potential functions in $\Omega \setminus \Sigma$ which are induced by appropriate input current fluxes do not have "too many" (or any) critical points in $\Omega \setminus \Sigma$. The verification of this fact typically involves a detailed analysis of the equipotential curves of the potential function.

To illustrate the central ideas in the above papers we will, for simplicity, consider the perfectly conducting case with $\gamma \equiv 1$, so that the function u in (2) is harmonic in $\Omega \setminus \Sigma$. We will consider input fluxes of the form $\psi_i = \delta_{P_0} - \delta_{P_i}$, for $i = 1, 2$, where P_0, P_1, and P_2 are distinct points on $\partial\Omega$, and where δ_{P_i} denotes a delta function on $\partial\Omega$ at P_i (physically a "point" input source of current at P_i.) Note that for this type of input flux the solution to (2) will not be an $H^1(\Omega)$ function, and so not obtained as the minimizer of (3). We rather interpret the solution u as a weak solution to (2), smooth except at the delta function current input, where u has a singularity of the form $\pm \ln |r|/\pi$, where r denotes distance to the input current point. It is also worth noting here that the solution u is continuous in Ω but will typically have $r^{1/2}$ type singularities at the ends of the cracks (see [35].)

We then have the following uniqueness result.

Theorem 2.1 *Let Σ and $\tilde{\Sigma}$ be two collections of cracks in Ω. Let u_1, u_2 (resp., $\tilde{u}_1, \tilde{u}_2$) be the functions which satisfy the boundary value problem (2) with $\gamma = 1$ on $\Omega \setminus \Sigma$ (resp., $\Omega \setminus \tilde{\Sigma}$) with input fluxes ψ_1, ψ_2. Let Γ be any open portion of $\partial\Omega$. If $u_1 = \tilde{u}_1$ and $u_2 = \tilde{u}_2$ on Γ then $\Sigma = \tilde{\Sigma}$.*

One of the main tools we need is a detailed analysis of the equipotential curves of the function u and how such curves can be extended. This is the focus of the following two lemmas.

In [35] the authors prove

Lemma 2.1 *Let u satisfy $\triangle u = 0$ in $\Omega \setminus \Sigma$ with u constant on each crack σ_k. Let ρ be a nonempty analytic curve in Ω with $\rho \cap \Sigma = \emptyset$ along which u is constant. Then there exists an analytic (open) curve ρ' with $\rho \subset \rho'$ such that*

- *u is constant on ρ',*
- *one endpoint of ρ' lies on $\partial\Omega$ or on σ_j for some j,*
- *the other endpoint of ρ' lies on $\partial\Omega$ or on σ_k for some k, $k \neq j$.*

The proof of Lemma 2.1 is similar to that of the next lemma, proved in [25]. Neither proof requires that the constants c_k assumed by u on the cracks be the energy minimizing constants.

Lemma 2.2 *Let u satisfy $\triangle u = 0$ in $\Omega \setminus \Sigma$ with u constant on each crack σ_k. Let ρ be a nonempty analytic curve in Ω with $\rho \cap \Sigma = \emptyset$ along which u is constant. Let x^* be some point in ρ at which $\nabla u(x^*) = 0$. Then there exists an analytic curve ρ' which has x^* as an interior point such that*

- *$\rho' \cap \rho = x^*$,*
- *u is constant on ρ'.*

Sketch of Proof: We can expand u in a Taylor series in r in polar coordinates near x^* to obtain $u(x) = u(x^*) + r^N(a\sin(N\theta) + b\cos(N\theta) + rA(r,\theta))$ for $N \geq 2$, $(r,\theta) \in [0,\epsilon) \times [0,2\pi]$, and A bounded. Note also that since $\nabla u(x^*) = 0$, we have $\frac{\partial u}{\partial r}(0,\theta) = 0$. Now we may assume via a rotational change of coordinates that ρ is tangential to the (half) line $\theta = 0$, so that $b = 0$ and $a \neq 0$, i.e., $u(x) = u(x^*) + r^N(a\sin(N\theta) + rA(r,\theta))$. Since $u(r,\theta)$ is analytic in r near $r = 0$ and $\frac{\partial u}{\partial r}(0,\theta) = 0$, u has an analytic extension to $r \in [-\epsilon,\epsilon]$, and hence the function $A(r,\theta)$ also has an extension in r to $[-\epsilon,\epsilon]$, We then have

$$u(x) = u(x^*) + r^N(a\sin(N\theta) + rA(r,\theta))$$

for $(r,\theta) \in [-\epsilon,\epsilon] \times [0,2\pi]$ with $N \geq 2$. The function $F(r,\theta) = a\sin(N\theta) + rA(r,\theta)$ satisfies $F(0,\pi/N) = 0$ and $\frac{\partial}{\partial\theta}F(0,\pi/N) = -aN$, and so by the implicit function theorem we can find a unique analytic function $\theta(r)$ such that $\theta(0) = \pi/N$ and $\{(r,\theta) : F(r,\theta) = 0\}$ coincides with $\{r,\theta(r)\}$ in some neighborhood of $(0,\pi/N)$. The curve $(r\cos(\theta(r)), r\sin(\theta(r))) + x^*$ satisfies the requirements in the statement of the Lemma.

The key fact noted in some variation in both [9] and [36] which allows us to prove a two-measurement uniqueness result is this:

Lemma 2.3 *Let u satisfy the boundary value problem (2) with $\gamma = 1$. Let $\sigma \in \Sigma$ and let ρ be a curve on which u is constant. Suppose that $\rho \cap \sigma = x^*$ with x^* an endpoint for ρ. Then ρ "can be extended", i.e., there is some curve ρ', with $\rho' \cap \rho \setminus \{x^*\} = \emptyset$, such that*

- *$\rho' \cap \sigma = y^*$, for some point $y^* \in \sigma$,*
- *u is constant on ρ'.*

Sketch of Proof: (following the idea in [9]). We may assume that $u = 0$ on σ. Let C be a simple C^2 closed curve which encloses σ but no other crack. We can find a conformal change of coordinates in which σ is mapped to $\partial B_1(0)$ and C is mapped to $\partial B_R(0)$ for $R > 1$. We will still use u to denote the potential function in the new coordinates. Then u is still harmonic and we have $u = 0$ on $\partial B_1(0)$; moreover, u is smooth up to $\partial B_1(0)$ and we can continue u as a harmonic function into the annulus $B_1(0)\setminus B_{1/R}(0)$ by defining

$u(z) = -u(\bar{z}^{-1})$. From Remark 1.1 we can see that

$$\int_{C'} \frac{\partial u}{\partial n}\, ds = 0 \tag{5}$$

for any closed curve C' contained in the annular region $B_R(0) \setminus B_{1/R}(0)$, and in particular for $C' = \partial B_1(0)$. In the new coordinates ρ intersects $\partial B_1(0)$ at some point and at this point we must have $\frac{\partial u}{\partial n} = 0$. From equation (5) we conclude that $\frac{\partial u}{\partial n}$ (which is continuous on $\partial B_1(0)$) must vanish at some other point y on $\partial B_1(0)$, corresponding to some point $y^* \in \sigma$. We then have $\nabla u(y) = 0$, and using the same reasoning as in the proof of Lemma 2.2 we can construct a level curve $\rho' \subset B_R(0) \setminus B_1(0)$ for u with $\rho' \cap \partial B_1(0) = y$. The "pullback" of this curve to the original coordinates yields a curve with the properties stated in this Lemma. Note that the "extension" of ρ may emanate from a point other than x^*; it may also emanate from x^*, but in that case it will extend ρ to the other side of σ.

Two additional facts that we need are

Lemma 2.4 *Let u satisfy the boundary value problem (2) with $\gamma = 1$. Let C be any simple closed curve in Ω and suppose u is constant on C. Then u is constant on Ω.*

Sketch of Proof: Let D denote the region enclosed by C and suppose that $u = c$ on C. Define a function $\tilde{u}$ as

$$\tilde{u}(x) = \begin{cases} c, & x \in D \\ u(x), & x \in \Omega \setminus D. \end{cases}$$

It's easy to verify that $\tilde{u} \in H^1(\Omega)$ and that $\tilde{u}$ is constant on each $\sigma \in \Sigma$. However if u is nonconstant on D then we have $Q(\tilde{u}) < Q(u)$, a contradiction (strictly speaking this energy argument should be performed locally, since with the input currents ψ_i, the solution u is not in $H^1(\Omega)$). We conclude that $u \equiv c$ in D, and by unique continuation and the fact that $\Omega \setminus \Sigma$ is connected we must have $u \equiv c$ in Ω.

Remark 2.1 *Note that the proof of Lemma 2.3 fails if the values assumed by u on the cracks are specified, rather than the energy minimizing constants, for* *then equation (5) may not hold. Also in this case the conclusion of Lemma 2.4 is plainly false: We could, for example, have $u \equiv 0$ on the curve C but $u \equiv 1$ on some crack $\sigma \subset D$. In this case the function $\tilde{u}$, constructed in the above proof, is not in the class of functions over which the relevant "minimization" takes place, and we cannot conclude that u is constant on D.*

Lemma 2.5 *Let u satisfy the boundary value problem (2) with $\gamma = 1$ and flux $\psi = \sum_{i=0}^{2} \beta_i \delta_{P_i}$ where not all $\beta_i = 0$ (but note $\sum_{i=0}^{2} \beta_i = 0$). Then u has no critical points ($\nabla u = 0$) in $\Omega \setminus \Sigma$.*

Proof of Lemma 2.5: We prove this by contradiction. Suppose that $\nabla u(x^*) = 0$ for some $x^* \in \Omega \setminus \Sigma$. Expanding u into a Taylor series in r and using the reasoning of Lemma 2.2 we can find transversal analytic curves ρ and ρ' such that x^* is an interior point for both curves (in particular $\rho \cap \rho' = x^*$).

From Lemma 2.1 we can extend both ends of ρ and ρ' until they either terminate on $\partial\Omega$ or on some crack σ. Let us suppose, for example, that one end of ρ can be extended as a level curve to some crack σ_{k_1}; let us still refer to the extended curve as ρ. By Lemma 2.3 we know that ρ can be extended again as a level curve, either terminating on $\partial\Omega$ or on some crack σ_{k_2}. If the latter occurs, this process can be repeated. It's clear that ρ must eventually terminate on $\partial\Omega$, for ρ cannot intersect the same crack twice or else we find some closed curve C on which u is constant and so by Lemma 2.4 u would be constant in Ω, a contradiction.

We conclude that both ends of ρ and ρ' can be extended as level curves for u to $\partial\Omega$. Let the points at which these level curves intersect $\partial\Omega$ be denoted by x_1, x_2, x_3, and x_4. Note that these points must be distinct or else some subdomain D would be enclosed with ∂D consisting entirely of pieces of ρ and ρ', and so Lemma 2.4 shows that u would be constant on Ω, a contradiction. Also, none of the x_j can coincide with a point P_i, $i = 0, 1, 2$, corresponding to which $\beta_i \neq 0$, since u has a logarithmic singularity at such a point. For simplicity let us suppose all β_i are nonzero.

The set $\partial\Omega \setminus (x_1 \cup x_2 \cup x_3 \cup x_4)$ contains four connected components. Now one of these components, call it S, does not contain any of P_0, P_1, or P_2, and hence $\partial u/\partial n \equiv 0$ on S. It is thus clear that S, together with some portion of ρ and ρ', enclose a region D on whose boundary u is either constant or has zero Neumann data. An argument similar to that in Lemma 2.4 (and again requiring that the constants on the cracks be chosen to "minimize" energy) shows that u must be constant in D, and hence in Ω, a contradiction.

Proof of Theorem 2.1: We first prove that if $u_i = \tilde{u}_i$ on Γ then $u_i = \tilde{u}_i$ in Ω. Let O be the (possibly empty) open region enclosed by $\Sigma \cup \tilde{\Sigma}$, those points in $\Omega \setminus (\Sigma \cup \tilde{\Sigma})$ from which it is possible to reach $\partial\Omega$ only by crossing Σ or $\tilde{\Sigma}$. Clearly $\Omega \setminus (O \cup \Sigma \cup \tilde{\Sigma})$ has only one connected component. Since u_i and $\tilde{u}_i$ have the same Cauchy data on Γ it follows by unique continuation that $u_i = \tilde{u}_i$ in $\Omega \setminus (O \cup \Sigma \cup \tilde{\Sigma})$. If O is non-empty then ∂O consists of pieces of Σ and $\tilde{\Sigma}$. On each piece of ∂O, u_i or $\tilde{u}_i$ is constant, and so u_i ($\tilde{u}_i$) assumes finitely many values on ∂O; indeed u_i ($\tilde{u}_i$) assumes at most $|\Sigma| + |\tilde{\Sigma}|$ values, where $|\Sigma|$ denotes the number of cracks in Σ. Since u_i ($\tilde{u}_i$) is continuous it now follows that it constant on each connected component of ∂O, and so by the maximum principle it is constant on each connected component of O itself. It follows that u_i ($\tilde{u}_i$) is constant in Ω, a contradiction. We conclude that O is empty. If O is empty we have $u_i = \tilde{u}_i$ on $\Omega \setminus (\Sigma \cup \tilde{\Sigma})$. It follows by continuity that $u_i = \tilde{u}_i$ in all of Ω.

If we assume that $\Sigma \neq \tilde{\Sigma}$ then we can, for example, find some curve ρ contained in $\tilde{\Sigma}$ with $\rho \cap \Sigma = \emptyset$. Since $u_1 = \tilde{u}_1$ and $u_2 = \tilde{u}_2$ in Ω the functions u_1 and u_2 must be constant on ρ (remember, $\tilde{u}_1$ and $\tilde{u}_2$ are by definition constant on ρ). Let x^* be a point in the interior of ρ and $\mathbf{n}$ a consistently oriented unit normal vector field on ρ. Note that $\frac{\partial u_2}{\partial \mathbf{n}}(x^*) \neq 0$, for if not then we would have $\nabla u_2(x^*) = 0$ (since ρ is a level curve for u_2) a contradiction to Lemma 2.5. Now let $u = u_1 - \alpha u_2$ where $\alpha = \frac{\partial u_1}{\partial \mathbf{n}}(x^*)/\frac{\partial u_2}{\partial \mathbf{n}}(x^*)$. Note that u satisfies the boundary value problem (2) with a flux of the form $\psi = \sum_{i=0}^{2} \beta_i \delta_{P_i}$ with not all β_i zero. We find that $\frac{\partial u}{\partial \mathbf{n}}(x^*) = 0$, so that $\nabla u(x^*) = 0$, a contradiction to Lemma 2.5. Thus we conclude that $\Sigma = \tilde{\Sigma}$.

The precise formulation of the result proved by Kim and Seo in [36] is

Theorem 2.2 *Let ψ_1, ψ_2 be two nonvanishing piecewise continuous functions on $\partial\Omega$ with $\int_{\partial\Omega} \psi_i \, ds = 0$ and with the property that for each real α the set $\{z \in \partial\Omega : \psi_1(z) - \alpha\psi_2(z) \geq 0\}$ is connected and ψ_1 is not identically equal to $\alpha\psi_2$. Suppose that Σ and $\tilde{\Sigma}$ are collections of C^2 cracks in Ω, and suppose γ is real-analytic. Let u_i, $i = 1, 2$, be the solution to the boundary value problem (2) with flux $\psi = \psi_i$ and let $\tilde{u}_i$ be the corresponding solution with Σ replaced by $\tilde{\Sigma}$. Then $u_i = \tilde{u}_i$ for $i = 1, 2$ on $\partial\Omega$ implies that $\Sigma = \tilde{\Sigma}$.*

Kim and Seo give an example of a suitable choice for ψ_1 and ψ_2. In fact, one can give a rather general class of suitable input fluxes. Let S_1, S_2, and S_3 be a decomposition of $\partial\Omega$ into three disjoint simple arcs. Let g_i, $i = 1, 2, 3$, be non-negative functions with $g_i > 0$ in S_i, $g_i = 0$ in $\partial\Omega \setminus S_i$ and $\int_{\partial\Omega} g_i \, ds = 1$. Then one can easily verify that choosing

$$\psi_1 = g_1 - g_2, \quad \psi_2 = g_1 - g_3 \tag{6}$$

satisfies the conditions in Theorem 2.2.

In [9] Alessandrini and Diaz Valenzuela prove a theorem very similar to Theorem 2.2 but under more general conditions. Specifically, the conductivity γ may be anisotropic and only L^∞, i.e., represented by the 2 by 2 matrix γ, with bounded measurable entries and $\xi^T \gamma(x) \xi \geq \lambda |\xi|^2$ for almost all $x \in \Omega$ and $\xi \in \mathbb{R}^2$. The input fluxes are chosen almost as in equation (6), except for the fact that it is only required that $g_i \geq 0$ in S_i. This is important in the sense that one may now approximate delta functions by taking the support of the g_i to "narrow" to a single point P_i. A collection of cracks Σ is defined to be a closed set which is a union of finitely many pairwise disjoint closed continua (a connected set containing at least two points) $\sigma_1, \ldots, \sigma_n$ such that each of the sets $\Omega \setminus \sigma_j$, $j = 1$ to n, is connected. Alessandrini and Diaz Valenzuela prove

Theorem 2.3 *Let Γ be a nonempty simple arc on $\partial\Omega$ and Σ and $\tilde{\Sigma}$ two collections of cracks. Let u_i, $i = 1, 2$ be the solution to (2) with input fluxes ψ_1, ψ_2 chosen in accordance to equation (6), and $\tilde{u}_i$ the corresponding solutions with $\tilde{\Sigma}$ replacing Σ. Then $u_1 = \tilde{u}_1$ and $u_2 = \tilde{u}_2$ on Γ implies that $\Sigma = \tilde{\Sigma}$.*

They also prove an analogous result for perfectly insulating cracks.

To prove Theorem 2.3 the authors use a quasi-conformal map to reduce the problem of characterizing critical points and the local behavior for solutions to (2) to equivalent problems for harmonic functions. Specifically, if D is a simply connected domain and u a solution to $\nabla \cdot \gamma \nabla u = 0$ in D then one can write $u + it = f \circ \xi$ where t is the associated *stream function*, ξ is a quasi-conformal mapping from D to $B_1(0) \subset \mathbb{R}^2$, and f is an analytic function. The key idea is that the geometric structure of the level lines for u will be the same as that of the harmonic function $\text{Re}(f)$. The authors define critical points for u as those points z for which $\nabla \text{Re}(f)(\xi(z)) = 0$ and show that the function u which satisfies (2) with the given input fluxes can have no critical points in $\Omega \setminus \Sigma$ (a specific example of an analysis characterizing the number of interior critical points for solutions to elliptic boundary value problems in terms of the number of sign changes in the boundary data. See [6] and [7].) The proof that $\Sigma = \tilde{\Sigma}$ if $u_i = \tilde{u}_i$ for $i = 1, 2$ is then similar to that for Theorem 2.1.

2.2 Stability in the Two Dimensional Case

Of theoretical and practical interest is the issue of how stably one can determine the shape and location of cracks inside a conducting body by using boundary data, since real data is invariably noisy.

In [8] the authors prove a stability estimate for the identification of a single insulating crack inside a two-dimensional conductive region Ω, which we now outline. A few technical definitions are required before stating the result.

Given a curve $c \subset \mathbb{R}^2$, a point $z \in c$, and $r > 0$ we will say that $c \cap B_r(z)$ is a *Lipschitz graph with norm* M if there is some cartesian coordinate system in which $c \cap B_r(z)$ can be represented as $\{(x, \phi(x)); -r < x < r\}$ where $||\phi'||_{L^\infty(-r,r)} \leq M$. Given a curve c with an endpoint z and $r > 0$ we will say that $c \cap B_r(z)$ is a *half Lipschitz graph with norm* M if there is some cartesian coordinate system in which $c \cap B_r(z)$ can be represented as $\{(x, \phi(x)); 0 \leq x < r\}$ where $||\phi'||_{L^\infty(-r,r)} \leq M$.

The stability result requires that for some positive constants L, M, and δ the domain Ω and crack σ satisfy the following conditions:

- length$(\partial\Omega) \leq L$;
- For each $z \in \partial\Omega$, $\partial\Omega \cap B_\delta(z)$ is a Lipschitz graph of norm M;
- length$(\sigma) \leq L$;
- dist$(\sigma, \partial\Omega) \geq \delta$.
- The crack σ is a simple curve and if V_1, V_2 are the endpoints of σ then $\sigma \cap B_{\delta/2}(V_i)$ is a half Lipschitz graph of norm M for $i = 1, 2$. Further, for each $z \in \sigma \setminus (B_{\delta/2}(V_1) \cup B_{\delta/2}(V_2))$, $\sigma \cap B_{\delta/2}(z)$ is a Lipschitz graph of norm M.

Let two input fluxes of the form $\psi_i = \eta_0 - \eta_i$, $i = 1, 2$, be applied, with $\eta_j \geq 0$ on $\partial\Omega$, $\int_{\partial\Omega} \eta_j \, ds = 1$, and $||\eta_j||_{L^2(\partial\Omega)} \leq M$ for some constant M and $j = 0, 1, 2$. The background conductivity γ may be anisotropic, i.e., represented by the 2 by 2 matrix γ, with bounded measurable entries and $\xi^T \gamma \xi \geq \lambda |\xi|^2$ for almost all $x \in \Omega$ and $\xi \in \mathbb{R}^2$. We know from Theorem 2.3 that we can uniquely determine any collection of cracks with these two input fluxes. We quantify the distance between two cracks σ and $\tilde{\sigma}$ using Hausdorff distance,

$$d_{\mathcal{H}}(\sigma, \tilde{\sigma}) = \max\{\sup_{x\in\sigma} \text{dist}(x, \tilde{\sigma}), \sup_{x\in\tilde{\sigma}} \text{dist}(x, \sigma)\}.$$

In [8] Alessandrini and Rondi show

Theorem 2.4 *Let u_i (resp., $\tilde{u}_i$) be the potential function on Ω with insulating crack σ (resp., $\tilde{\sigma}$) for input flux ψ_i, $i = 1, 2$. Suppose Γ is a simple arc on $\partial\Omega$ with* length$(\Gamma) \geq \delta$. *There exists a positive function ω defined on $(0, \infty)$ such that if*

$$\max_{i=1,2} ||u_i - \tilde{u}_i||_{L^\infty(\Gamma)} \leq \epsilon$$

then

$$d_{\mathcal{H}}(\sigma, \tilde{\sigma}) \leq \omega(\epsilon) \quad .$$

The function ω satisfies $\omega(\epsilon) \leq K(\ln|\ln\epsilon|)^{-\alpha}$ for $0 < \epsilon < 1/e$ and $\alpha, K > 0$, where α, K depend only the constants L, δ, and M.

If one is willing to make further a priori assumptions about the nature of the crack then the stability estimates can be considerably improved. In [4] the problem of the stability of identifying linear (line segment) cracks which are perfectly conducting is considered. A priori it is assumed that

- Ω is bounded and simply connected in $\mathbb{R}^2$ with length$(\partial\Omega) \leq L$ for some constant L.
- There is some constant $\delta > 0$ such that for all $z \in \partial\Omega$ there exists two circles of radius δ which are tangent to $\partial\Omega$ at z, one circle contained in $\bar{\Omega}$, the other in $\mathbb{R}^2 \setminus \Omega$.
- If $z = z(s)$ parameterizes $\partial\Omega$ then $||z||_{C^{2,\alpha}} \leq M$ for some M and $0 < \alpha < 1$.
- length$(\sigma) \geq \delta$ and dist$(\sigma, \partial\Omega) \geq \delta$.

The input fluxes are of the form $\psi_i = \eta_0 - \eta_i$, $i = 1, 2$ where the η_i are non-negative, $\int_{\partial\Omega} \eta_i \, ds = 1$ and $\text{supp}(\eta_i) \subset \partial\Omega \cap B_h(P_i)$ with $h < \delta/2$ and P_0, P_1, P_2 are distinct points on $\partial\Omega$ with $|P_i - P_j| \geq \delta$. In this case we have

Theorem 2.5 *Let u_i (resp., $\tilde{u}_i$) be the potential function on Ω with conductive linear crack σ (resp., $\tilde{\sigma}$) for input flux ψ_i, $i = 1, 2$. If Γ is a simple arc on $\partial\Omega$ with* $\text{length}(\Gamma) \geq \delta$ *then there exists constants C and h_0, depending only on the prior constants L, δ and M, such that for $h < h_0$*

$$d_{\mathcal{H}}(\sigma, \tilde{\sigma}) \leq C \sum_{j=1}^{2} \|u_i - \tilde{u}_i\|_{L^2(\gamma)}.$$

The requirement $h < h_0$ quantifies the condition that the input electrodes be sufficiently "concentrated." Moreover, as $h \to 0$ the constant C remains bounded, so the theorem is also valid for delta function current inputs.

2.3 Uniqueness Results in the Three Dimensional Case

Some work has been done to establish uniqueness results for two-dimensional cracks inside a three-dimensional conductive body. In [5] Alessandrini and DiBenedetto study uniqueness results for a finite collection of such surface cracks inside a three-dimensional object; they consider both perfectly conducting and perfectly insulating cracks.

Let Ω be a bounded region in $\mathbb{R}^3$ with $C^{1,\alpha}$ boundary for some $\alpha \in (1/2, 1)$. A crack σ is a simply connected, Lipschitz, non-self-intersecting surface lying at a positive distance from $\partial\Omega$ and such that $\partial\sigma$ is also Lipschitz. We use Σ to denote a finite collection of such disjoint cracks. The background conductivity of Ω is taken to be identically one.

For the conductive problem the input fluxes are of the form $\delta_P - \delta_Q$ where P and Q are distinct points on the boundary of $\partial\Omega$. The potential u will be harmonic in $\Omega \setminus \Sigma$ with $u = c_k$ on crack σ_k. It is also required that u satisfy the "zero flux" condition

$$\int_{\sigma_k} \left[\frac{\partial u}{\partial \nu}\right] ds = 0$$

for each crack σ_k, where $[\frac{\partial u}{\partial \nu}]$ denotes the jump in $\frac{\partial u}{\partial \nu}$ across σ (this condition determines the constants c_k up to a common additive constant). In this setting Alessandrini and DiBenedetto prove

Theorem 2.6 *Let $\Gamma \subset \partial\Omega$ be a set open in the relative topology of $\partial\Omega$ and Σ a finite collection of perfectly conducting cracks. Let P_0, P_1 and P_2 be distinct points on $\partial\Omega$ and u_i the potential functions on the domain $\Omega \setminus \Sigma$ with input fluxes $\psi_i = \delta_{P_0} - \delta_{P_i}$, $i = 1, 2$. Let $\tilde{\Sigma}$ denote a second finite collection of perfectly conducting cracks and $\tilde{u}_i$ the corresponding potential functions on $\Omega \setminus \tilde{\Sigma}$. Then $u_i = \tilde{u}_i$ on Γ for $i = 1, 2$ implies that $\Sigma = \tilde{\Sigma}$.*

As noted earlier, unlike the two-dimensional case, in three dimensions there is no simple duality between the perfectly conducting and perfectly insulating cases. However, in [5] Alessandrini and DiBenedetto also prove a uniqueness result for collections of insulating cracks which are planar. We say that a crack σ_i is *planar* if it is an open portion of a two-dimensional plane, simply connected with Lipschitz boundary.

Theorem 2.7 *Let $\Gamma \subset \partial\Omega$ be a set open in the relative topology of $\partial\Omega$. Let P_1, P_2, Q_1, Q_2 be distinct points on $\partial\Omega$ which are not coplanar and let u_i be the potential functions on the domain $\Omega \setminus \Sigma$ with input fluxes $\psi_i = \delta_{P_i} - \delta_{Q_i}$, $i = 1, 2$, where Σ is a finite collection of insulating planar cracks. Let $\tilde{u}_i$ denote the corresponding potential functions on $\Omega \setminus \tilde{\Sigma}$ for some other finite collection of insulating planar cracks. Then $u_i = \tilde{u}_i$ on Γ for $i = 1, 2$ implies that $\Sigma = \tilde{\Sigma}$.*

The authors also establish a stability result for conducting cracks; see section 2.4 below.

In [34] the author gives a relatively simple proof that given a three dimensional region Ω containing a single two-dimensional crack σ, the full Dirichlet-to-Neumann operator uniquely determines σ for insulating boundary conditions. In fact, the author shows more: let the boundary conditions on the crack be of the form $\frac{\partial u}{\partial \nu}_+ + b_+ u_+ = 0$, $\frac{\partial u}{\partial \nu}_- + b_- u_- = 0$ in which subscripts $\pm$ on u and $\frac{\partial u}{\partial \nu}$ signify the traces on each side of σ. The subscript $+$ signifies the trace on the side of σ into which the normal ν points, and $b_+ \leq 0$, $b_- \geq 0$ are unknown functions on each side of the crack. The author shows that the forward problem is well-posed and that knowledge of the Dirichlet-to-Neumann operator determines not only the crack, but the functions b_+ and b_- as well. Inclusion of the functions b_+ and b_- embodies the case in which one need not have a perfectly conducting or insulating crack, but possibly some other conduction condition. Taking $b_+ = b_- = 0$ models a perfectly insulating crack; the limit $|b_\pm| \to \infty$ corresponds to a perfectly insulating crack.

In [11] the authors give a simple constructive proof that a single planar two-dimensional crack can be identified, under a certain nondegeneracy condition, by a single input flux and the corresponding voltage measurements. We note that even for a planar crack a single pair of measurements is not always sufficient for identification , *i.e.*, the nondegeneracy condition may exceptionally fail. The ideas in [11] form the basis for a complete reconstruction algorithm [17], which we describe in the section on reconstruction algorithms below.

2.4 Stability in the Three Dimensional Case

In [5] the authors also prove a log-log stability result for perfectly conducting planar cracks in a three-dimensional conductor Ω. It is assumed that $\Omega \subset \mathbb{R}^3$ is bounded and convex with C^2 boundary, that σ lies in some plane π, and that $\partial\sigma$ is C^2. It is also assumed that for some $\delta > 0$,

$$\mathrm{diam}(\sigma) \geq \delta, \quad \mathrm{dist}(\sigma, \partial\Omega) \geq \delta, \quad \mathrm{dist}(P_i, P_j) \geq \delta, \ i \neq j \ ,$$

with input fluxes $\delta_{P_0} - \delta_{P_i}$, $i = 1, 2$ as above, and that the resulting potentials are measured on $\Gamma \subseteq \partial\Omega$, where Γ is open in the relative topology of $\partial\Omega$. It is required that Γ be sufficiently large, in the sense that there exists $y_0 \in \Gamma$ such that $\partial\Omega \cap B_\delta(y_0) \subset \Gamma$, and that $\mathrm{dist}(\{P_0, P_1, P_2\}, \Gamma) \geq \delta$. With these assumptions it is shown that

Theorem 2.8 *Let u_i and $\tilde{u}_i$, $i = 1, 2$ denote the potential in Ω with perfectly conducting planar cracks σ and $\tilde{\sigma}$, respectively, with fluxes $\delta_{P_0} - \delta_{P_i}$. Then*

$$d_{\mathcal{H}}(\sigma, \tilde{\sigma}) \leq \omega(\max_{i=1,2} \|u_i - \tilde{u}_i\|_{L^\infty(\Gamma)})$$

where $d_{\mathcal{H}}$ denotes Hausdorff distance and ω is a continuous, non-negative, non-decreasing function satisfying $\omega(s) \leq K(\ln|\ln s|)^{-\alpha}$ for $s \in [0, 1/e)$ and some constants K, $\alpha > 0$ (K and α depend on the a priori data $|\partial\Omega|_{C^2}$, $|\partial\sigma|_{C^2}$, $|\partial\tilde{\sigma}|_{C^2}$, and δ.)

3 Reconstruction Algorithms

Much work has been done on the very practical problem of reconstructing the interior conductivity of an object from electrostatic boundary measurements, but as with uniqueness and stability, we should expect that superior results will be obtained by incorporating a priori information about the expected features of the object, in this case, cracks.

The first reconstruction algorithm specifically designed for locating cracks in a two-dimensional conductor was developed by Santosa and Vogelius in [44]. The algorithm assumes that the crack is linear and perfectly conducting; the latter assumption is not restrictive, given the duality between the conducting and insulating problems. As discussed above, in this case we have a Lipschitz stability estimate for the location of the crack. For this algorithm a linear crack σ is specified by giving the cartesian coordinates of one endpoint, the angle of the crack with respect to the horizontal axis, and the length of the crack, a total of four parameters. The algorithm uses current input fluxes of the form $\delta_P - \delta_Q$, where P and Q are distinct points on the boundary of the region Ω; the induced potential is then measured on $\partial\Omega$. One would expect generically that the position of the crack would be overdetermined by this data.

The algorithm, however, distills the boundary data down to just four numbers by integrating the data against specified test functions which depend on the current estimated position of the crack. Specifically, let u^σ denote the solution to $\triangle u^\sigma = 0$ in $\Omega \setminus \sigma$, with u^σ constant on σ, $\frac{\partial u^\sigma}{\partial \nu} = \delta_P - \delta_Q$ on $\partial\Omega$, and $\int_{\partial\Omega} u^\sigma \, ds = 0$. Note that u^σ has logarithmic singularities at P and Q. Let us use u_0 to denote the harmonic function on Ω with this same Neumann data (the response of an "uncracked" domain.) Note that u_0 is in principle known (and indeed, if Ω is a circle we can write u_0 in closed form.) It's not hard to see that the quantity $u^\sigma - u_0$ is smooth on $\partial\Omega$, and we will in fact work with $u^\sigma - u_0$ on $\partial\Omega$ (so an uncracked domain should give a zero response for any input flux).

In order to define the test functions let us take, for the moment, a cartesian coordinate system in which a crack σ lies with one endpoint at the origin, at a zero angle with respect to the horizontal axis, with length L (so that the other endpoint of σ is at coordinates $(L, 0)$.) We denote $z = (x, y)$ (or

$z = x + iy$) and define functions

$$
\begin{aligned}
w_1(z) &= \mathrm{Im}[z], \quad w_2(z) = \mathrm{Im}[z^2], \\
w_3(z) &= \begin{cases} \mathrm{Re}[(z-L)\sqrt{z(z-L)}], & \mathrm{Re}(z) > L/2, \\ -\mathrm{Re}[(z-L)\sqrt{z(z-L)}], & \mathrm{Re}(z) < L/2 \end{cases} \\
w_4(z) &= \begin{cases} \mathrm{Re}[\sqrt{z(z-L)}], & \mathrm{Re}(z) > L/2, \\ -\mathrm{Re}[\sqrt{z(z-L)}], & \mathrm{Re}(z) < L/2 \end{cases}
\end{aligned}
$$

The functions w_3 and w_4 extend to $\mathrm{Re}(z) = L/2$ by continuity. Both w_1 and w_2 are smooth everywhere; w_3 has a square root singularity at $z = 0$ and w_4 has a square root singularity at $z = L$. One can check that all w_i are harmonic in $\Omega \setminus \sigma$, and all satisfy

$$\int_{\partial\Omega} \frac{\partial w_i}{\partial \nu}\, ds = 0, \quad \int_{\sigma} \left[\frac{\partial w_i}{\partial \nu}\right] ds = 0.$$

The algorithm distills the quantity $(u^\sigma - u_0)|_{\partial\Omega}$ into the four numbers

$$F_i(\sigma, w^\sigma) = \int_{\partial\Omega} (u^\sigma - u_0)(z) \frac{\partial w_i^\sigma}{\partial \nu}(z)\, ds_z, \tag{7}$$

for $i = 1$ to 4 where w_i^σ denotes the function w_i adapted to the crack σ (that is, $w_i^\sigma = w_i \circ \psi$ where ψ is the mapping which transforms σ to $(0, L)$ on the x axis). The rationale for this choice of test functions is described below.

Let $\bar{\sigma}$ denote the actual location of the linear crack and note that we can determine $F_i(\bar{\sigma}, w^\sigma)$ for any linear crack σ by applying the specified current flux, measuring the potential $u^{\bar{\sigma}}$ on $\partial\Omega$, subtracting u_0 and then computing the F_i using (7). One would expect generically that the four equations $F_i(\sigma, w^\sigma) = F_i(\bar{\sigma}, w^\sigma)$ in four unknowns (the coordinates describing σ) would have a solution at $\sigma = \bar{\sigma}$ which is at least locally unique. The system of equations $F_i(\sigma, w^\sigma) = F(\bar{\sigma}, w^\sigma)$ could be solved using any standard root-finding technique, e.g., Newton's method.

However, the algorithm has an additional important feature, a feature which is a major reason for these specific choices for the w_i. In order to introduce this feature, we should first note that for any given crack location σ, certain input fluxes will yield very small values for $u^\sigma - u_0$ on $\partial\Omega$. Suppose, for example, that Ω is the unit ball and consider a perfectly conducting crack

σ of length L, vertically oriented in Ω anywhere on the line $x = 0$. In this case it's easy to see that an input flux $\delta_P - \delta_Q$ with $P = (1, 0)$ and $Q = (-1, 0)$ will yield $u^\sigma = u_0$ throughout Ω, since σ lies on an equipotential surface for u_0. In short, such an input flux yields no useful information for recovering σ from boundary data. Note also that σ can be moved up and down the line $x = 0$ and produce no change in the boundary measurements. Moving the points P and Q slightly may yield non-zero values for $u^\sigma - u_0$, but the values will be so small that measurement error will corrupt the data, and we expect reconstructions which use this data will be compromised. Moreover, even relatively large changes in the crack position will likely produce only small changes in the boundary data.

On the other hand, if we place the electrodes at positions $(0, 1)$ and $(0, -1)$ then the crack σ cuts orthogonally across equipotential lines for u_0, and so we expect $u^\sigma - u_0$ to be large and maximally sensitive to changes in the position of σ.

The algorithm in [44] is based on Newton's method for solving $F_i(\sigma, w^\sigma) = F_i(\bar{\sigma}, w^\sigma)$, but also attempts to adaptively change the input flux in a way that maximizes the sensitivity of $u^\sigma - u_0$ to changes in the crack position after each iteration of Newton's method. The manner in which the electrode locations are updated is detailed below.

With the given choices for F_i one can verify that at $\sigma = \bar{\sigma}$ the four by four Jacobian matrix for the system $F_i(\sigma, w^\sigma) - F_i(\bar{\sigma}, w^\sigma) = 0$ is lower triangular, regardless of the input flux used. We should expect that Newton's method (which involves implicitly inverting the Jacobian) will be well-conditioned, or alternatively, that the estimate of the crack location will be most stable with respect to the boundary data, when the diagonal elements of the Jacobian are as large as possible (which tends to improve the conditioning of the associated linear system of equations). Thus at each stage of Newton's method the algorithm adapts the input flux pattern to maximize certain diagonal entries of the lower triangular Jacobian. Indeed, the authors show that at $\sigma = \bar{\sigma}$ the first and second entries of the diagonal are given by m and $2m$ where

$$m = (\xi - w_1)(P) - (\xi - w_1)(Q)$$

where the electrodes are located at points P and Q on $\partial\Omega$ and ξ satisfies $\triangle\xi = 0$ in $\Omega \setminus \bar{\sigma}$ with $\xi = 0$ on $\bar{\sigma}$ and $\frac{\partial \xi}{\partial \nu} = \frac{\partial w_1}{\partial \nu}$ on $\partial\Omega$. Note that in an iterative scheme (an approximation to) the function ξ can be computed from the current estimated crack σ. One can thus maximize m and the associated diagonal entries of the Jacobian by choosing P to maximize the quantity $\xi - w_1$ on $\partial\Omega$ and Q to minimize $\xi - w_1$ on $\partial\Omega$.

The general outline of the algorithm is this. Let $\mathbf{F}(\sigma, w^\sigma) = (F_1(\sigma, w^\sigma), F_2(\sigma, w^\sigma), F_3(\sigma, w^\sigma), F_4(\sigma, w^\sigma))$.

1. Make initial guess $\sigma = \sigma_0$ at the crack location, set $k = 0$.

2. Based on the current estimated crack location σ_k, select the maximally sensitive electrode locations P and Q as defined above.

3. Apply the appropriate currents, measure the resulting potential on $\partial\Omega$, and use equation (7) and the w_i to compute $\mathbf{F}(\bar{\sigma}, w^{\sigma_k})$.

4. Compute $\mathbf{F}(\sigma_k, w^{\sigma_k})$. If the residual $\|\mathbf{F}(\sigma_k, w^{\sigma_k}) - \mathbf{F}(\bar{\sigma}, w^{\sigma_k})\|$ is sufficiently small (in say, the L^2 norm) then terminate with estimate $\bar{\sigma} = \sigma_k$.

5. Compute the Jacobian and make the appropriate Newton update to compute a new estimated crack location σ_{k+1}. Set $k = k + 1$ and return to step 2.

This algorithm proved quite successful on both computationally generated data and on data collected from an experimental apparatus; see [39].

This algorithm was also adapted to seek out multiple cracks in a conductor in [26]. In this case the algorithm, in seeking out a collection of n cracks, applies n distinct fluxes at each stage of Newton's method. The fluxes are of the form $\delta_P - \delta_{Q_i}$, $i = 1$ to n. The points P and the Q_i are chosen by a procedure similar to the single crack case, in which we seek to maximize diagonal entries on the relevant Jacobian matrix in order to stabilize the estimates of the crack locations. The algorithm was tested on both computationally generated data and experimentally gathered data [28].

One of the difficulties in the multiple crack version of the algorithm is that of determining how many cracks might be present (one must choose this a priori). As originally proposed the algorithm uses an ad hoc procedure for adjusting the number of estimated cracks as the algorithm runs. However, in [10] a modification to the algorithm is proposed, in which the number of cracks is automatically adjusted using a Bayesian statistical approach, which also serves to help regularize the inversion.

Other approaches have also been used for the recovery of cracks in conductors. The so-called "reciprocity gap" principle has been used to recover both linear cracks in two-dimensional conductors and planar cracks in three-dimensional conductors [13, 17].

The basis of the reciprocity gap approach is as follows. Let Ω be a bounded domain in $\mathbb{R}^3$ with unit conductivity and σ a planar insulating crack contained in Ω. Suppose that the plane in which σ lies is described by the equation $\mathbf{n} \cdot \mathbf{x} = c$, where $\mathbf{n} = (n_1, n_2, n_3)$ denotes a unit normal vector to the plane and $\mathbf{x} = (x_1, x_2, x_3)$. Let u be the electrical potential induced in $\Omega \setminus \sigma$ by input current g, so that $\triangle u = 0$ in $\Omega \setminus \sigma$ with $\frac{\partial u}{\partial \mathbf{n}} = 0$ on σ and $\frac{\partial u}{\partial \nu} = g$ on $\partial\Omega$ (ν an outward unit normal vector field on $\partial\Omega$). We require $\int_{\partial\Omega} g\, ds = 0$, and we can normalize u by $\int_{\partial\Omega} u\, ds = 0$. We use f to denote the measured boundary potential, so $f = u|_{\partial\Omega}$. Let v denote a $C^2(\bar{\Omega})$ harmonic function on Ω and define

$$RG_{[g,f]}(v) = \int_{\partial\Omega} (gv - \frac{\partial v}{\partial \nu} f)\, ds, \tag{8}$$

the so-called reciprocity gap functional. Note that given the input current g and response f we can compute $RG_{[g,f]}(v)$ for any given harmonic function v. One can easily verify using the divergence theorem that

$$RG_{[g,f]}(v) = \int_{\sigma} [u] \frac{\partial v}{\partial \mathbf{n}}\, ds \tag{9}$$

where $[u]$ denotes the jump in u in the direction of the normal vector field $-\mathbf{n}$.

In [13] the authors show that if the input flux g is chosen such that $\int_\sigma [u]\, ds \neq 0$, then a normal vector to the plane containing σ is given by $\mathbf{L} = (L_1, L_2, L_3)$ with $L_i = RG_{[g,f]}(v_i)$, where v_i denotes the harmonic function $v_i(\mathbf{x}) = x_i$. Of course then we have $\mathbf{n} = \mathbf{L}/||\mathbf{L}||$. Although the condition $\int_\sigma [u]\, ds \neq 0$ is not guaranteed, it is generically expected for a "typical" input flux.

Since we have identified $\mathbf{n}$ we can, by an appropriate change of coordinates, assume that the plane containing σ is of the form $x_3 = c$. They then show that (again provided $\int_\sigma [u]\, ds \neq 0$) one can obtain c as $c = RG_{[g,f]}(p)/||\mathbf{L}||$ where $p(x_1, x_2, x_3) = (x_3^2 - x_2^2)/2$, which is harmonic. Thus if the input flux is such that $\int_\sigma [u]\, ds \neq 0$, we can identify the plane in which the crack lies. One can perform similar computations to show that one may identify the line on which a linear crack lies in a two-dimensional conductor.

Having determined the plane in which the crack lies, one may attempt to recover the actual shape of the crack. In [17] the authors approach the problem as follows. Let us assume that after appropriate scaling, translation, and rotation the plane containing σ is given by $x_3 = 0$ and that σ is contained in the square $S = \{(x_1, x_2, 0) \in \mathbb{R}^3; -1 < x_1, x_2 < 1\}$. Define harmonic functions

$$\phi^i_{p,q}(x_1, x_2, x_3) = \frac{1}{\pi\sqrt{p^2+q^2}}\psi^i_{p,q}(x_1, x_2)\sinh(\pi x_3\sqrt{p^2+q^2})$$

for $i = 1$ to 4 where

$$\begin{aligned} \psi^1_{p,q}(x_1, x_2) &= \cos(p\pi x_1)\cos(q\pi x_2), \quad \psi^2_{p,q}(x_1, x_2) = \cos(p\pi x_1)\sin(q\pi x_2) \\ \psi^3_{p,q}(x_1, x_2) &= \sin(p\pi x_1)\cos(q\pi x_2), \quad \psi^4_{p,q}(x_1, x_2) = \sin(p\pi x_1)\sin(q\pi x_2) \end{aligned}$$

Then

$$RG_{[g,f]}(\phi^i) = \int_S [\tilde{u}]\psi^i\, ds \tag{10}$$

where $[\tilde{u}]$ denotes the extension of $[u]$ by zero from σ to S. It's not hard to see then that we can recover the Fourier coefficients of $[\tilde{u}]$ by computing $RG_{[g,f]}(\phi^i)$ for all positive integers p and q, with $i = 1$ to 4, and so recover $[\tilde{u}]$. We expect that $[\tilde{u}] \neq 0$ on any open portion of σ and indeed, this is proven in [13], subject again to the condition that $\int_\sigma [u]\, ds \neq 0$. We can thus identify the crack as the support of $[\tilde{u}]$. Analogous formulae hold in the two-dimensional case.

However, any estimate of $[\tilde{u}]$ based on a truncated Fourier series will certainly have as support all of S. In [17] the authors construct an estimate of σ as follows. Let $[\tilde{u}]_n$ denote the estimate of $[\tilde{u}]$ constructed by truncating the Fourier series constructed from equation (10) at $p, q \leq n$. For $\epsilon > 0$ define

$$\sigma_{n\epsilon} = \{x \in S; |[\tilde{u}]_n(x)| > \epsilon\}.$$

We take $\sigma_{n\epsilon}$ as an approximation to σ. The authors prove convergence results which state that provided a certain "stress intensity factor" does not vanish on $\partial\sigma$, we have

Theorem 3.1 *Given any $\delta > 0$, there exist two positive constants c and $\tilde{c}$ and some positive real number ϵ_0 such that for $\epsilon \leq \epsilon_0$ and $n > \tilde{c}\epsilon^{-(2+\delta)}$ we have*

$$d_{\mathcal{H}}(\sigma_{n\epsilon}, \sigma) < c\epsilon^2 \quad ,$$

where $d_{\mathcal{H}}$ is Hausdorff distance.

The authors provide a variety of numerical computations to illustrate the effectiveness of the algorithm, in both two and three dimensions. A major limitation of this very direct and elegant approach is that it appears impossible to adapt to the identification of multiple cracks that are not all coplanar.

The reciprocity gap principle has also been applied to the crack identification problem in elastostatics, in which the forward problem is governed by the linear equations of elasticity; see [14, 19].

Another approach for determining cracks has been proposed in [32]. The authors describe an efficient procedure for recovering a piecewise linear crack inside a two-dimensional polygonal domain, in which the "crack" may even self-intersect (enclosing a void) or penetrate the exterior boundary of the region. The approach is to reformulate the conduction problem using a Schwarz-Christoffel transformation, and reduce the crack identification problem to system of non-linear algebraic equations which can be solved efficiently. The authors propose several distinct algorithms, depending on the nature of the crack to be identified.

In addition to the above techniques, more traditional regularized least-squares techniques have been investigated, see e.g., [42] and the references therein. The main novelty in [42] is the design of an efficient boundary integral equation formulation for the forward problem (and for computing derivatives with respect to crack parameters) thus minimizing the cost of the optimization.

Two additional approaches to crack identification have emerged in recent work. In [18] the authors use complex analytic techniques to detect and estimate crack locations from boundary data. Moreover, this technique allows one to estimate the crack location(s) even if one has measurements of the induced potentials on only a portion of the boundary. The algorithm described in [23] also enjoys this advantage.

The basis of the technique in [18] is as follows. Suppose $\gamma \equiv 1$ and that only a single insulating, interior crack σ is present. Let a current g be imposed on $\partial\Omega$. The potential u satisfies $\triangle u = 0$ in $\Omega \setminus \sigma$ with $\frac{\partial v}{\partial \nu} = g$ on $\partial\Omega$ and $\frac{\partial v}{\partial \nu} = 0$ on σ. Suppose we have measurements of u on some segment Γ of $\partial\Omega$. Let v denote the harmonic conjugate to u in $\Omega \setminus \sigma$. The function $F = u + iv$ is then analytic in $\Omega \setminus \sigma$.

Now since $\nabla v = (\nabla u)^{\perp}$, we have $\frac{\partial v}{\partial s} = \frac{\partial u}{\partial \nu} = g$, so that we can compute v on $\partial\Omega$ as

$$v(x) = \int_a^x g(y)ds_y$$

for $x \in \partial\Omega$ (with the normalization $v(a) = 0$ for some fixed $a \in \partial\Omega$). We thus know the value of $\mathrm{Im}(F) = v$ on all of $\partial\Omega$ and the value of $\mathrm{Re}(F) = u$ on $\Gamma \subseteq \partial\Omega$, with F analytic on $\Omega \setminus \sigma$. To solve the inverse problem we may then seek a crack σ and bounded analytic function F on $\Omega \setminus \sigma$ such that $\mathrm{Im}(F)$ and $\mathrm{Re}(F)$ take on the known values on $\partial\Omega$ (or Γ) and on Γ, respectively. In practice, of course, the latter two constraints can be achieved only to within some specified tolerance.

The approach taken in [18] consists of finding a meromorphic function ϕ on Ω with at most N poles (N a positive integer which can be adjusted) which minimizes $\sup_\Gamma |\mathrm{Re}(\phi) - u|$ subject to the condition that $\sup_\Gamma |\mathrm{Im}(\phi) - v|$ be smaller than some prescribed tolerance. The authors give a procedure for approximately solving this problem, using ideas from AAK approximation theory (see [1]). The idea is that the poles of ϕ inside Ω somehow approximate the location of σ, at least as N increases. The authors provide some analysis of what kind of convergence can be expected, and give numerical examples. It should be mentioned that for fixed (small) N, related "pole-finding" algorithms have earlier been studied in [41].

Finally, an interesting new approach to crack detection is proposed in [23]. This approach also has the advantage that one need not specify the number of cracks a priori, and this method allows one to reconstruct crack estimates by taking measurements only on some portion of $\partial\Omega$. The method is a modification of ideas developed in [21, 22] and [37].

Let Γ be a subset of $\partial\Omega$ with positive measure and define $L^2_{\#}(\Gamma)$ to be those functions in $L^2(\Gamma)$ with zero integral over Γ. Let Σ be a collection of insulating cracks in Ω. For a function $g \in L^2_{\#}(\Gamma)$ we can solve the problem $\triangle u = 0$ in $\Omega \setminus \Sigma$, with $\frac{\partial u}{\partial \nu} = 0$ on Σ and $\partial\Omega \setminus \Gamma$, and $\frac{\partial u}{\partial \nu} = g$ on Γ. The solution is determined up to an additive constant only, but we can fix a unique solution by requiring $u|_\Gamma \in L^2_{\#}(\Gamma)$. Define an operator Λ on $L^2_{\#}(\Gamma)$ by $\Lambda(g) = u|_\Gamma$, the "restricted" Neumann-to-Dirichlet operator. Let Λ_0 be the restricted Neumann-to-Dirichlet operator on $L^2_\#(\Gamma)$ for the case in which no cracks are present. Of course we seek to use the operator Λ (or really, our ability to evaluate this operator for select input currents) to determine Σ.

In [23] the authors provide a factorization $\Lambda - \Lambda_0 = K^*K$, where K maps $L^2_\#(\Gamma)$ to a certain Sobolev space defined on $\Omega \setminus \Sigma$. This can be used to demonstrate that the operator $\Lambda - \Lambda_0$ is symmetric, positive semidefinite, and compact. As such we can find a basis of orthogonal eigenfunctions v_j for $\Lambda - \Lambda_0$ on $L^2_\#(\Gamma)$. Moreover, the operator $(\Lambda - \Lambda_0)^{1/2}$ is well-defined.

Now consider some suitably smooth curve σ_0 in Ω, and let n denote a unit normal field on σ_0. Define

$$v_1(x) = \frac{1}{2\pi}\int_{\sigma_0} \phi(y)\frac{\partial}{\partial n_y}\ln\left(\frac{1}{|x-y|}\right)\,ds_y$$

for $x \in \Omega \setminus \sigma_0$. Here ϕ is a smooth density which is positive except at the endpoints of σ_0, where it vanishes. Let v_0 be a harmonic function on Ω with $\frac{\partial v_0}{\partial \nu} = \frac{\partial v_1}{\partial \nu}$ on $\partial\Omega$; such a function exists since $\int_{\partial\Omega} \frac{\partial v_1}{\partial \nu}\,ds = 0$. For a suitable choice of c the function $v = v_1 - v_0 - c$ (or rather, its trace on Γ) will be in $L^2_\#(\Gamma)$. The authors prove that

Theorem 3.2 *The trace $v|_\Gamma$ is in the range of $(\Lambda - \Lambda_0)^{1/2}$ if and only if $\sigma_0 \subseteq \Sigma$.*

They also provide a quantitative method, based on knowledge of the eigenfunctions v_j and eigenvalues, for determining whether a function lies (to some given precision) in the range of $(\Lambda - \Lambda_0)^{1/2}$.

This leads to a simple method for reconstructing cracks: divide Ω into a large number of small rectangles; in each rectangle consider a small linear "test" crack σ_0; rotate this test crack through various angles about its center and determine if the corresponding function v lies in the range of $(\Lambda - \Lambda_0)^{1/2}$, to within some tolerance. If so, consider σ_0 part of the collection of cracks Σ we seek to image. The authors also use "test dipoles" (the zero length limit of a short crack) to image Σ. Although computationally intensive, the algorithm yields good results, with no a priori assumptions required concerning the number of cracks. This is an example of an imaging method that reconstructs objects by testing whether certain "signatures" are in the range of a (measured) linear operator. Methods of this type are often associated with the name "linear sampling".

4 Final Remarks

The elliptic boundary value problems corresponding to the perfectly insulating and the perfectly conducting cracks we have considered here may be obtained as the limit of problems in which the cracks are approximated by "thin domains" inside which the conductivity goes to zero or infinity at an appropriate rate (relative to the thickness). The crack identification problems we have discussed may thus be seen as important special examples of a more general class of problems involving volumetrically small inhomogeneities. There is a whole body of work associated with such problems. In particular, various authors have developed special purpose algorithms for these identification problems, of a nature similar to some of the algorithms described here. It is well outside the scope of this brief survey to attempt any review of the literature on these subjects. In the absence of such a review the interested reader may, as a point of departure, consult [30] and [24], and the references therein.

References

[1] Adamjan, V.M., Arov, D.Z., and Krein, M.G., *Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem*, Math. USSR Sbornik, **15**, 1971, pp. 31-73.

[2] Alessandrini, G., *Examples of instability in inverse boundary-value problems*, Inverse Probl, **13** (4), 1997, pp. 887-897.

[3] Alessandrini, G., *Stable Determination of a crack from boundary measurement*, Proc Roy Soc Edinburgh Sect A, 123, 1993, pp. 497-516.

[4] Alessandrini, A., Beretta, E., and Vessella, S., *Determining linear cracks by boundary measurements: Lipschitz stablity*, Siam J Math Anal, **27** (2), 1996, pp. 361-375.

[5] Alessandrini, G., and DiBenedetto, E., *Determining 2-dimensional crack in 3-dimensional bodies: Uniqueness and stability*, Indiana U Math J, **46** (1), 1997, pp. 1-82.

[6] Alessandrini, G., and Magnanini, R., *The index of isolated critical points and solutions of elliptic equations in the plane*, Ann Scuola Norm Sup Pi Cl Sci, **19** (4), 1992, pp. 567-589.

[7] Alessandrini, G., and Magnanini, R., *Elliptic equations in divergence form, geometric critical points of solutions, and Stekloff eigenfunctions*, Siam J Math Anal, **25**, 1994, pp.1259-1268.

[8] Alessandrini, G., and Rondi, L., *Stable determination of a crack in a planar inhomogeneous conductor*, Siam J Math Anal **30** (2), 1998, pp. 326-340.

[9] Alessandrini, G., and Diaz Valenzuela, A., *Unique determination of multiple cracks by two measurements*, Siam J Cont Opt, **34** (3), 1996, pp. 913-921.

[10] Andersen, K., Brooks, S., and Hansen, M., *A Bayesian approach to crack detection in electrically conducting media*, Inverse Probl, **17**, 2001, pp. 121-136.

[11] Andrieux, S., and Ben Abda, A., *Identification de fissures planes par une donneé de bord unique; un procédé direct de localisation et d'identification*, , C.R. Acad. Sci., Paris I, 1992, **315**, pp 1323-1328.

[12] Andrieux, S., Ben Abda, A., and Jaoua, M., *On the inverse emergent plane crack problem*, Math Method Appl Sci, **21** (10), 1998, pp. 895-906.

[13] Andrieux, S., and Ben Abda, A., *Identification of planar cracks by complete overdetermined data: inversion formulae*, Inverse Probl, **12**, 1996, pp. 553-563.

[14] Andrieux, S., Ben Abda, A., and Bui, H.D., *Reciprocity principle and crack identification*, Inverse Probl, **15** (1), 1999, pp. 59-65.

[15] Aparicio, N.D., and Pidcock, M.K., *On a class of free boundary problems for the Laplace equation in two dimensions*, Inverse Probl, **14** (1), 1998, pp. 9-18.

[16] Aparicio, N.D., and Pidcock, M.K., *The boundary inverse problem for the Laplace equation in two dimensions*, Inverse Probl, **12** (5), 1996, pp. 565-577.

[17] Bannour, T., Ben Abda, A., and Jaoua, M., *A semi-explicit algorithm for the reconstruction of 3D planar cracks*, Inverse Probl, **13** (4), 1997, pp. 899-917.

[18] Baratchart, L., Leblond, J., Mandréa, F., and Saff, E.B., *How can the meromorphic approximation help to solve some 2D inverse problems for the Laplacian?*, Inverse Probl, **15** (1), 1999, pp. 79-90.

[19] Ben Abda, A., Ben Ameur, H., and Jaoua, M., *Identification of 2D cracks by elastic boundary measurments*, Inverse Probl, **15** (1), 1999, pp. 67-77.

[20] Ben Abda, A., Chaabane, S., Dabaghi, F. El, and Jaoua, M., *On a non-linear geometrical inverse problem of Signorini type: Identifiablity and stability*, Math Method Appl Sci, **21** (15), 1998, pp. 1379-1398.

[21] Brühl, M. *Explicit characterization of inclusions in electrical impedance tomography*, preprint.

[22] Brühl, M., and Hanke, M., *Numerical implementation of two non-iterative methods for locating inclusions by impedance tomography*, Inverse Probl, to appear.

[23] Brühl, M., Hanke, M., and Pidcock, M., *Crack detection using electrostatic measurements*, M2AN Math. Model. Numer. Anal., **35**, 2001, pp. 595-605.

[24] Brühl, M., Hanke, M., and Vogelius, M.S., *A direct impedance tomography algorithm for locating small inhomogeneities*, Numerische Mathematik, to appear.

[25] Bryan, K., and Vogelius, M., *A uniqueness result concerning the identification of a collection of cracks from finitely many electrostatic boundary measurements*, Siam J Math Anal, **23** (4), 1992, pp. 950-958.

[26] Bryan, K., and Vogelius, M., *A computational algorithm to determine crack locations from electrostatic boundary measurements—the case of multiple cracks*, Int J Engr Sci, **32** (4), 1994, pp. 579-603.

[27] Bryan, K., and Vogelius, M., *Effective behavior of clusters of microscopic cracks inside a homogeneous conductor*, Asymptotic Anal, **16** (2), 1998, pp. 141-179.

[28] Bryan, K., Liepa, V., and Vogelius, M., *Reconstruction of multiple cracks from experimental, electrostatic boundary measurements*, proceedings of the conference on Inverse Problems and Optimal Design in Industry, July 8-10, 1993, Philadelphia PA.

[29] Bui, H.D., Constantinescu, A., and Maigre, H., *Inverse scattering of a planar crack in 3D acoustics: closed form solution for a bounded body*, CR Acad Sci II B, **327** (10), 1999, pp. 971-976.

[30] Capdeboscq, Y., and Vogelius M.S., *A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction*, M2AN Math. Model. Numer. Anal., to appear.

[31] Diaz Valenzuela, A., *Unicita' e stabilita per il problema inverse del crack perfettamenta isolante*, thesis, Universita' di Trieste, Trieste, Italy, 1993.

[32] Elcrat, A.R., and Hu, C.L., *Determination of surface and interior cracks from electrostatic boundary measurements using Schwarz-Christoffel transformations*, Int J Engr Sci, **34** (10), 1996, pp. 1165-1181.

[33] Elcrat, A.R., Isakov, V., and Neculoiu, O., *On finding a surface crack from boundary measurements*, Inverse Probl, **11** (2), 1995, pp. 343-351.

[34] Eller, M., *Identification of cracks in three-dimensional bodies by many boundary measurements*, Inverse Probl, **12** (4), 1996, pp. 395-408.

[35] Friedman, A., and Vogelius, M., *Determining cracks by boundary measurements*, Ind U Math J, **38**, 1989, pp. 527-556.

[36] Kim, H., and Seo, J., *Unique determination of a collection of a finite number of cracks from two boundary measurements*, Siam J. Math. Anal. **27**, 1996, pp. 1336-1340.

[37] Kirsch, A., *Characterization of the shape of a scattering obstacle using the spectral data of the far field operator*, Inverse Probl, **14**, 1998, pp. 1489-1512.

[38] Kolehmainen V., Arridge S.R., et al., *Recovery of region boundaries of piecewise constant coefficients of an elliptic PDE from boundary data*, Inverse Probl, **15** (5), 1999, pp. 1375-1391.

[39] Liepa, V., Santosa, F., and Vogelius, M., *Crack determination from boundary measurements—reconstruction using experimental data*, J. Nondestructive Evaluation, **12**, 1993, pp.163-174.

[40] Mellings, S.C., and Aliabadi, M.H., *Three-dimensional flaw identification using inverse analysis*, Int J Engr Sci, **34** (4), 1996, pp. 453-469.

[41] Miller, K., *Stabilized numerical analytic prolongation with poles*, SIAM J Appl Math, **18**, 1970, pp. 346-363.

[42] Nishimura, N. and Kobayashi, S., *Determination of cracks having arbitrary shapes with the boundary integral-equation method*, Eng Anal Bound Elem, **15** (2), 1995, pp. 189-195.

[43] Rondi, L., *Optimal stability estimates for the determination of defects by electrostatic measurements*, Inverse Probl **15** (5), 1999 pp. 1193-1212.

[44] Santosa, F., and Vogelius, M., *A computational algorithm to determine cracks from electrostatic boundary measurements*, Int J Engr Sci, **29**, 1991, pp. 917-937.