Rose-Hulman Undergraduate Mathematics Journal

Volume 14
Issue 2

Article 11

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Rose-Hulman Undergraduate Mathematics Journal

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Volume 14, no. 2, Fall 2013

Sponsored by

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Seth Hausken   Jared Skinner

Abstract. The directed graph of a commutative ring is a graph representation of its additive and multiplicative structure. Using the mapping $(a, b) \to (a + b, a \cdot b)$ one can create a directed graph for every finite, commutative ring. We examine the properties of directed graphs of commutative rings, with emphasis on the information the graph gives about the ring.

Acknowledgements: We would like to acknowledge the University of St. Thomas for making this research possible and Dr. Michael Axtell for his guidance throughout.

1 Introduction

When studying group theory, especially early on in such an exploration, Cayley tables are used to give a simple visual representation to the structure of the group. Cayley tables can also be used when studying rings, but since rings have both an additive and multiplicative structure, two Cayley tables are necessary to fully represent a ring. It is desirable to create a visual representation of a ring that maintains the structure while being a single representation instead of two. We turn to graph theory to create a directed graph representation of a ring, as first proposed by Lipkovski in [8].

Definition By the directed graph, or digraph for brevity, of R, denoted $\Psi(R)$, we mean the graph with $V(\Psi(R)) = R \times R$, and for distinct $(a, b), (c, d) \in R \times R$, there is a directed edge, denoted $(a, b) \to (c, d)$, connecting (a, b) to (c, d) if and only if $a + b = c$ and $a \cdot b = d$.

Example 1.1 *The digraph of $\mathbb{Z}_4$ consists of vertices with entries from $\mathbb{Z}_4$ with addition and multiplication modulo 4 [See Figure 1].*

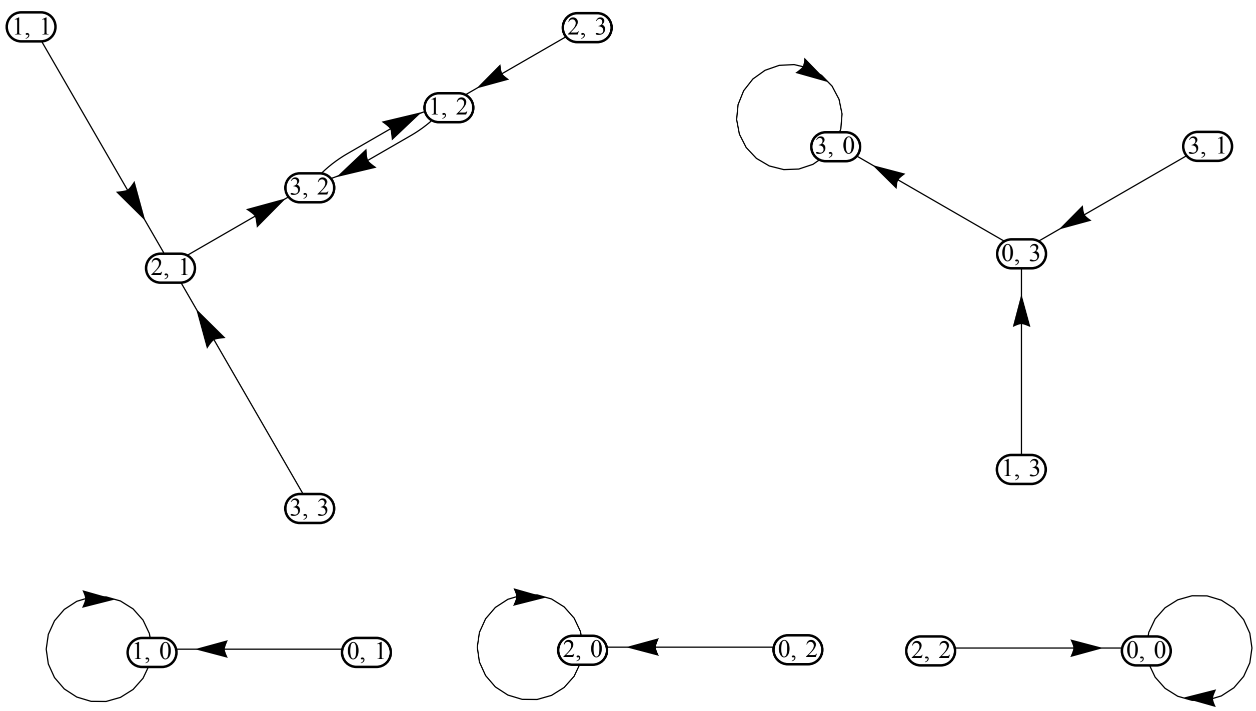

Figure 1: $\Psi(\mathbb{Z}_4)$

The digraph of a ring holds the additive and multiplicative structure of the ring in a simpler, more compressed fashion. Unlike other common graph representations of rings, such as the zero-divisor graph [1], the digraph attempts to retain all the elements and structure of the ring while still being fairly straight forward. Since the digraph does retain the entire structure of the ring, the digraph should contain information about algebraic structures, such as ideals and zero-divisors.

2 Background

In this paper, we use concepts from both ring theory and graph theory. Both these topics are far too dense to delve into with great detail here. Instead, we define some of the basic concepts needed for this paper, then further define concepts as they become relevant. For further explanation of ring theory, see [2], [5], [6], [4]. For further explanation of graph theory, see [3].

Definition [4] A *ring* is a set with two binary operations, addition and multiplication, such that for all a, b, and $c \in R$:

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. There is an additive identity 0 in R. That is, there is an element 0 in R such that $a + 0 = a$ for all $a \in R$.
4. There is an element $-a$ in R such that $a + (-a) = 0$.
5. $a(bc) = (ab)c$
6. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$

A given ring R is called commutative if multiplication in the ring is commutative. That is, for all $a, b \in R$, $ab = ba$. A given ring R is said to have identity if there exists $1 \in R$ such that $a \cdot 1 = a$ for all $a \in R$. By a *ring*, we generally mean a commutative ring with identity, denoted R, unless otherwise specified. While it is only feasible to graph finite rings, results are given for all rings, unless otherwise specified.

The notion of a subring is exactly as one would expect, a ring that is contained within another ring and has the same structure as the containing ring. Next, we define a special type of subring called an ideal.

Definition [4] A *subring* of a ring R is called an *ideal* of R is for every $r \in R$ and every $a \in A$, ra and ar are in A.

By an ideal, we mean a proper ideal of a ring, that is an ideal that is neither the trivial ring nor the entire ring R.

Next, we define a zero-divisor, which is a particular type of element of a ring with peculiar and interesting properties.

Definition A *zero-divisor* is an element a of a commutative ring R such that there is a nonzero element b in R such that $ab = 0$.

Note that this implies that 0 is a zero-divisor in every non-trivial ring. Some texts define zero-divisors as strictly nonzero elements, but we adopt the above definition from [2] and [6]. We adopt the notation $Z(R)$ for the set of all zero-divisors of a commutative ring from [6]. The set of units of R is denoted $U(R)$.

The remaining definitions are from graph theory.

Definition [3] A *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set V^2 of unordered pairs of V. The set V is the set of *vertices* of G and the set E is the set of *edges* of G. An edge (x, y) joins vertices x and y.

For a graph G, the set of vertices is denoted $V(G)$ and the set of edges is denoted $E(G)$. We say G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. If G' contains all edges of G that join two vertices in $V(G')$, then G' is an *induced subgraph.* For our purposes, all subgraphs will be induced, so we will use the term subgraph to mean induced subgraph. The digraph of an ideal I of a ring R, denoted $\Psi(I)$, is a subgraph of $\Psi(R)$ where $V(\Psi(I)) = I \times I$.

Definition [3] If the edges of a graph G are ordered pairs of vertices, then G is called a *directed graph.* An ordered pair (a, b) is directed from a to b.

Definition [3] If $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ are vertices in $\Psi(R)$, then $(a_1, b_1)—(a_2, b_2)— \cdots —(a_n, b_n)$ denotes a *walk* in $\Psi(R)$ from vertex (a_1, b_1) to vertex (a_n, b_n), where (a_i, b_i) is *adjacent* to (a_{i+1}, b_{i+1}) and (a_{i-1}, b_{i-1}) for $1 \leq i \leq n-1$ and — denotes a connection between two vertices without regard to direction.
If $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ are distinct, then the walk is a *path.*

For directed graphs, it becomes necessary to make a distinction between a walk (path) and a *directed walk (directed path).*

Definition If $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ are vertices in $\Psi(R)$, then $(a_1, b_1) \to (a_2, b_2) \to \cdots \to (a_n, b_n)$ denotes a *directed walk* in $\Psi(R)$ from vertex (a_1, b_1) to vertex (a_n, b_n), where (a_i, b_i) is *directionally adjacent* to (a_{i+1}, b_{i+1}) for $1 \leq i \leq n-1$. If $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ are distinct, then the directed walk is a *directed path.*

Definition A *connected* graph is one in which there is a path between two distinct vertices. A *connected component* of a graph G, denoted $\mathcal{C}$, is a maximal connected subgraph of G.

3 Paths

We begin by determining the behavior of paths and walks in the digraph of a ring. It is necessary to begin with such results as they are used later when studying the algebraic structures represented in digraphs.

Definition Let $(a, b), (x, y) \in \Psi(R)$. Suppose $(a, b) \to \cdots \to (x, y)$, then (x, y) is said to be *downstream* from (a, b), and (a, b) is said to be *upstream* from (x, y).

Definition If $(x, y) \to (z, w)$, we say that (x, y) *points to* (z, w).

Definition Given a vertex v, the number of vertices that point at v is the *incoming degree of* v. The number of vertices that v points to is the *outgoing degree of* v.

It should be noted that because $(x, y) \to (x + y, xy)$, the outgoing degree of (x, y) will always be one.

Definition A *cycle of length* n is a directed path such that $(x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_n, y_n) \to (x_1, y_1)$.

Lemma 3.1 *Let $\mathcal{K}$ and $\mathcal{L}$ be cycles in $\Psi(R)$ for some ring R. If there exists a vertex v such that $v \in V(\mathcal{K})$ and $v \in V(\mathcal{L})$, then $\mathcal{V}$ and $\mathcal{L}$ are the same cycle.*

Proof Since $v \in V(\mathcal{K})$, v points to some $k \in V(\mathcal{K})$ and since $v \in V(\mathcal{L})$, v points to some $l \in V(\mathcal{L})$. But v only points to one vertex, since the outgoing degree of v is one. Thus $k = l \in V(\mathcal{K})$ and $k = l \in V(\mathcal{L})$. Continuing in this fashion, we get that all vertices downstream from v are elements of both $V(\mathcal{K})$ and $V(\mathcal{L})$. But all vertices in $V(\mathcal{K})$ and $V(\mathcal{L})$ are downstream from v, since they are both cycles. Thus $V(\mathcal{K}) = V(\mathcal{L})$ and $\mathcal{K}$ and $\mathcal{L}$ are the same cycle.

This result implies that either cycles in a digraph are distinct, or they are the exact same cycle.

Lemma 3.2 *If $\mathcal{K}$ and $\mathcal{L}$ are distinct cycles in $\Psi(R)$ for some ring R, then there is no path from k to l for any $k \in V(\mathcal{K})$, $l \in V(\mathcal{L})$.*

Proof By contradiction assume that k is downstream from l for some $k \in V(\mathcal{K})$ and $l \in V(\mathcal{L})$, then since l is in a cycle, l points at some other element of $\mathcal{L}$. Also, since the outgoing degree of l is one, l cannot point to any element outside of $\mathcal{L}$. Thus, k is not downstream from l.

So, suppose both k and l are downstream from some vertex x, external to $\mathcal{K}$ and $\mathcal{L}$. This implies that x, or some element downstream from x, has an outgoing degree greater than 1, a contradiction. Thus, k and l cannot be downstream from x.

Therefore, there is no path from k to l.

The following result comes from [8] and is included with additional proof.

Theorem 3.3 *Let $\mathcal{C}$ be a connected component of the digraph $\Psi(R)$, then:*

1. *$\mathcal{C}$ cannot contain more than one cycle*
2. *If R is finite then $\mathcal{C}$ contains exactly one cycle*

Proof Suppose $\mathcal{C}$ contains more than one cycle. Let K and L be two such cycles in $\mathcal{C}$. Since K and L are both in $\mathcal{C}$, there exists a path from some $k \in V(K)$ to $l \in V(L)$, a contradiction of lemma 3.2. Therefore, $\mathcal{C}$ cannot contain more than one cycle.

Now, assume R is finite. Suppose $\mathcal{C}$ contains no cycle. Since R is finite, $\mathcal{C}$ has a finite number of vertices. Let, $|V(\mathcal{C})| = k$. Choose some arbitrary $(x, y) \in \mathcal{C}$. Take a directed walk of length $k + 1$ starting at (x, y), this defines a specific walk since every vertex has an outgoing degree of one. Since this walk must be internal to $\mathcal{C}$, some vertex must have been crossed twice, thus a cycle exists. Thus, every connected component must have at least one cycle.

Theorem 3.4 *Let $(x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_n, y_n)$ be a directed path of length n in $\Psi(R)$, then $x_n = x_1 + \sum_{i=1}^{n-1} y_i$ and $y_n = y_1 \prod_{i=1}^{n-1} x_i$.*

Proof The proof follows by induction: By definition $x_2 = x_1 + y_1$ and $y_2 = x_1 y_1$, thus the proposition is true for $n = 2$. We assume the proposition is true for n = k, where k is a fixed but arbitrary positive integer. i.e. $x_k = x_1 + \sum_{i=1}^{k-1} y_i$ and $y_k = y_1 \prod_{i=1}^{k-1} x_i$. We then show the proposition to be true for n = k+1. By definition, $x_k + y_k = x_{k+1}$ and, $x_k y_k = y_{k+1}$.
By the inductive hypothesis, $x_{k+1} = x_1 + y_k + \sum_{i=1}^{k-1} y_i$ and $y_{k+1} = y_1 x_k \prod_{i=1}^{k-1} x_i$. Therefore, $x_n = x_1 + \sum_{i=1}^{n-1} y_i$ and $y_n = y_1 \prod_{i=1}^{n-1} x_i$

Corollary 3.5 *Suppose $(x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_n, y_n) \to (x_1, y_1)$ is a cycle of length n in $\Psi(R)$, then $\sum_{i=1}^{n} y_i = 0$.*

Proof By Theorem 3.4, for a path of length $n + 1$, $x_{n+1} = x_1 + \sum_{i=1}^{n} y_i$. Also $(x_{n+1}, y_{n+1}) = (x_1, y_1)$ since $(x_n, y_n) \to (x_1, y_1)$. This implies that $x_1 = x_1 + \sum_{i=1}^{n} y_i$. Therefore, $\sum_{i=1}^{n} y_i = 0$.

Corollary 3.6 *Suppose* $(x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_n, y_n) \to (x_1, y_1)$ *is a cycle of length* n *in* $\Psi(D)$*, where* D *is an integral domain, then* $\prod_{i=1}^{n} x_1 = 1$.

Proof By Theorem 3.4, for a path of length $n+1$, $y_{n+1} = y_1 \prod_{i=1}^{n} x_i$. So, $(x_{n+1}, y_{n+1}) = (x_1, y_1)$ since, $(x_n, y_n) \to (x_1, y_1)$. This implies that $y_1 = y_1 \sum_{i=1}^{n} x_i$. Also, since D is an integral domain, cancellation holds. Therefore, $\prod_{i=1}^{n} x_i = 1$.

Theorem 3.7 *Let* $\Psi(R)$ *be a digraph of a ring* R*. Suppose there exists a set of elements* $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ *such that,* $(x_i, y_i) \to (x_{i+1}, y_{i+1}) \to \cdots \to (x_n, y_n) \to (a, u)$*, where* $u \in U(R)$ *and* $a \in R$ *then* $x_i, y_i \in U(R)$*, for all* i*.*

Proof By Theorem 3.4, $y_{n+1} = y_1 \prod_{i=1}^{n} x_i$. For any path such that $(x_i, y_i) \to \cdots \to (a, u)$, $u = y_i \prod_{j=i}^{n} x_j$. Then, since $u \in U(R)$, $1 = (y_i \prod_{j=i}^{n} x_j)u^{-1}$. Since y_i is fixed but arbitrary, x_i $y_i \in U(R)$, for all i.

Theorem 3.8 *Let* $\mathcal{C} \subset \Psi(R)$ *for some ring* R *where* $\mathcal{C} = (x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_n, y_n) \to (x_1, y_1)$*, such that* $y_n \in U(R)$*. Then,* $x_i, y_i \in U(R)$ *for all* i*.*

Proof By Theorem 3.4, $y_n = y_n \prod_{i=1}^{n} x_i$. Since $y_n \in U(R)$, $1 = \prod_{i=1}^{n} x_i$. Thus, $x_i \in U(R)$ for all i.

Now, by Theorem 3.4, $y_n = y_j \prod_{i=j}^{n-1} x_i$ for all $j \in \mathbb{Z}$ such that $1 \leq j \leq n-1$. Since $x_i \in U(R)$ for all i and $y_n \in U(R)$, $y_n \cdot \left(\prod_{i=j}^{n-1} x_i\right)^{-1} \in U(R)$. Thus, $y_j \in U(R)$ for all j.

4 Digraphs of Ideals

Next, we discuss ideals. In a perfect world, the subgraph of an ideal within the digraph of a ring would be a union of connected components within the digraph of a ring. This is sometimes the case, but there exists another case where the subgraph of the ideal has vertices connected to it from outside the ideal. We discuss in each case what kind of ideal is a union of connected components, and when it is not we describe the vertices connected to the subgraph of the ideal.

Lemma 4.1 *Let I be an ideal of a ring R. No vertex with one entry not from I and one entry from I connects to $\Psi(I)$.*

Proof Suppose there exists some $c, d \in R$ such that $c \in I$ and $d \notin I$ and $(c, d) \to (a, b)$ where $a, b \in I$. Then $c + d = a$ by definition of Ψ, implying $d = a - c$ since R is a ring. Since $a - c \in I$ by closure of I, $d \in I$, a contradiction. Therefore, no vertex with one entry not from I and one entry from I connects to $\Psi(I)$.

Theorem 4.2 *Let I be an ideal of a ring R. If I is prime then $\Psi(I)$ is a union of connected components of $\Psi(R)$.*

Proof Let R be a commutative ring. Let I be a prime ideal of R. Suppose there exists some $c, d \in R$ such that $c, d \notin I$ and $(c, d) \to (a, b)$ where $a, b \in I$. Then $c \cdot d = b$ by definition of Ψ. Thus $c \cdot d \in I$. Therefore, since I is prime, either $c \in I$ or $d \in I$, a contradiction. Therefore, no vertex with both entries not in I connects to $\Psi(I)$.

Now, by Lemma 4.1 no vertex with one entry not from I and one entry from I connects to $\Psi(I)$. Thus no vertex not in $I \times I$ connects to the digraph of the ideal. Also, since I is closed under addition and multiplication, vertices downstream from a vertex which is an element of $I \times I$ must also be in $I \times I$. So, $\Psi(I)$ is a union of connected components in $\Psi(R)$.

It is necessary to have a union of connected components for the digraph of a prime ideal, not simply one connected component. For each $a \in I$, $(a, 0)$ will cycle on itself, so there will be at least $|I|$ components, since no component has more than one cycle (Theorem 3.3).

Example 4.3 *Consider the ring $\mathbb{Z}_4$. Since $\mathbb{Z}_4$ is a principal ideal ring, it has one distinct ideal, namely* (2)*. The digraph $\Psi((2))$ is made up of two connected components in $\Psi(\mathbb{Z}_4)$ [See Figure 2].*

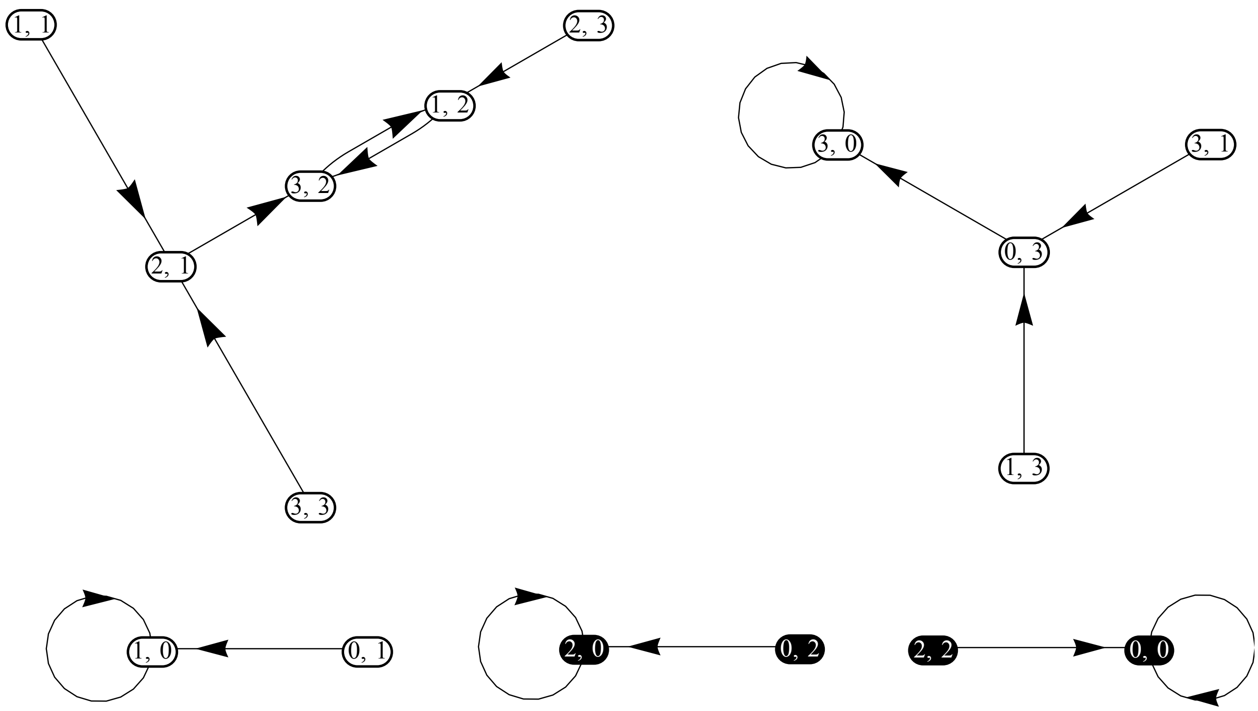

Figure 2: $\Psi(\mathbb{Z}_4)$ with $\Psi((2))$ highlighted

Corollary 4.4 *Let $Z(R)$ be the set of zero-divisors of R. If $Z(R)$ is an ideal, $\Psi(Z(R))$ is a union of connected components of $\Psi(R)$.*

Proof Let $Z(R)$ be an ideal. Let $c \cdot d \in Z(R)$ such that $c, d \notin Z(R)$. Then $(c \cdot d) \cdot x = 0$ for some $x \in R \backslash \{0\}$, implying $c \cdot (d \cdot x) = 0$ by associativity. Note that $d \cdot x \neq 0$ since $d \notin Z(R)$. Then, since $c \cdot (d \cdot x) = 0$, $c \in Z(R)$, a contradiction. Therefore, $Z(R)$ is prime, and by Theorem 4.2, $\Psi(Z(R))$ is a union of connected components.

For a non-prime ideal, K, of R, $\Psi(K)$ need not be a union of connected components in $\Psi(R)$.

Example 4.5 *Consider* (4) *in* $\mathbb{Z}_{12}$. *Unlike with prime ideals,* $\Psi((4))$ *has many vertices connected to it which are not in* $(4) \times (4)$ *[See Figure 3].*

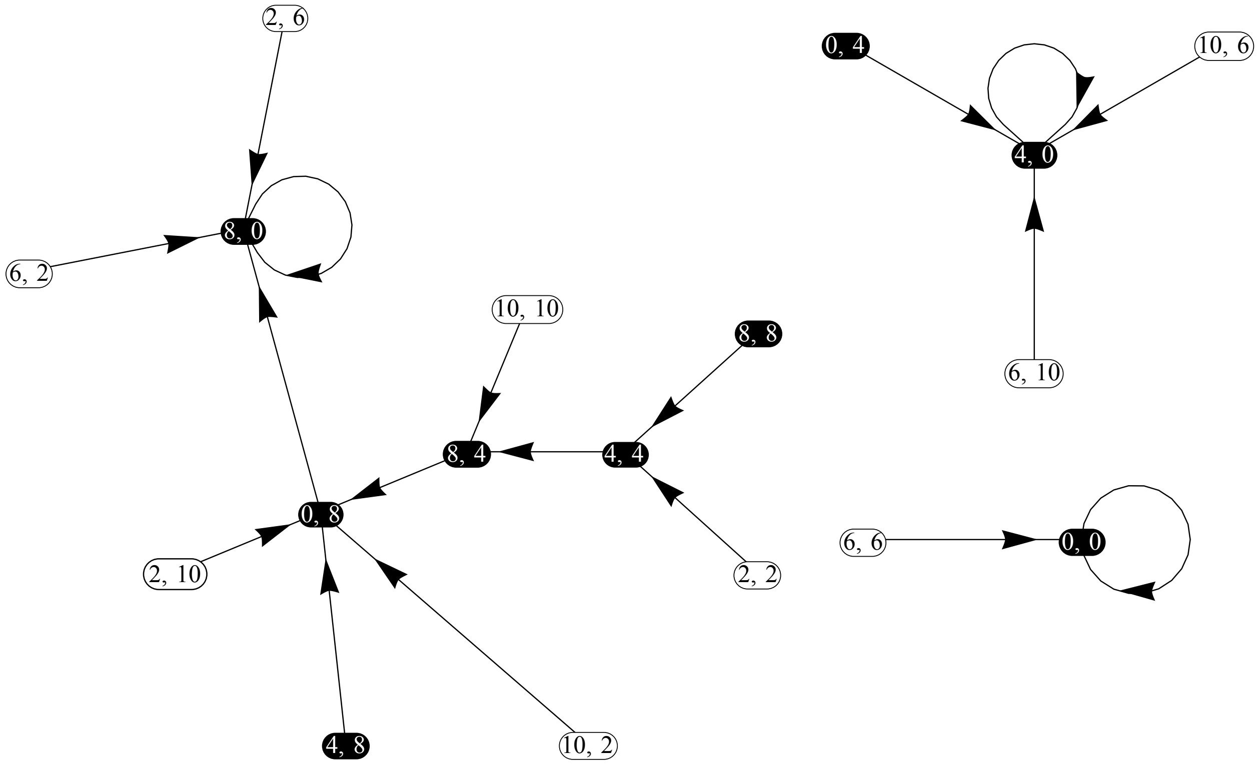

Figure 3: Section of $\Psi(\mathbb{Z}_{12})$ with $\Psi((4))$ highlighted

The following theorem presents a case where an ideal need not be prime to have a digraph that is a union of connected components of $\Psi(R)$.

Theorem 4.6 *Let $P_1, P_2, \ldots, P_n$ where P_i is a prime ideal of R and let $I = \bigcap_{i=1}^{n} P_i$. Then $\Psi(I)$ is a union of connected components.*

Proof Suppose there is a vertex $(a, b) \notin V(\Psi(I))$ connected to $\Psi(I)$. Then, since the digraph of any P_i is a union of connected components, (a, b) must be a vertex in all $\Psi(P_i)$. Therefore, a and b are in $\bigcap_{i=1}^{n} P_i$, which is I. Thus $a, b \in I$, a contradiction.

Corollary 4.7 *The* nil radical *of a ring, denoted nil(R), forms a union of connected components in $\Psi(R)$.*

Proof By definition, the nil radical is the intersection of all prime ideals of a given ring R [2]. Therefore, the result follows from Theorem 4.6.

Corollary 4.8 *The* Jacobson radical *of a ring, denoted $J(R)$, forms a union of connected components in $\Psi(R)$.*

Proof By definition, the Jacobson radical is the intersection of all maximal ideals of a given ring R [2]. Maximal ideals are prime ideals. Therefore, the result follows from Theorem 4.6.

Example 4.9 *Consider* $(6) = (2) \cap (3)$ *in* $\mathbb{Z}_{18}$*. The ideal is non-prime, is contained in* (2) *and* (3)*, which are both prime, and is a union of connected components [See Figure 4].*

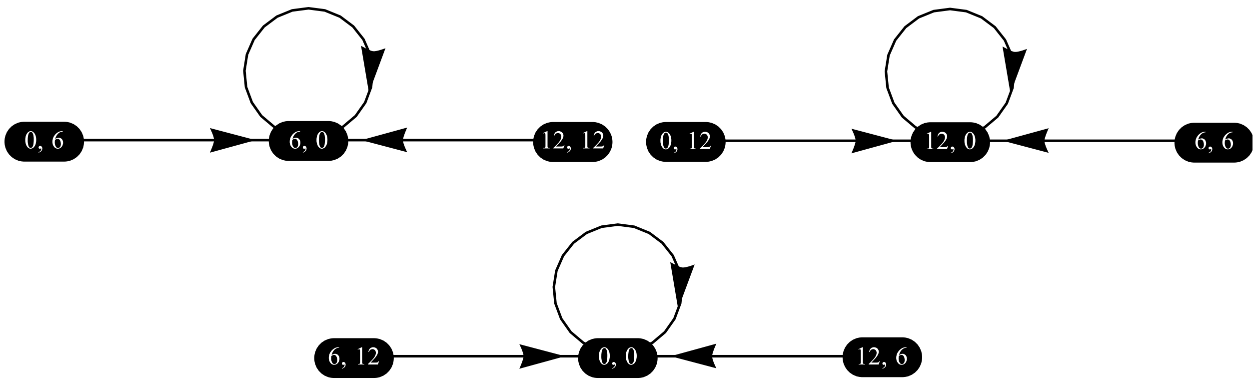

Figure 4: $\Psi((6))$ from $\Psi(\mathbb{Z}_{18})$

Unfortunately, the condition for the ideals to be prime in Theorem 4.6 is necessary.

Example 4.10 *Consider* (12) *in* $\mathbb{Z}_{24}$*. The ideal is contained within two prime ideals, namely* (2) *and* (3)*, but is not equal to their intersection. The ideal is also contained in* (2) *and* (6)*, and is equal to their intersection, but* (6) *is not prime. As such, one would rightfully expect* $\Psi((12))$ *to not be a union of connected components [See Figure 5].*

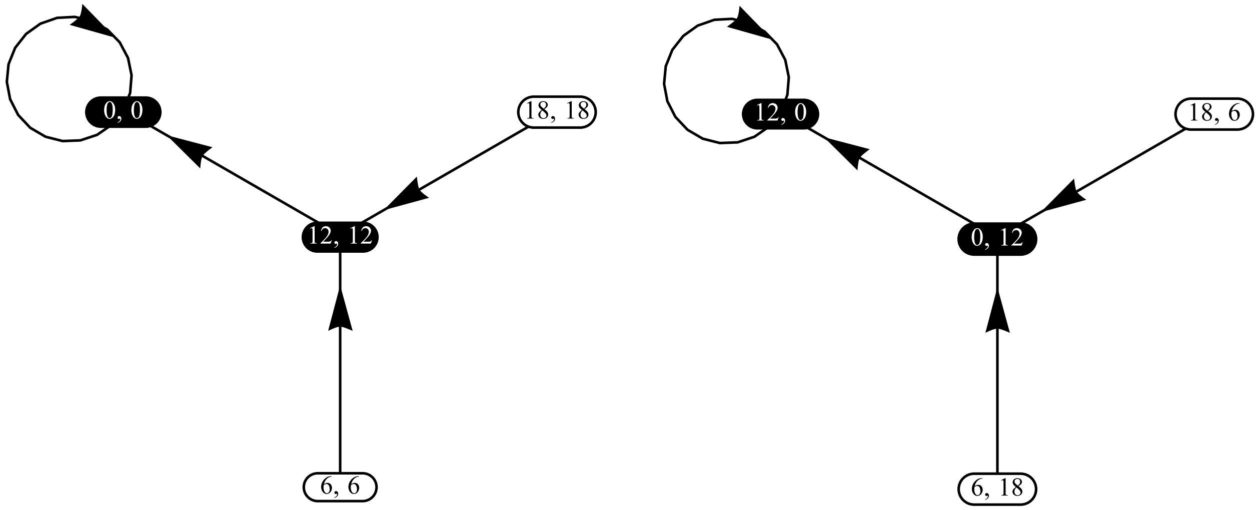

Figure 5: Section of $\Psi(\mathbb{Z}_{24})$ containing $\Psi((12))$

Lemma 4.11 *Let a and b be nilpotent elements of a ring R. Then $(a,b) \to \cdots \to (c,0)$ for some $c \in nil(R)$.*

Proof Clearly, since $nil(R)$ is closed under addition and multiplication $(a,b) \to \cdots \to (c,x)$ where $c \in nil(R)$. Since a and b are nilpotent, $a^n = b^m = 0$. Taking (a,b) downstream $n+m$ steps $(a,b) \to \cdots \to (c, a^{x_1}b^{y_1} + a^{x_2}b^{x_2} + \ldots)$ where $x_i + y_i = n+m+1$. We must show that each summand is equal to zero. If $x_i \geq n$, then $a^{x_i} = 0$ and, hence, $a^{x_i}b^{y_i} = 0$. If $x_i < n$ then $y_i > m+1$. Hence, $b^{y_i} = 0$, implying that $a^{x_i}b^{y_i} = 0$. Therefore all summands $a^{x_1}b^{y_1} + a^{x_2}b^{x_2} + \ldots$ are zero, and thus $a^{x_1}b^{y_1} + a^{x_2}b^{x_2} + \ldots = 0$. Therefore $(a,b) \to \cdots \to (c,0)$

Theorem 4.12 *Let R be a finite ring and let $\mathcal{C}(nil(R))$ be the set of connected components of $\Psi(nil(R))$. Then $|\mathcal{C}(nil(R)| = |nil(R)|$.*

Proof Since $(x,0)$ forms a cycle for every $x \in nil(R)$, and all cycles of length one are of this form (since if the vertex (a,b) forms a one-cycle, then $a+b=a$, which implies that $b=0$) there are at least $|nil(R)|$ components in $\Psi(nil(R))$. Also, since R is finite, there is no connected component with no cycle by Theorem 3.3. It suffices to show that $\Psi(nil(R))$ contains no cycle of length greater than one. Suppose that $\Psi(nil(R))$ contains a cycle of length $n > 1$. Let $C = (a_1,b_1) \to \cdots \to (a_n,b_n)$ be said cycle. Without loss of generality, consider (a_1,b_1). By Lemma 4.11, $(a_1,b_1) \to \cdots \to (c_1,0)$ where $c_1 \in nil(R)$. Since $(c_1,0)$ cycles back to itself, C contains two cycles, which is a contradiction of Theorem 3.3. Therefore, there is no cycle of length greater than one in $\Psi(nil(R))$.

For ideals which are contained in prime ideals, the following result holds.

Theorem 4.13 *If I is a non-prime ideal contained in the prime ideals $P_1, P_2, \ldots, P_n$, then I is only connected to by vertices with entries from the set $\bigcap_{i=1}^{n} P_i \backslash I$*

Proof Since the radical, $Rad(I)$, is equal to the intersection of all prime ideals containing I, $\bigcap_{i=1}^{n} P_i \backslash I = Rad(I) \backslash I$. By Theorem 4.6, $\Psi(Rad(I))$ is a union of connected components. Clearly, $I \subset Rad(I)$. Thus, $\Psi(I) \subset \Psi(Rad(I))$. Therefore all connections to $\Psi(I)$ are vertices with entries from $Rad(I)$. Suppose a vertex (a, p) connects to $\Psi(I)$ where $a \in I$ and $p \in Rad(I) \backslash I$. Then $a + p = \bar{a}$ for some $\bar{a} \in I$. By subtraction in R, we achieve $p = \bar{a} - a$, thus $p \in I$, a contradiction. Therefore all connections to $\Psi(I)$ are vertices with entries from the set $Rad(I) \backslash I$.

Example 4.14 *Consider* (4) *an ideal of* $\mathbb{Z}_8$*. The ideal* (4) *is contained in the ideal* (2)*, which is prime. The subgraph* $\Psi((4))$ *is connected to by vertices with entries* 2 *and* 6*, which are in* (2) *and not in* (4) *[See Figure 6].*

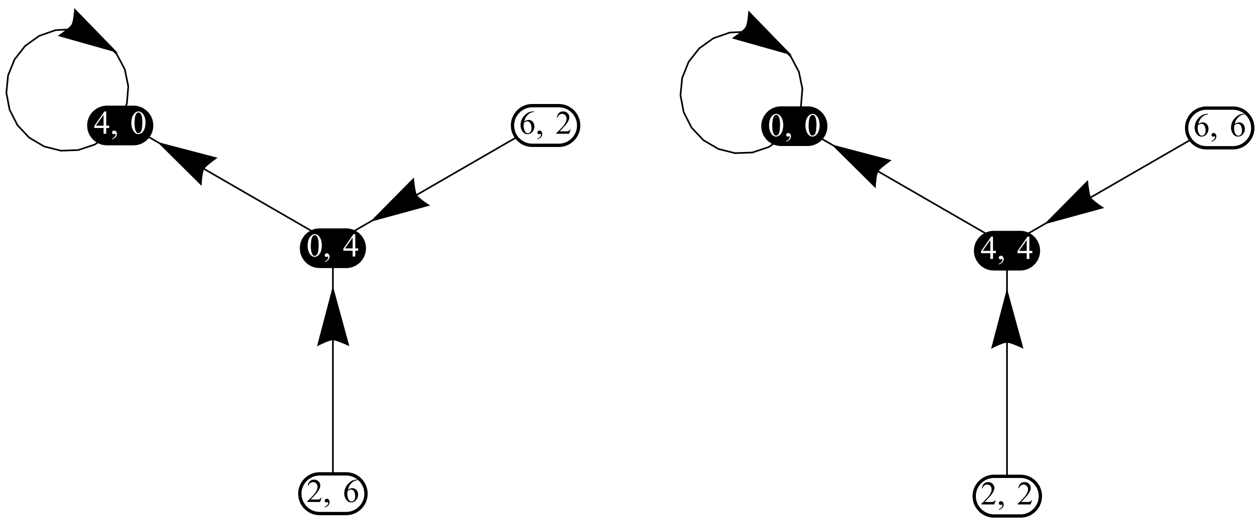

Figure 6: Section of $\Psi(\mathbb{Z}_8)$ containing $\Psi((4))$

It is important to note that the digraph of an ideal of a ring does not have vertices with a unit from R as an entry. This follows from the fact that we are studying proper ideals only. For the next result, we need a definition.

Definition A ring R is called Noetherian if for every chain $I_1 \subset I_2 \subset I_3 ...$ of ideals in R, there is an integer n such that $I_i = I_n$ for all $i \geq n$.

Theorem 4.15 *Let* $\Psi(R)$ *be the digraph of the Noetherian ring* R*. A minimal union of connected components of* $\Psi(R)$ *containing the digraph of an ideal of* R *has no vertices of the form* (u, x) *or* (x, u) *for any* $u \in U(R)$*.*

Proof Let I be an ideal of R. If I is prime, then $\Psi(I)$ is a union of connected components of $\Psi(R)$. Therefore, no vertices of $\Psi(I)$ have entry u. If I is not prime, it is contained in a maximal ideal, call it J, since R is Noetherian. This implies $\Psi(I)$ is contained within $\Psi(J)$, which is a union of connected components, since maximal ideals are prime. Therefore, since $\Psi(J)$ has no vertices with entry u and since all connections to $\Psi(I)$ are vertices from $\Psi(J)$, no vertices connected to $\Psi(I)$ have entry u.

5 Zero-Divisors

Next, we discuss the topic of zero-divisors within the digraph of a ring. We establish how to find all zero-divisors within the digraph and give a method for determining whether a digraph is the graph of a domain. To begin, we define annihilating pairs.

Definition If $a \cdot b = 0$ where $a, b \neq 0$ then a and b are an *annihilating pair*. Such elements are also elements of $Z(R)$. In fact, the set of all elements which occur in annihilating pairs is the set $Z(R)\backslash\{0\}$.

Proposition 5.1 *Let $x, y \in R$. Then $(x, y) \to (x + y, 0)$ and $x, y \neq 0$ if and only if x and y are an annihilating pair.*

Proof ($\Rightarrow$) Let $(x, y) \to (x + y, 0)$ and $x, y \neq 0$. Then $x \cdot y = 0$. This implies that x and y are an annihilating pair.

($\Leftarrow$) Let $x, y \in R$ be an annihilating pair. This implies that $(x, y) \to (x + y, 0)$, and by definition $x, y \neq 0$.

The above proposition implies that every $x \in Z(R)\backslash\{0\}$ appears as a component in a vertex (x, y) such that $(x, y) \to (x + y, 0)$. Thus, the above proposition allows one to find all of the nonzero zero-divisors of a ring simply by locating all of the elements $(x, y) \in \Psi(R)$ where $x, y \neq 0$, and $(x, y) \to (a, 0)$. This may not look too simple at first, but remember that $(a, 0)$ is the only element that points to itself.

In addition, one is able to locate all the annihilating pairs in the ring simply by looking at what points at $(a, 0)$ for all $a \in R$.

Theorem 5.2 *Let $\Psi(R)$ be the digraph of some Noetherian ring R, and let $(a, 0) \in \Psi(R)$ such that $a \notin U(R)$. If $(x, y) \to \cdots \to (a, 0)$ then $x, y \notin U(R)$.*

Proof A maximal ideal cannot have any units in it. By Theorem 4.2, since a maximal ideal is prime its digraph is the union of connected components. Since $a \notin U(R)$ and R is Noetherian, a is contained in at least one maximal ideal. This means that the connected component containing $(a, 0)$ is part of the digraph of a maximal ideal. Thus, no unit can be connected to $(a, 0)$. Therefore, if $(x, y) \to \cdots \to (a, 0)$ then $x, y \notin U(R)$.

For finite rings, the last result implies that any vertices upstream from some $(a, 0)$ where $a \in Z(R)$ have entries from $Z(R)$. This comes from the fact that non-units in a finite ring are zero-divisors [5].

Proposition 5.3 *Let $\mathcal{C}$ be a cycle in the digraph of a ring R. Let $\mathcal{C} = (a_1, b_1) \to \cdots \to (a_n, b_n) \to (a_1, b_1)$. If $(a_i, b_i) \in \mathcal{C}$ is a vertex such that $b_i \in Z(R)$, then $b_j \in Z(R)$ for all b_j in the cycle.*

Proof For any $a, b \in R$, if a or b is a zero-divisor, then their product is a zero-divisor. Hence, since all b_i are a product of a zero-divisor, all b_i are zero-divisors.

Corollary 5.4 *Let R be a ring with a cycle $\mathcal{C}$. If R is local, then:*

1. *If (a_i, b_i) a vertex in $\mathcal{C}$ is of the form where $a_i, b_i \in Z(R)$, then all vertices in $\mathcal{C}$ have zero-divisors as entries.*
2. *If (a_i, b_i) a vertex in $\mathcal{C}$ is of the form where $a_i \notin Z(R)$ and $b_i \in Z(R)$, then all vertices in $\mathcal{C}$ have a non-zero-divisor in the first entry and a zero divisor in the second.*

Proof Suppose $a_i \in Z(R)$. In a local ring, $Z(R)$ is an ideal. Hence $Z(R)$ is closed under addition. Therefore, for every vertex downstream from (a_i, b_i), the first entry is a zero-divisor. Also by Proposition 5.3 , the second entries are also zero-divisors.

Now suppose $a_i \notin Z(R)$. Suppose by way of contradiction that $a_{i+1} \in Z(R)$. Then since $a_{i+1}, b_{i+i} \in Z(R)$, all vertices downstream from (a_i, b_i) have zero-divisor entries by above. But since $\mathcal{C}$ is a cycle, (a_i, b_i) is downstream from itself, so $a_i \in Z(R)$, a contradiction. Therefore, if $a_i \notin Z(R)$, then the first entry in all vertices are not in $Z(R)$. Also by Proposition 5.3, the second entries of all vertices in the cycle are zero-divisors.

Just like the last result, in the finite case, this corollary implies that a unit entry in the first position in the cycle implies units in all the first entries.

Theorem 5.5 *Let $\Psi(R)$ be the unlabeled digraph of some ring R, then R is an integral domain if and only if:*

1. *There is exactly one connected component consisting of exactly one vertex.*
2. *All other one-cycles have an incoming degree of two.*

Proof ($\Rightarrow$) Suppose R is an integral domain. By Proposition 5.1 and since R has no zero-divisors, no (a, b) is directly upstream from a vertex $(c, 0)$ for all $a, b, c \in R$ and $a \neq 0$. Since all one cycles are of the form $(a, 0)$ for all $a \in R$, this implies that one cycles have no vertices directly upstream except $(0, a)$, making the incoming degree of all one cycles two. In the case of $(0, 0)$, $(0, a) = (a, 0)$, making the incoming degree of $(0, 0)$ one. So, there is one vertex, namely $(0, 0)$, which is itself a connected component, and all other one cycles ($(a, 0)$) have incoming degree of two.

($\Leftarrow$) Suppose the digraph of R has one connected component with one vertex and all other one-cycles have incoming degree two.

Let $(x, y) \to (x, y)$ be a connected component with one vertex. Then $x + y = x$, which implies $y = 0$. If $x \neq 0$, then $(0, x) \to (x, 0)$, a contradiction of our component having only one vertex. Thus, $x = 0$. Thus, a connected component consisting of one vertex must be the vertex $(0, 0)$.

Now, assume R is not an integral domain. Then there exists $c, d \in R \backslash \{0\}$ such that $c \cdot d = 0$. Thus, $(c, d) \to (c + d, 0)$. We know $(c + d, 0)$ forms a one-cycle. If $c + d = 0$ then we have contradicted the previous paragraph. Thus, $c + d \neq 0$. Thus, $(c + d, 0)$ has incoming degree of more than 2 since $(c, d) \to (c+d, 0)$, $(c+d, 0) \to (c+d, 0)$, and $(0, c+d) \to (c+d, 0)$, a contradiction. Hence, R is an integral domain.

6 Isomorphisms

Another important and oft studied concept in ring theory is the ring isomorphism. One would naively expect that if two rings have the same digraph that those rings are isomorphic and vice versa. Unfortunately, this is not the case, for which we include pertinent counterexamples. As a reminder, we begin with the definition of a ring isomorphism.

Definition [4] A *ring isomorphism* is a mapping ϕ from a ring R to a ring S that has the following properties:

1. $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$

2. ϕ is one-to-one

3. ϕ is onto

Definition Let $\Psi(R)$ and $\Psi(S)$ be the digraphs of rings R and S, then $\Psi(R)$ and $\Psi(S)$ are said to be *isomorphic*, denoted $\Psi(R) \simeq \Psi(S)$, if there exists a bijection $f : V(\Psi(R)) \to V(\Psi(S))$ such that any two vertices $u, v \in V(\Psi(R))$ are adjacent if and only if $f(u), f(v) \in V(\Psi(S))$ are adjacent, [3].

$\Psi(R)$ and $\Psi(S)$ are said to be *directly isomorphic*, denoted $\Psi(R) \succapprox \Psi(S)$, if $\Psi(R)$ and $\Psi(S)$ are isomorphic, and when $a, b \in V(\Psi(R))$ and $a \to b$ then $f(a) \to f(b)$.

Since a digraph may have multiple copies of a connected component [See Figure 7], there are multiple possible isomorphisms from one digraph to another. Because of this, there is not a clear way of using a graph isomorphism to induce an isomorphism between rings.

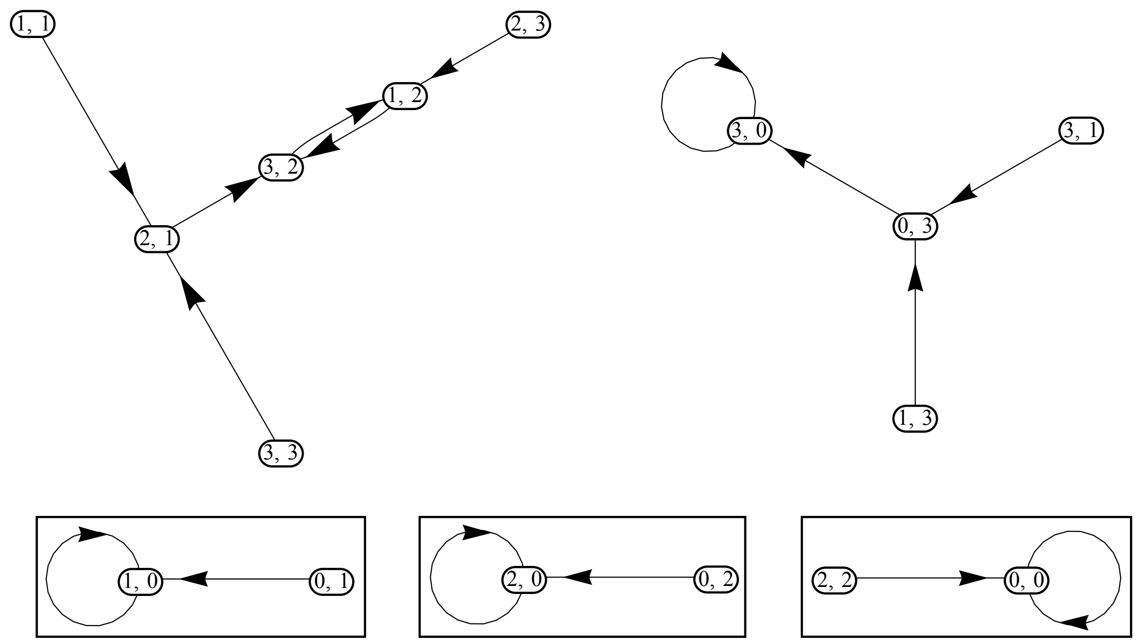

Figure 7: $\Psi(\mathbb{Z}_4)$

Theorem 6.1 *Let $\Psi(R)$ and $\Psi(S)$ be the digraphs of two rings R and S. If $R \cong S$, then $\Psi(R) \succsim \Psi(S)$.*

Proof Suppose $f : R \to S$ is a ring isomorphism, then for all $x, y \in R$, $f(x \cdot y) = f(x) \cdot f(y)$ and $f(x + y) = f(x) + f(y)$. This implies that if $(x, y) \to (a, b)$, then $(f(x), f(y)) \to (f(x) + f(y), f(x) \cdot f(y)) = (f(x + y), f(x \cdot y)) = (f(a), f(b))$. Thus, $\Psi(R) \succsim \Psi(S)$.

Unfortunately, the converse is false, as illustrated in the following examples.

Example 6.2 *Consider the two rings, $\mathbb{Z}_4$ and $\mathbb{Z}_2[i]$. As rings, they are certainly not isomorphic. The additive structure of $\mathbb{Z}_4$ is cyclic, while the additive structure of $\mathbb{Z}_2[i]$ is not. The multiplicative structures of the rings are the same, as illustrated by the Cayley tables shown below.*

$\mathbb{Z}_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\mathbb{Z}_2[i]$	0	1	$i+1$	i
0	0	0	0	0
1	0	1	$i+1$	i
$i+1$	0	$i+1$	0	$i+1$
i	0	i	$i+1$	1

The digraphs of these two rings are isomorphic, however [See Figure 8].

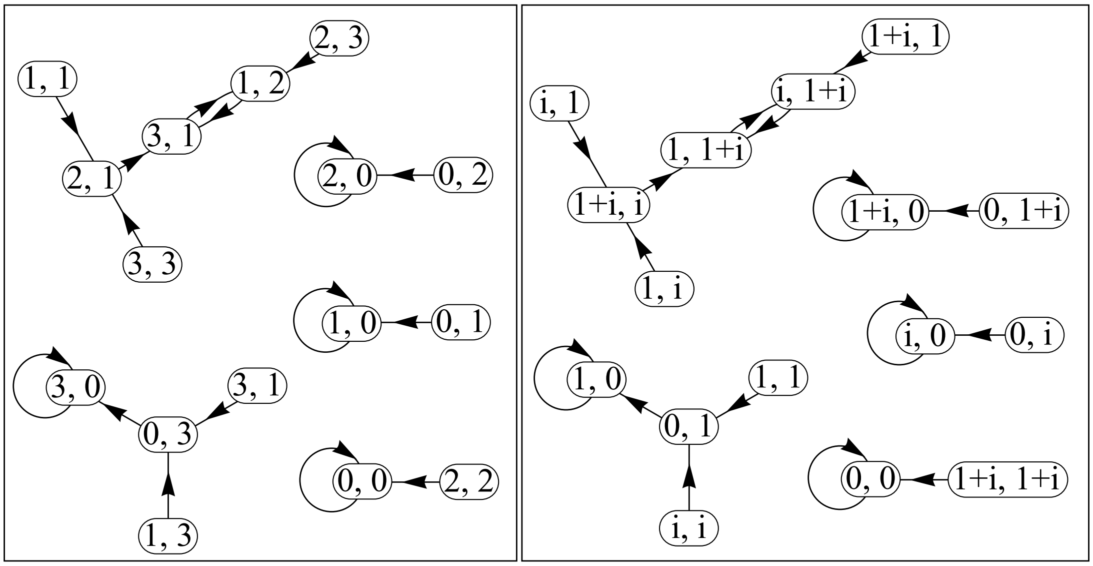

Figure 8: $\Psi(\mathbb{Z}_4)$ on the left and $\Psi(\mathbb{Z}_2[i])$ on the right

Example 6.3 *Consider the two rings of matrices,*
$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} \right\}$ *and*
$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}$, *with addition and multiplication modulo four. Clearly, these two rings are not isomorphic, since R has an additive structure generated by $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, whereas the additive structure of S is not cyclic. The multiplicative structure of both rings is given by the zero-product, so as in the previous example, two rings with different additive structure and the same multiplicative structure have isomorphic digraphs [See Figure 9] $\left(\text{for } \Psi(R),\ a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and for } \Psi(S),\ a = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right)$.*

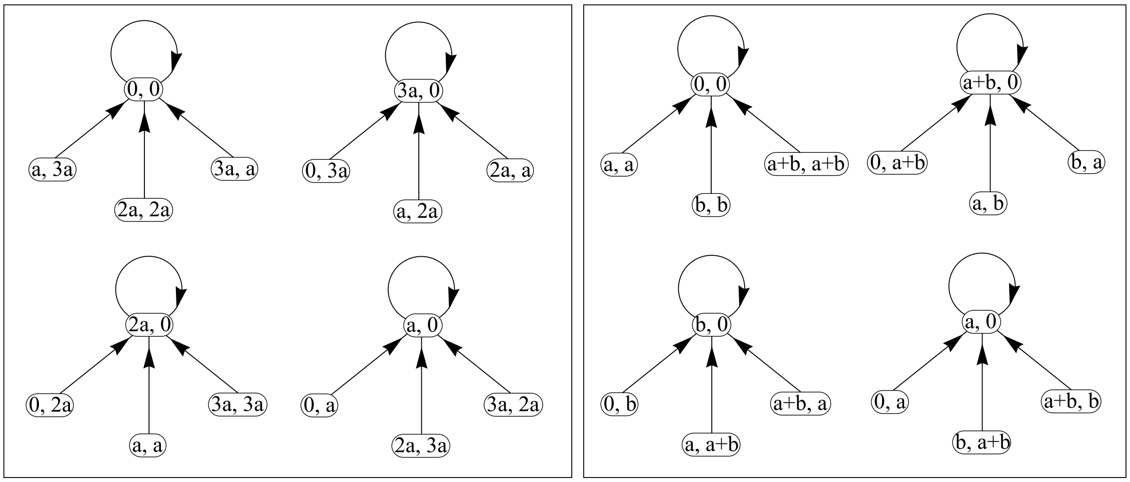

Figure 9: $\Psi(R)$ on the right and $\Psi(S)$ on the left

7 Direct Products

It is a very natural question to ask what happens when one makes the digraph of a direct product. In this section we explore the nature of the digraph of a direct product and its connection to the digraph of its factors. As a reminder, we begin with the definition of a direct product.

Definition [4] Let R_1,R_2,...,R_n be a finite collection of rings. The *direct product* of R_1,...,R_n, denoted $R_1 \times R_2 \times \ldots \times R_n$ is the set of all n-tuples for which the ith component is an element of R_i and the operations are componentwise.

Definition Let $\Psi(R \times S)$ be the digraph of the direct product of rings R and S. The subgraph $\Psi(R \times \{0\})$ [$\Psi(\{0\} \times S)$ resp.] is called the *canonical subgraph of R [S resp.] in* $\Psi(R \times S)$ and is denoted $\Psi'(R)$ [$\Psi'(S)$ resp.].

Theorem 7.1 *Let $\Psi(R \times S)$ be the digraph of the finite ring $R \times S$, and let $\mathcal{C}$ be the union of all connected components whose intersection with $\Psi'(S)$ is non-empty. Then, $\mathcal{C} \simeq \Psi(S)$ if and only if for all $v \in V(\Psi(R))$ and $v \neq (0,0)$, $v \nrightarrow (0,0)$.*

Proof ($\Rightarrow$) Suppose $\mathcal{C} \simeq \Psi(S)$. This implies the $|\mathcal{C}| = |\Psi(S)|$ and since $|\Psi'(S)| = |\Psi(S)|$, $|\Psi'(S)| = |\mathcal{C}|$, which implies $\Psi'(S) = \mathcal{C}$. This means that there are no vertices outside of $\Psi'(S)$ connected to $\Psi'(S)$.

Every element of $\Psi'(S)$ looks like $((0,a),(0,b))$ where $a, b \in S$. A vertex external to $\Psi'(S)$, say $((x_R, x_S),(y_R, y_S))$ can only point at a vertex of $\Psi'(S)$ if $x_R + y_R = 0$ and $x_R \cdot y_R = 0$. This means that $(x_R, y_R) \to (0,0)$. From above there are no vertices outside of $\Psi'(S)$, which implies no vertices point to $(0,0)$.

($\Leftarrow$) Now suppose that there does not exist a $(x, y) \in V(\Psi(R))$, other than $(0, 0)$, such that $(x, y) \to (0, 0)$. This implies that $x + y \neq 0$ or $x \cdot y \neq 0$ for all $x, y \in R \backslash \{0\}$. Since every element of $\Psi'(S)$ looks like $((0, a), (0, b))$, this implies that there cannot be a $((x_R, x_S), (y_R, y_S)) \in \Psi(R \times S)$ who is external to $\Psi'(S)$ such that $((x_R, x_S), (y_R, y_S)) \to ((0, a), (0, b))$.

For the next results, we need another theorem from graph theory.

Definition A *source* in $\Psi(R)$ is a vertex with incoming degree zero. The set of all sources is denoted $\mathcal{S}(\Psi(R))$.

Theorem 7.2 *Let $\Psi(R \times S)$ be the digraph of the ring $R \times S$, then $((x, a), (y, b)) \in \mathcal{S}(\Psi(R \times S))$ if and only if $(x, y) \in \mathcal{S}(\Psi(R))$ or $(a, b) \in \mathcal{S}(\Psi(S))$.*

Proof ($\Rightarrow$) Let $((x, a), (y, b)) \in \mathcal{S}(\Psi(R \times S))$ and suppose $(x, y) \notin \mathcal{S}(\Psi(R))$ and $(a, b) \notin \mathcal{S}(\Psi(S))$. This implies that there exists $(x', y') \in \Psi(R)$ and $(a', b') \in \Psi(S)$ such that $(x', y') \to (x, y)$ and $(a', b') \to (a, b)$ (respectively). Then, $x' + y' = x$, $x' \cdot y' = y$, $a' + b' = a$ and $a' \cdot b' = b$. By the definition of $R \times S$, $(x', a'), (y', b') \in R \times S$ and $(x', a') + (y', b') = (x' + y', a' + b') = (x, a)$, $(x', a') \cdot (y', b') = (x' \cdot y', a' \cdot b') = (y, b)$. This implies that $((x', a'), (y', b')) \to ((x, a), (y, b))$, which implies that $((x, a), (y, b)) \notin \mathcal{S}(R \times S)$. Thus, by contradiction $(x, y) \in \mathcal{S}(\Psi(R))$ or $(a, b) \in \mathcal{S}(\Psi(S))$.

($\Leftarrow$) Without loss of generality, suppose $(x, y) \in \mathcal{S}(\Psi(R))$ and suppose $((x, a), (y, b)) \notin \mathcal{S}(\Psi(R \times S))$. This implies that there exists $((x', a'), (y', b')) \in \Psi(R \times S)$ such that $((x', a'), (y', b')) \to ((x, a), (y, b))$. By the definition of $R \times S$, then, $x', y', x, y \in R$ and $a', b', a, b \in S$. Since $((x', a'), (y', b')) \to ((x, a), (y, b))$, $(x', a') + (y', b') = (x' + y', a' + b') = (x, a)$ and $(x', a') \cdot (y', b') = (x' \cdot y', a' \cdot b') = (y, b)$. This implies that $x' + y' = x$, and $x' \cdot y' = y$. Which implies that $(x', y') \to (x, y)$. So, $(x, y) \notin \mathcal{S}(\Psi(R))$. Thus, by contradiction, $((x, a), (y, b)) \in \mathcal{S}(\Psi(R \times S))$.

Therefore, $((x, a), (y, b)) \in \mathcal{S}(\Psi(R \times S))$ if and only if $(x, y) \in \mathcal{S}(\Psi(R))$ or $(a, b) \in \mathcal{S}(\Psi(S))$.

Corollary 7.3 *If R and S are finite rings then $|\mathcal{S}(\Psi(R \times S))| = |\mathcal{S}(\Psi(R))| \cdot |S|^2 + |\mathcal{S}(\Psi(S))| \cdot |R|^2 - |\mathcal{S}(\Psi(R))| \cdot |\mathcal{S}(\Psi(S))|$.*

Proof By Theorem 7.2, for every element $(x, y) \in \mathcal{S}(\Psi(R))$, $((x, a), (y, b) \in \mathcal{S}(\Psi(R \times S))$ for every $a, b \in S$. The same is true for all sources in S. Also by Theorem 7.2, these are the only sources in $\Psi(R \times S)$. This accounts for $|\mathcal{S}(\Psi(R))| \cdot |S|^2 + |\mathcal{S}(\Psi(S))| \cdot |R|^2$ sources in $\mathcal{S}(\Psi(R \times S))$ with redundancy. By the inclusion/exclusion principle, the above quantity has double counted all sources of the form $((x, a), (y, b))$ where $(x, y) \in \mathcal{S}(\Psi(R))$ and $(a, b) \in \mathcal{S}(\Psi(S))$. Therefore, $|\mathcal{S}(\Psi(R \times S))| = |\mathcal{S}(\Psi(R))| \cdot |S|^2 + |\mathcal{S}(\Psi(S))| \cdot |R|^2 - |\mathcal{S}(\Psi(R))| \cdot |\mathcal{S}(\Psi(S))|$.

8 Future Work

The following are conjectures which have yet to be proven regarding digraphs of rings. Included are any necessary definitions.

Definition An ideal I of a ring R is said to be *primary* if whenever $ab \in I$, either $a \in I$ or $b^n \in I$, $n \in \mathbb{N}$.

Definition Let Q be a primary ideal of the ring R. The *degree of primality* of Q is the length of the minimal path from any prime ideal to Q.

Since the $Rad(Q)$ is the smallest prime ideal containing Q, this definition amounts to saying the degree of primality is the length of the shortest path from $Rad(Q)$ to Q.

Definition Let Q be a primary ideal of the ring R. The *degree* of Q is the least power n for which every element of $Rad(Q)$ is in Q. (i.e. $Rad(Q)^n = \{r^n | r \in Rad(Q)\} \subseteq Q$).

It is curious that both of the previous definitions involve the radical of a primary ideal. A potential connection exists between these two concepts.

Conjecture 1 *Let Q be a primary ideal. If the degree of primality of Q is n and the degree of Q is m, then $n + 1 = m$.*

We also have a conjecture related to isomorphisms based on the counterexamples that we have found[See Example 6.2 and 6.3]. We have not found a counterexample to the following conjecture.

Conjecture 2 *If $\Psi(R) \gtrsim \Psi(S)$ then the additive structure of R and S are the same.*

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