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PIGEON-HOLING MONODROMY GROUPS

NILES G. JOHNSON

ABSTRACT. A simple tiling on a sphere can be used to construct a tiling on a d-fold branched cover of the sphere. By lifting a so-called equatorial tiling on the sphere, the lifted tiling is locally kaleidoscopic, yielding an attractive tiling on the surface. This construction is via a correspondence between loops around vertices on the sphere and paths across tiles on the cover. The branched cover and lifted tiling give rise to an associated monodromy group in the symmetric group on d symbols. This monodromy group provides a beautiful connection between the cover and its base space. Our investigation of will focus on consideration of all possible low genus branched covers for a sphere, and therefore all locally kaleidoscopic tilings of low genus surfaces. It will be carried out through the classification of their associated monodromy groups. To this end, the relationship between classifications of branched covers and classifications of monodromy groups will be stressed.

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1. INTRODUCTION

1.1. **Covering spaces.** The idea of a covering space is that one topological space 'folds up nicely' onto another. To speak of a covering space, then, involves two spaces and a surjective map from one to the other. To say that the map is 'nice' means that in addition to being surjective, it is locally a homeomorphism for every point in its domain (the covering space). To be precise:

Definition. A map p from a space S to a space X is a **homeomorphism** if

(1) p is 1-1
(2) p is onto
(3) p is continuous
(4) p^{-1} is continuous

Definition. A space S is a **cover** of a space X if there exists a continuous and surjective map

$$p : S \to X$$

such that for all $x \in X$, there is an open set, V, containing x such that $p^{-1}(V)$ is a disjoint union:

$$p^{-1}(V) = \coprod_{i=1}^{n} U_i,$$

and p is a homeomorphism of U_i onto V for each i.

In this case, we say that S covers X and that p is the covering map. More specifically, for each point $x \in X$, we speak of its pre-image, $p^{-1}(x) \subset S$, as the *fiber* of x and say that V is *evenly covered* by $p^{-1}(V)$. Our interests will be focused on path-connected covers. For such spaces each sufficiently small open set in the covered space has the same number of disjoint sets in its pre-image [3]. The cardinality of the fiber of x is then the same for all $x \in X$, and is called the *degree* of the cover S over X.

An important example of a covering space is the covering of a circle by the real line. One can imagine coiling the real line around a circle of radius $\frac{1}{2\pi}$ (whose circumference will be 1) so that two points on the line will be identified if and only if the distance between them on the line (their difference) is exactly an integer. Explicitly, the covering map is:

$$p : \mathbb{R} \to X = \{x \in \mathbb{C} : |x| = \frac{1}{2\pi}\}$$

$$\text{where } p(t) = \frac{1}{2\pi} e^{2\pi i t} = \frac{1}{2\pi}(\cos 2\pi t + i \sin 2\pi t).$$

We notice that p is both continuous and onto, and, for example, given any $x \in X$, the open set

$$V_x = \{v \in X : |x - v| < \frac{1}{2\pi}\}$$

is evenly covered by a disjoint union of homeomorphic images in $\mathbb{R}$. Thus, the map p is indeed a covering map. The degree of this cover is the cardinality of the integers, since $\mathbb{R}$ is path-connected and the fiber of each point in X is isomorphic to $\mathbb{Z}$.

One may note, however, that not all spaces cover our circle in this 'even' way. A good example of this is to consider a 'figure eight' shape. The figure eight consists of two circles meeting at a point, and certainly each circle will cover our circle, but the point they share causes trouble. To see why, let us suppose that there is a cover of the circle, X, by the figure eight, $F8$, and call the shared point f_0. Let $z \in X$ denote $p(f_0)$, where p is the covering map in question. Now as a point of X, z has a fiber in $F8$. Unfortunately, every neighborhood of f_0 is path-connected. Even worse, however, is the realization that no neighborhood of f_0 is homeomorphic to a circle or to a line, but that all the connected neighborhoods of z are homeomorphic to one of them. Any potential covering map breaks down at f_0. The reason for this is clear; the circles that make up $F8$ need to be disjoint for a cover to 'work', but they are not.

1.2. **Branched covers.** Though such a situation does look bleak, it is still of some interest. Such 'almost-coverings' are called branched covers, and will be our main focus. From here on we will apply our study almost exclusively to Riemann surfaces, defined as follows:

Definition. A **Riemann surface** X is a topological space in which each point has a neighborhood which maps onto the open unit disk in the complex plane via an analytic homeomorphism. More precisely, for each $x \in X$ there is a coordinate homeomorphism $\phi_x : U_x \to V_x$ from an open neighborhood U_x of x to an open set V_x in $\mathbb{C}$ such that compositions, $\phi_x \circ \phi_y^{-1} : \phi_y(U_x \cap U_y) \to \phi_x(U_x \cap U_y)$ are analytic, i.e., are infinitely differentiable with respect to a complex variable in $\mathbb{C}$.

When one Riemann surface analytically covers another, $p : \widetilde{X} \to X$, then at each point $\tilde{x} \in \widetilde{X}$ we require that the covering map induce an analytic map from an open disk centered at 0 to a like disk, fixing 0, as follows. We may assume that the coordinate homeomorphisms $\psi_{\tilde{x}}, \tilde{x} \in \widetilde{X}$ and $\phi_x, x \in X$ satisfy $\psi_{\tilde{x}}(\tilde{x}) = 0$ and $\phi_x(x) = 0$. Then, setting $x = p(\tilde{x})$, the map $\phi_x \circ p \circ \psi_{\tilde{x}}^{-1} : V_{\tilde{x}} \to V_x$ maps 0 to 0. We want this map to be analytic when restricted to a small disc about 0. The situation is pictured in the diagram below.

$$\begin{array}{ccc} U_{\tilde{x}} & \xrightarrow{\;\;p\;\;} & U_x \\ \psi_{\tilde{x}}\big\downarrow & & \phi_x\big\downarrow \\ V_{\tilde{x}} & \xrightarrow{\phi_x \circ p \circ \psi_{\tilde{x}}^{-1}} & V_x \end{array}$$

A critical property of this induced map, $\phi_x \circ p \circ \psi_{\tilde{x}}^{-1}$, is its automatic analyticity. It allows a power series expansion of the map and a local approximation of the map as a power map

$$z \mapsto z^e, \text{ where } e \text{ is a positive integer,}$$

(see [2]). When this integer is greater than one, we have a situation similar to that of the figure eight covering the circle. At almost every point the induced map is a degree e cover of the disk by the disk, with the sole exception being the origin. There is only one point in the fiber of the origin, whereas there are e points in the fiber of every other point on the disk.

Definitions. The inverse image of the origin of the covered disk lies on the covering Riemann surface and is called a **ramification point**. The origin of the covered disk lies on the covered Riemann surface and is called a **branch point**.

Definition. A Riemann surface S is a **branched cover** of a Riemann surface X if there is an analytic map

$$p : S \to X$$

and a discrete set of branch points $B \subset X$ such that the restricted map

$$p : (S - R) \to (X - B), \text{ where } R = p^{-1}(B)$$

is a covering map.

Remark. Even though there may be points of R which are not ramification points, we need to remove the full inverse image of the branch points to get a covering space.

1.3. **The tiling on S.** As suggested by the title, we will limit our investigation to branched covers of a sphere. On the sphere, we will consider the branch points all to be arranged along a great circle and connected along the great circle by edges so that the sphere is divided into an upper hemisphere, X_u, and a lower hemisphere, X_l, each tiled by a single polygon. One can imagine for convenience that the polygons are regular. We will call this an *equatorial* tiling.

Since the branch points are all on the equator, and since X_u is simply connected, then the inverse image of the interior of X_u is a disjoint union of open discs in S. It may be shown that the closure of each of these open sets is mapped homeomorphically onto the upper half plane and so is homeomorphic to a closed disc or polygon. Call these closures *upper tiles* or tiles of type X_u. Likewise we construct from X_l *lower tiles* or tiles of type X_l in S. If we consider the interior of the arcs along the equator, determined by the branch points, we see that the inverse image of each of these is a disjoint union of open arcs, each of whose closures maps homeomorphically onto a closed arc on the equator. Call these closed arcs in S *edges*. The inverse image of the branch points will be called *vertices*. Thus we can construct a tiling on S by taking the upper and lower tiles as our polygons. An upper tile and a lower tile will meet along exactly one edge, unless the ramification index is 1 where their boundary edges meet. At a given vertex the number of tiles is $2e$, where e is the degree of ramification at the vertex. As one circles a vertex, the tiles alternate between upper and lower tiles.

Definition. In general, given a set $A \subset X$, a set $\widetilde{A}$ **of type A** in S is a set such that $p : \widetilde{A} \to A$ is a homeomorphism, where p is the branched covering map of the spaces in question.

1.4. **Lifting loops to paths.** When considering branched covers, it is often convenient to remove the ramification points (full inverse image of the branch points) from the cover and the branch points from the covered space in order to work with an unbranched cover. For our purposes, for a branched cover $p : S \to X$, we will denote by $X - B$, the surface with punctures at the branch points, by X° and $S - R = S - p^{-1}(B)$ by S° so that $p : S^\circ \to X^\circ$ is a connected covering space. We still have tiles on X° and S° except that all vertices have been removed and we can only get from one tile to another by crossing an edge.

Fact. Given a loop, γ in X°, based at an interior point $x_0 \in X_u$, and a point $\widetilde{x}_i \in p^{-1}(x_0)$, there is a unique path $\tilde{\gamma}$ in S° based at $\widetilde{x}_i$ that covers γ, i.e., $\tilde{\gamma}(0) = \widetilde{x}_i$ and $p(\tilde{\gamma}(t)) = \gamma(t)$ (see [3]).

When we consider liftings of loops, we will not be so concerned with their mid-sections as with their endpoints. Since homotopic loops lift to paths that are homotopic with fixed endpoints [3], we need only consider homotopy classes of loops and their liftings. The lifting of a representative element γ is constructed by considering the edges that γ crosses and their order, and then by constructing a path $\widetilde{\gamma}_i$ in S° which begins at $\widetilde{x}_i$ and crosses the appropriate edge types of tiles on S°. One might believe that this constructed path is in fact unique up to homotopy because on any given tile there is a unique edge of a given type.

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2. The Monodromy Representation

The importance of the endpoints of lifted loops is seen in the monodromy representation of the fundamental group $\pi_1(X^\circ, x_0)$ of the covered surface, X°. The endpoints carry information about how tiles on the covering surface, S° are connected to each other and how ramification points can be added to S° to form a branched cover of the sphere $p : S \to X$.

2.1. **Construction of the monodromy group.** The monodromy group of a branched cover, of degree d,

$$p : S \to X$$

is a representation of the fundamental group of X°, $\pi_1(X^\circ, x_0)$, in the symmetric group on d elements, Σ_d, where d is the degree of the cover S over X. It is denoted $M(S/X)$.

Constructing the monodromy representation consists of identifying, via a homomorphism, each element of the fundamental group with an element of the symmetric group. For a given homotopy class in the fundamental group, then, we must find some elements and a way to permute them. The elements will be the points in the fiber of the base point, x_0, and the permutation will be via a lifting of a representative element.

Let $I = \{\widetilde{x}_1, \ldots, \widetilde{x}_d\} = p^{-1}(x_0)$ denote the fiber of x_0. Given a class $[\gamma] \in \pi_1(X^\circ, x_0)$, let $\widetilde{\gamma}_i$ denote the lifting of the representative element γ to a path in S beginning at $\widetilde{x}_i$. Each such path has an endpoint; let it be denoted by $\widetilde{\gamma}_i(1)$. Now we are in a position to construct a permutation of the points in I as follows:

$$\begin{array}{ccc} \widetilde{x}_1 & \mapsto & \widetilde{\gamma}_1(1) \\ & \vdots & \\ \widetilde{x}_i & \mapsto & \widetilde{\gamma}_i(1) \\ & \vdots & \\ \widetilde{x}_d & \mapsto & \widetilde{\gamma}_d(1) \end{array}$$

Because loops ending at the same point must begin at the same point, the above map is bijective. Because homotopic loops have liftings with common endpoints, a different choice of representative will have no effect on this permutation. Let σ_γ denote the permutation assigned to γ as above.

Remark. Where permutations are involved, there is the possibility of confusion as to the order of multiplication. In this document we will always use the 'left to right' convention. The symbol $\sigma\tau$ will mean "first apply the permutation σ, then apply the permutation τ." For example:

$$(1, 2, 3) * (1, 4) = (1, 2, 3, 4)$$

$$(1, 4) * (1, 2, 3) = (1, 4, 2, 3)$$

Now the map $\mu : \pi_1(X^\circ, x_0) \to \Sigma_d$ defined by

$$[\gamma] \mapsto \sigma_\gamma$$

is a homomorphism since a composition of two loops lifts to a composition of their two individual liftings, and so does indeed give a representation of $\pi_1(X^\circ, x_0)$ in Σ_d. Remembering the tiling on X°, we note (see next section) that the generators of $\pi_1(X^\circ, x_0)$ are single loops (in a specific direction—call it 'counterclockwise') around single vertices. Furthermore, since each tile is simply connected, the exact position of the base point in X_u can be ignored, and we can consider the symbols permuted by $M\left(\mathrm{S}/\mathrm{X}\right)$ to be the tiles of type X_u on S.

2.2. Properties of the monodromy group.

Proposition 2.1. *The monodromy group of a covering is unique up to conjugation in Σ_d. For, if we relabel the vertices in the base fibre I by means of a permutation $\rho \in \Sigma_d$, i.e., order the points of I by $\widetilde{x}'_i = \widetilde{x}_{\rho i}$, then the monodromy representation, with respect to the new labelling, is given by $\gamma \to \rho\sigma_\gamma\rho^{-1}$. The corresponding monodromy groups are conjugate by ρ.*

Pick a point x_0 in the open upper hemisphere and order the branch points $y_i, 1 \leq i \leq t$ in counter clockwise order, looking from above. Draw a system of loops $\gamma_1, \ldots, \gamma_t$, intersecting only at x_0, so that γ_i moves in a straight path from x_0 almost to y_i, encircles y_i once in a counter clockwise direction and then retraces it path back to x_0. It is well known that the γ_i generate $\pi_1(X^\circ, x_0)$ and have one relation. The product of the generators of $\pi_1(X^\circ, x_0)$, in the order that their associated vertices appear when travelling along the boundary of the tile X_u , is homotopic to the identity loop, i.e., $\gamma_1\gamma_2\cdots\gamma_t = 1$. Likewise, the product of their monodromic representations in order yields the identity permutation.

Proposition 2.2. *Let the notation be as immediately above. Then,*

$$\gamma_1\gamma_2\cdots\gamma_t = 1,$$
$$\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_t} = 1.$$

Also we have:

$$\pi(X^\circ, x_0) = \langle\gamma_1, \gamma_2, \ldots, \gamma_t : \gamma_1\gamma_2\cdots\gamma_t = 1\rangle,$$
$$M\left(S/X\right) = \langle\sigma_{\gamma_1}, \sigma_{\gamma_2}, \ldots, \sigma_{\gamma_t}\rangle.$$

If the cover, S, is path-connected, then given any two tiles on S, there must be a permutation of $M\left(\mathrm{S}/\mathrm{X}\right)$ permuting the point $\widetilde{x}_i$ of type x_0 in the first tile to the corresponding point $\widetilde{x}_j$ in the second tile. For, we may draw a path δ from the $\widetilde{x}_i$ to $\widetilde{x}_j$ in S° avoiding the branch points. Then the lift of the projection $p_*(\delta)$ in X°, starting at $\widetilde{x}_i$ is just δ. Thus $\mu(p_*(\delta)) \cdot i = j$ and so $M\left(\mathrm{S}/\mathrm{X}\right)$ is a transitive subgroup of Σ_d.

Proposition 2.3. *Let $S \to X$ be a connected branched covering space. Then, $M\left(S/X\right)$ is a transitive subgroup of Σ_d.*

Definition. Let $M \subseteq \Sigma_d$ be a subgroup. Let $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ be a t-tuple of elements of M. Then $(\sigma_1, \sigma_2, \ldots, \sigma_t)$ is called a t-generating vector for M if and only if

$$\sigma_1\sigma_2\cdots\sigma_t = 1, \tag{1}$$
$$M = \langle\sigma_1, \sigma_2, \ldots, \sigma_t\rangle. \tag{2}$$

A key theorem proved in the last century is the Riemann existence theorem which specifies whether a cover exists with a specified monodromy. Slimmed down for our purposes we have:

Proposition 2.4. *Let X be the Riemann sphere and let $B = \{y_1, \dots, y_t\} \subset X$ be a set of t distinct points in X. Let M be a transitive subgroup of Σ_d with a t-generating vector $(\sigma_1, \sigma_2, \dots, \sigma_t)$. Then there is a branched cover $p : S \to X$, of degree d, with branch set lying in B, such that in the monodromy representation $\pi(X^\circ, x_0) \to \Sigma_d$, $\gamma \to \sigma_\gamma$ we get*

$$\sigma_{\gamma_i} = \sigma_i.$$

2.3. **Generators of the monodromy group.** The generators of $M(\mathrm{S}/\mathrm{X})$ contain all the information needed to determine the covering space S. Our question of what branched covers of the sphere are possible can be translated now to a question of group theory: what generating vectors are possible. To answer this question, one simply has to consider each symmetric group in turn and ask about all the possible ordered sets of permutations. A given set will be a generating vector if the product of the permutations in order is the identity and they generate a transitive subgroup of the symmetric group. Now as simple as this process is, it has no real way of being finished and it is a bit redundant. The redundancy comes because some pair of generating vectors may generate essentially the same surface.

The search can be shortened by determining an equivalence relation on the generating vectors based on an equivalence of their associated covering spaces.

We begin this task by considering how a generating vector determines a cover. In particular, let us consider a single permutation, that is, a single component of a generating vector. This component is a permutation corresponding to a generator of $\pi_1(X^\circ, x_0)$, which is a single loop around a single vertex, v, of X°. We can write any such permutation as a product of disjoint cycles. If we recall that the symbols being permuted correspond to tiles of type X_u, and that the order of a cycle is its length, we conclude that the number of disjoint cycles in the product must correspond to the number of distinct vertices of type v. The tiles meeting at each such distinct vertex are determined by the symbols permuted by the corresponding disjoint cycle. In other words, the ramification index of this vertex is given by the cycle length of the corresponding cycle. The degree of the cover, d, is given by the number of elements on which the generating vector acts. The number of vertices on X (that is, the number of removed branch points) is t, the number of components of the generating vector.

We have defined the vertices of X and of S so that they only occur at branch points and ramification points, respectively (or rather, at the holes created by the removal of such points). For each edge and tile on X, there are exactly d edges and tiles of that type on S. Since we are considering X to consist of two tiles with the edges and vertices situated along a great circle, the Euler characteristic of S is completely determined by the generating vector. The number of faces (tiles) is $2 \times d$, the number of edges is $t \times d$, and the number of vertices is given by the total number of disjoint cycles of the generators.

Note that when we consider the Euler characteristic or genus of these surfaces, we mean their completions to a sphere with branch points and a branched cover, not X° and S°.

The Riemann-Hurwitz formula is:

$$\chi_S = d\chi_X - \sum_{r \in R} (e_r - 1)$$

where R is the set of ramification points and e_r is the ramification index of $r \in$ R. Note the formula is correct if we add unramified points to R since $e_r - 1$ is zero for unramified points.

We can write this formula in a different way. For a branch point y_i let $(e_1, \dots, e_{s_i})$ be the cycle structure of σ_{γ_i}, where s_i is the number of points in $p^{-1}(y_i)$. Then

$$\sum_{j=1}^{s_i}(e_j - 1) = \left(\sum_{j=1}^{s_i} e_j\right) - s_i = d - s_i.$$

It follows that

$$\chi_S = d\chi_X - td + \sum_{i=1}^{t} s_i = d(\chi_X - t) + \sum_{i=1}^{t} s_i$$

Since a surface of genus g has Euler characteristic

$$\chi = 2 - 2g$$

and a sphere has genus zero, we can solve for the genus of the branched cover determined by a given generating vector ('branching data')

$$g_S = 1 - d + \frac{1}{2}\sum_{r\in R}(e_r - 1) = 1 + \frac{1}{2}\left(d(t-2) - \sum_{i=1}^{t} s_i\right).$$

For a given degree and number of branch points, t, we have an upper bound on g_S:

$$g_{S\,max} = 1 - d + t\frac{d-1}{2} = \frac{(d-1)(t-2)}{2} \tag{3}$$

The maximum possible genus is the greatest integer less than or equal to $g_{S\,max}$. Likewise, there is a lower bound for the minimum genus:

$$g_{S\,min} = 1 - d + \frac{t}{2} \tag{4}$$

The minimum possible genus is either zero or the smallest integer greater than or equal to $g_{S\,min}$, which ever is larger.

3. The Equivalence Relation on Generating Vectors

3.1. **Introduction and motivation.** Having discussed how the generating vector of a monodromy group determines a branched cover of the sphere, we can examine how topological conjugacy of the branched covers translates to equivalence of generating vectors. To say that two surfaces S, S' are topologically conjugate means that they have the same arrangement of tiles and vertices; they differ only by the names associated to those tiles and vertices. Specifically there should be a homeomorphism $h : S \to S'$ mapping tiles to tiles, edges to edges, and vertices to vertices.

3.2. **Preparing the pigeon holes.** Recall that the monodromy representation assigns a label to each tile of the cover and that generators of $\pi_1(X^\circ, x_0)$ are represented as permutations of those tiles. If two covers differ only by a relabelling of the tiles for the monodromy representation, then they are topologically conjugate. To relabel the tiles corresponds to conjugating the monodromy group by some element of the symmetric group.

Another simple homeomorphism of the surface is to rotate each tile of type X_u in the 'counterclockwise' direction and rotate each tile of type X_l in the opposite direction by a cyclic permutation of the vertices of each tile. This corresponds to a rotation of the sphere through the axis perpendicular to the plane separating the upper and lower tiles. The permutation is a cyclic permutation of the components of the generating vector of $M(\mathrm{S}/\mathrm{X})$.

Remark. Though it will not be a part of our equivalence relation, a more subtle homeomorphism of the cover is through a braid automorphism, which interchanges two vertices of the cover in a way compatible with the projection to the covered surface. Given a generating vector, a braid automorphism interchanges two adjacent coordinates and conjugates one by the other—leaving the overall product unaffected. This type of homeomorphism is the only one keeping our equivalence relation from full-blown topological equivalence [1].

$$(\sigma_1, \dots, \sigma_a, \sigma_b, \dots, \sigma_t) \mapsto (\sigma_1, \dots, \sigma_b, \sigma_b^{-1}\sigma_a\sigma_t, \dots, \sigma_t)$$

The equivalence relation we will impose is as follows: two generating vectors,

$$GV_1 = (\sigma_1, \dots, \sigma_a, \sigma_b, \dots, \sigma_t)$$

and

$$GV_2 = (\tau_1, \dots, \tau_a, \tau_b, \dots, \tau_t)$$

are equivalent if one of the following conditions is met any nonzero number of times

(1) They differ by a cyclic permutation of components:

$$GV_2 = (\sigma_2, \dots, \sigma_a, \sigma_b, \dots, \sigma_b, \sigma_1)$$

(2) They (and, consequently, their generated monodromy groups) are conjugate:

there exists $\rho \in \Sigma_d$ such that

$$GV_2 = \rho^{-1} GV_1 \rho$$

3.3. **Implementation.** The search for representative generating vectors for every possible class was performed by a MAGMA program in a fairly straightforward manner. Given a degree of the cover, d, and a number of branch points, t, MAGMA rolled through all the possible ordered subsets of Σ_d with $t-1$ elements like an odometer rolls through all the numbers between 0 and 999…99. The last element is calculated by MAGMA to be the inverse of the product of the first $t-1$ elements. The remainder of the program consisted of checking that the generated group was transitive and checking for equivalences of monodromy groups.

A lexicographic ordering was imposed on Σ_d based on the cycle structure of each permutation, and then generating vectors were considered only if their elements were in ascending order. The permutations of Σ_d were also divided into conjugacy classes and the first elements were selected from the set of representatives of those classes. These requirements did not eliminate any equivalences fully, but were used to severely limit the generating vectors under consideration.

The cyclic permutations of each candidate were then checked against a list of representatives for conjugacy. This involved representing each cyclic permutation of the candidate as an element of the t-fold direct product of Σ_d and using MAGMA's 'IsConjugate' command to test conjugacy to each element from the list of representatives.

If no cyclic permutation of the candidate was conjugate to a vector already on the list, MAGMA's 'IsTransitive' command was used to test whether the generated group was indeed transitive on Σ_d. If so, the candidate was added to the list as a generating vector not equivalent to any other vector on the list.

The program takes as input the degree of the cover and has the number of branch points built in. A separate program is required for each number of branch points, T, and is in the MAGMA script "GenClasses$T$.mgm" and supporting scripts. GenClasses3.mgm through GenClasses6.mgm currently exist; a few lines of code can be added to GenClasses$T$.mgm to make GenClasses$(T+1)$.mgm.

For a number of branch points T, the program in "GenSets$T$.mgm" and supporting scripts classifies all generating vectors of the same cycle structure and generating the same transitive subgroup of Σ_d as equivalent. This equivalence relation is somewhat less fine than that imposed by topological equivalence. The GenSets programs work similarly to the GenClasses programs.

A third program has been written as a step toward classifying the generating vectors by topological equivalence of their corresponding branched covers. The program identifies some of the duplicate representatives in the output of the GenClasses scripts [4]. It is in the MAGMA script "Unbraid.mgm" and supporting scripts. If two generating vectors are equivalent (in the sense of topological conjugacy) after elimination of a single element from each, then they must differ by a braid automorphism moving one component through the vector, as below:

$$\begin{gathered}(\sigma_1, \ldots, \sigma_a, \sigma_b, \sigma_c, \ldots, \sigma_t)\\ \downarrow\\ (\sigma_1, \ldots, \sigma_b, \sigma_b^{-1}\sigma_a\sigma_b, \sigma_c, \ldots, \sigma_t)\\ \downarrow\\ (\sigma_1, \ldots, \sigma_b, \sigma_c, \sigma_c^{-1}\sigma_b^{-1}\sigma_a\sigma_b\sigma_c, \ldots, \sigma_t)\end{gathered}$$

The Unbraid script uses this method to identify some pairs that are equivalent through an automorphism:

$$\begin{aligned}\text{if } (\sigma_1, \ldots, \widehat{\sigma_a}, \sigma_b, \sigma_c, \ldots, \sigma_t) &= (\sigma_1, \ldots, \sigma_b, \sigma_c, \widehat{\sigma_a'}, \ldots, \sigma_t), \text{ then}\\ \sigma_a' &= \sigma_c^{-1}\sigma_b^{-1}\sigma_a\sigma_b\sigma_c\\ \text{since } \sigma_1 \ldots \sigma_a\sigma_b\sigma_c \ldots \sigma_d = 1 &= \sigma_1 \ldots \sigma_b\sigma_c\sigma_a' \ldots \sigma_d\end{aligned}$$

Data from this program is in the Dups (for duplicates) series of text files.

4. Questions and Conjectures

Given a number of branch points and degree of a branched cover, is the genus maximum as calculated in (3) always attained by some cover? We conjecture that it will be, but a good reason why is not obvious. One must ensure that there is a list of the required length made up of permutations from the required symmetric group and all having the longest cycle length possible.

How can one identify (fully) when two generating vectors differ by a braid automorphism? This is possible by running the Unbraid MAGMA script on the set of all possible generating vectors, but it will be a costly calculation as far as time and computer memory are concerned.

References

[1] S. Allen Broughton, private communication.
[2] Fulton, William. *Algebraic Topology*. Springer (1995).
[3] Munkres, James R. *Topology*, Second Edition. Prentice Hall (2000).
[4] Steve Young, private communication on the "Unbraid" script.

5. DATA

5.1. **Detailed data.** This data was generated with the GenClasses scripts and reflects the number of generating vectors up to topological conjugacy for a given number of branch points, degree of cover, genus of cover, genus of Galois Cover, and monodromy group.

A Galois Cover is an extension of a branched cover, S, to a regular cover of both the branched cover, S, and it's covered space, X. We will not discuss it further except to say that it's genus is given by a formula similar to the Riemann-Hurwitz equation and is dependent entirely on the cycle structure of the generating vector for the branched cover. The genus of the Galois Cover (abbreviated G.C. Genus) has been included in the table to distinguish between generating vectors of differing cycle structures.

The monodromy groups are identified as MAGMA identifies them, "TransitiveGroup(d,n)" is the MAGMA command for the n^{th} transitive subgroup of the symmetric group on d letters (abbreviated "TrnGp(d,n)"). A list of MAGMA's descriptions of the relevant groups is provided in the appendix.

No. Branch Pts.	Degree	Genus	G.C. Genus	Monodromy Group, Order	No. Classes
3	3	0	0	TrnGp(3 , 2) , 6	1
3	3	1	1	TrnGp(3 , 1) , 3	1
3	4	0	0	TrnGp(4 , 2) , 4	1
3	4	0	0	TrnGp(4 , 3) , 8	1
3	4	0	0	TrnGp(4 , 4) , 12	1
3	4	0	0	TrnGp(4 , 5) , 24	1
3	4	0	1	TrnGp(4 , 4) , 12	1
3	4	1	1	TrnGp(4 , 1) , 4	1
3	4	1	3	TrnGp(4 , 5) , 24	1
3	5	0	0	TrnGp(5 , 2) , 10	1
3	5	0	0	TrnGp(5 , 4) , 60	1
3	5	0	1	TrnGp(5 , 3) , 20	2
3	5	0	4	TrnGp(5 , 5) , 120	1
3	5	0	21	TrnGp(5 , 5) , 120	1
3	5	0	5	TrnGp(5 , 4) , 60	1
3	5	0	6	TrnGp(5 , 5) , 120	1
3	5	0	9	TrnGp(5 , 5) , 120	1
3	5	0	11	TrnGp(5 , 5) , 120	2
3	5	0	16	TrnGp(5 , 5) , 120	2
3	5	1	19	TrnGp(5 , 5) , 120	1
3	5	1	4	TrnGp(5 , 3) , 20	2
3	5	1	4	TrnGp(5 , 4) , 60	1
3	5	1	24	TrnGp(5 , 5) , 120	2
3	5	1	9	TrnGp(5 , 4) , 60	2
3	5	1	29	TrnGp(5 , 5) , 120	1
3	5	2	2	TrnGp(5 , 1) , 5	1
3	5	2	13	TrnGp(5 , 4) , 60	1

No. Branch Pts.	Degree	Genus	G.C. Genus	Monodromy Group, Order	No. Classes
3	6	0	0	TrnGp(6 , 2) , 6	1
3	6	0	0	TrnGp(6 , 3) , 12	1
3	6	0	0	TrnGp(6 , 4) , 12	1
3	6	0	0	TrnGp(6 , 7) , 24	1
3	6	0	0	TrnGp(6 , 8) , 24	1
3	6	0	0	TrnGp(6 , 12) , 60	1
3	6	0	1	TrnGp(6 , 5) , 18	1
3	6	0	1	TrnGp(6 , 6) , 24	1
3	6	0	1	TrnGp(6 , 10) , 36	2
3	6	0	3	TrnGp(6 , 8) , 24	1
3	6	0	3	TrnGp(6 , 11) , 48	1
3	6	0	4	TrnGp(6 , 10) , 36	1
3	6	0	4	TrnGp(6 , 12) , 60	1
3	6	0	4	TrnGp(6 , 13) , 72	2
3	6	0	4	TrnGp(6 , 14) , 120	1
3	6	0	91	TrnGp(6 , 16) , 720	4
3	6	0	6	TrnGp(6 , 14) , 120	1
3	6	0	49	TrnGp(6 , 15) , 360	1
3	6	0	49	TrnGp(6 , 16) , 720	2
3	6	0	10	TrnGp(6 , 15) , 360	2
3	6	0	139	TrnGp(6 , 16) , 720	4
3	6	0	11	TrnGp(6 , 14) , 120	1
3	6	0	16	TrnGp(6 , 15) , 360	1
3	6	0	61	TrnGp(6 , 16) , 720	3
3	6	0	19	TrnGp(6 , 14) , 120	1
3	6	0	19	TrnGp(6 , 15) , 360	2
3	6	0	151	TrnGp(6 , 16) , 720	4
3	6	0	25	TrnGp(6 , 15) , 360	1
3	6	0	121	TrnGp(6 , 16) , 720	9
3	6	0	40	TrnGp(6 , 15) , 360	2
3	6	0	169	TrnGp(6 , 16) , 720	7
3	6	1	1	TrnGp(6 , 1) , 6	1
3	6	1	1	TrnGp(6 , 4) , 12	1
3	6	1	3	TrnGp(6 , 6) , 24	1
3	6	1	3	TrnGp(6 , 7) , 24	1
3	6	1	3	TrnGp(6 , 11) , 48	1
3	6	1	46	TrnGp(6 , 15) , 360	2
3	6	1	4	TrnGp(6 , 5) , 18	1
3	6	1	4	TrnGp(6 , 9) , 36	1
3	6	1	4	TrnGp(6 , 10) , 36	1
3	6	1	5	TrnGp(6 , 12) , 60	1
3	6	1	49	TrnGp(6 , 15) , 360	1
3	6	1	9	TrnGp(6 , 11) , 48	1
3	6	1	9	TrnGp(6 , 12) , 60	2
3	6	1	9	TrnGp(6 , 14) , 120	1
3	6	1	139	TrnGp(6 , 16) , 720	4
3	6	1	11	TrnGp(6 , 14) , 120	1
3	6	1	55	TrnGp(6 , 15) , 360	4
3	6	1	13	TrnGp(6 , 12) , 60	1
3	6	1	16	TrnGp(6 , 13) , 72	2

No. Branch Pts.	Degree	Genus	G.C. Genus	Monodromy Group, Order	No. Classes
3	6	1	16	TrnGp(6 , 14) , 120	2
3	6	1	64	TrnGp(6 , 15) , 360	8
3	6	1	151	TrnGp(6 , 16) , 720	3
3	6	1	24	TrnGp(6 , 14) , 120	2
3	6	1	73	TrnGp(6 , 15) , 360	4
3	6	1	121	TrnGp(6 , 16) , 720	9
3	6	1	40	TrnGp(6 , 15) , 360	2
3	6	1	169	TrnGp(6 , 16) , 720	8
3	6	2	2	TrnGp(6 , 1) , 6	1
3	6	2	4	TrnGp(6 , 5) , 18	1
3	6	2	5	TrnGp(6 , 6) , 24	1
3	6	2	21	TrnGp(6 , 14) , 120	1
3	6	2	151	TrnGp(6 , 16) , 720	4
3	6	2	29	TrnGp(6 , 14) , 120	1
3	6	2	169	TrnGp(6 , 16) , 720	7
3	7	0	0	TrnGp(7 , 2) , 14	1
3	7	0	1	TrnGp(7 , 3) , 21	2
3	7	0	1	TrnGp(7 , 4) , 42	2
3	7	0	3	TrnGp(7 , 5) , 168	2
3	7	0	8	TrnGp(7 , 5) , 168	4
3	7	0	136	TrnGp(7 , 6) , 2520	2
3	7	0	10	TrnGp(7 , 5) , 168	2
3	7	0	901	TrnGp(7 , 7) , 5040	1
3	7	0	649	TrnGp(7 , 7) , 5040	1
3	7	0	15	TrnGp(7 , 5) , 168	2
3	7	0	271	TrnGp(7 , 7) , 5040	1
3	7	0	526	TrnGp(7 , 6) , 2520	2
3	7	0	274	TrnGp(7 , 6) , 2520	5
3	7	0	22	TrnGp(7 , 5) , 168	4
3	7	0	409	TrnGp(7 , 6) , 2520	1
3	7	0	1429	TrnGp(7 , 7) , 5040	4
3	7	0	1177	TrnGp(7 , 7) , 5040	10
3	7	0	1051	TrnGp(7 , 7) , 5040	25
3	7	0	799	TrnGp(7 , 7) , 5040	5
3	7	0	547	TrnGp(7 , 6) , 2520	4
3	7	0	547	TrnGp(7 , 7) , 5040	2
3	7	0	421	TrnGp(7 , 6) , 2520	7
3	7	0	421	TrnGp(7 , 7) , 5040	7
3	7	0	169	TrnGp(7 , 6) , 2520	1
3	7	0	169	TrnGp(7 , 7) , 5040	1
3	7	0	1321	TrnGp(7 , 7) , 5040	9
3	7	0	691	TrnGp(7 , 7) , 5040	1
3	7	0	442	TrnGp(7 , 6) , 2520	4
3	7	0	316	TrnGp(7 , 6) , 2520	7
3	7	0	451	TrnGp(7 , 6) , 2520	1
3	7	0	199	TrnGp(7 , 6) , 2520	2
3	7	0	1471	TrnGp(7 , 7) , 5040	3
3	7	0	1345	TrnGp(7 , 7) , 5040	7
3	7	0	1219	TrnGp(7 , 7) , 5040	12

No. Branch Pts.	Degree	Genus	G.C. Genus	Monodromy Group, Order	No. Classes
3	7	0	967	TrnGp(7 , 7) , 5040	1
3	7	0	841	TrnGp(7 , 7) , 5040	14
3	7	0	589	TrnGp(7 , 6) , 2520	1
3	7	0	589	TrnGp(7 , 7) , 5040	4
3	7	0	337	TrnGp(7 , 6) , 2520	3
3	7	0	337	TrnGp(7 , 7) , 5040	3
3	7	0	211	TrnGp(7 , 6) , 2520	5
3	7	0	211	TrnGp(7 , 7) , 5040	5
3	7	0	346	TrnGp(7 , 6) , 2520	2
3	7	0	1111	TrnGp(7 , 7) , 5040	3
3	7	0	481	TrnGp(7 , 7) , 5040	4
3	7	0	484	TrnGp(7 , 6) , 2520	4
3	7	0	241	TrnGp(7 , 6) , 2520	3
3	7	0	1387	TrnGp(7 , 7) , 5040	5
3	7	0	1261	TrnGp(7 , 7) , 5040	18
3	7	0	1135	TrnGp(7 , 7) , 5040	2
3	7	0	1009	TrnGp(7 , 7) , 5040	10
3	7	0	757	TrnGp(7 , 7) , 5040	3
3	7	0	631	TrnGp(7 , 6) , 2520	1
3	7	0	631	TrnGp(7 , 7) , 5040	6
3	7	0	505	TrnGp(7 , 6) , 2520	1
3	7	0	505	TrnGp(7 , 7) , 5040	1
3	7	0	379	TrnGp(7 , 6) , 2520	14
3	7	0	379	TrnGp(7 , 7) , 5040	3
3	7	1	3	TrnGp(7 , 3) , 21	2
3	7	1	514	TrnGp(7 , 6) , 2520	18
3	7	1	1531	TrnGp(7 , 7) , 5040	11
3	7	1	8	TrnGp(7 , 4) , 42	2
3	7	1	1279	TrnGp(7 , 7) , 5040	8
3	7	1	649	TrnGp(7 , 7) , 5040	3
3	7	1	271	TrnGp(7 , 6) , 2520	3
3	7	1	17	TrnGp(7 , 5) , 168	2
3	7	1	19	TrnGp(7 , 5) , 168	2
3	7	1	24	TrnGp(7 , 5) , 168	4
3	7	1	661	TrnGp(7 , 6) , 2520	2
3	7	1	409	TrnGp(7 , 6) , 2520	8
3	7	1	1681	TrnGp(7 , 7) , 5040	2
3	7	1	1555	TrnGp(7 , 7) , 5040	10
3	7	1	31	TrnGp(7 , 5) , 168	2
3	7	1	1429	TrnGp(7 , 7) , 5040	17
3	7	1	1177	TrnGp(7 , 7) , 5040	24
3	7	1	1051	TrnGp(7 , 7) , 5040	40
3	7	1	556	TrnGp(7 , 6) , 2520	8
3	7	1	1321	TrnGp(7 , 7) , 5040	27
3	7	1	691	TrnGp(7 , 7) , 5040	2
3	7	1	577	TrnGp(7 , 6) , 2520	12
3	7	1	451	TrnGp(7 , 6) , 2520	25
3	7	1	1597	TrnGp(7 , 7) , 5040	12
3	7	1	1471	TrnGp(7 , 7) , 5040	14
3	7	1	1345	TrnGp(7 , 7) , 5040	18

No. Branch Pts.	Degree	Genus	G.C. Genus	Monodromy Group, Order	No. Classes
3	7	1	1219	TrnGp(7 , 7) , 5040	25
3	7	1	841	TrnGp(7 , 7) , 5040	10
3	7	1	346	TrnGp(7 , 6) , 2520	6
3	7	1	1489	TrnGp(7 , 7) , 5040	14
3	7	1	1111	TrnGp(7 , 7) , 5040	10
3	7	1	481	TrnGp(7 , 6) , 2520	5
3	7	1	481	TrnGp(7 , 7) , 5040	5
3	7	1	619	TrnGp(7 , 6) , 2520	7
3	7	1	1639	TrnGp(7 , 7) , 5040	3
3	7	1	1513	TrnGp(7 , 7) , 5040	10
3	7	1	1387	TrnGp(7 , 7) , 5040	26
3	7	1	1261	TrnGp(7 , 7) , 5040	34
3	7	1	1009	TrnGp(7 , 7) , 5040	10
3	7	2	1657	TrnGp(7 , 7) , 5040	12
3	7	2	1531	TrnGp(7 , 7) , 5040	20
3	7	2	12	TrnGp(7 , 4) , 42	2
3	7	2	649	TrnGp(7 , 6) , 2520	24
3	7	2	33	TrnGp(7 , 5) , 168	4
3	7	2	40	TrnGp(7 , 5) , 168	4
3	7	2	1699	TrnGp(7 , 7) , 5040	10
3	7	2	1321	TrnGp(7 , 7) , 5040	38
3	7	2	691	TrnGp(7 , 6) , 2520	7
3	7	2	586	TrnGp(7 , 6) , 2520	20
3	7	2	1741	TrnGp(7 , 7) , 5040	9
3	7	2	1489	TrnGp(7 , 7) , 5040	24
3	7	2	481	TrnGp(7 , 6) , 2520	8
3	7	3	3	TrnGp(7 , 1) , 7	3
3	7	3	49	TrnGp(7 , 5) , 168	2
3	7	3	721	TrnGp(7 , 6) , 2520	9
4	3	0	1	TrnGp(3 , 2) , 6	2
4	3	1	2	TrnGp(3 , 2) , 6	2
4	3	2	2	TrnGp(3 , 1) , 3	2
4	4	0	1	TrnGp(4 , 3) , 8	2
4	4	0	3	TrnGp(4 , 5) , 24	3
4	4	0	4	TrnGp(4 , 5) , 24	4
4	4	0	5	TrnGp(4 , 5) , 24	6
4	4	1	1	TrnGp(4 , 2) , 4	2
4	4	1	2	TrnGp(4 , 3) , 8	2
4	4	1	3	TrnGp(4 , 3) , 8	2
4	4	1	3	TrnGp(4 , 4) , 12	3
4	4	1	4	TrnGp(4 , 4) , 12	4
4	4	1	5	TrnGp(4 , 4) , 12	8
4	4	1	6	TrnGp(4 , 5) , 24	3
4	4	1	7	TrnGp(4 , 5) , 24	4
4	4	1	8	TrnGp(4 , 5) , 24	8
4	4	2	11	TrnGp(4 , 5) , 24	8
4	4	2	2	TrnGp(4 , 1) , 4	1
4	4	2	3	TrnGp(4 , 3) , 8	2

No. Branch Pts.	Degree	Genus	G.C. Genus	Monodromy Group, Order	No. Classes
4	4	2	9	TrnGp(4 , 5) , 24	3
4	4	2	10	TrnGp(4 , 5) , 24	4
4	4	3	13	TrnGp(4 , 5) , 24	2
4	4	3	3	TrnGp(4 , 1) , 4	3
4	5	0	1	TrnGp(5 , 2) , 10	4
4	5	0	6	TrnGp(5 , 4) , 60	9
4	5	0	11	TrnGp(5 , 4) , 60	9
4	5	0	16	TrnGp(5 , 4) , 60	12
4	5	0	16	TrnGp(5 , 5) , 120	8
4	5	0	19	TrnGp(5 , 5) , 120	5
4	5	0	21	TrnGp(5 , 4) , 60	5
4	5	0	21	TrnGp(5 , 5) , 120	6
4	5	0	26	TrnGp(5 , 5) , 120	10
4	5	0	29	TrnGp(5 , 5) , 120	5
4	5	0	31	TrnGp(5 , 5) , 120	15
4	5	0	36	TrnGp(5 , 5) , 120	16
4	5	0	41	TrnGp(5 , 5) , 120	16
4	5	1	6	TrnGp(5 , 3) , 20	12
4	5	1	10	TrnGp(5 , 4) , 60	10
4	5	1	15	TrnGp(5 , 4) , 60	15
4	5	1	20	TrnGp(5 , 4) , 60	20
4	5	1	25	TrnGp(5 , 4) , 60	25
4	5	1	31	TrnGp(5 , 5) , 120	16
4	5	1	34	TrnGp(5 , 5) , 120	15
4	5	1	36	TrnGp(5 , 5) , 120	18
4	5	1	37	TrnGp(5 , 5) , 120	10
4	5	1	39	TrnGp(5 , 5) , 120	10
4	5	1	41	TrnGp(5 , 5) , 120	45
4	5	1	44	TrnGp(5 , 5) , 120	20
4	5	1	46	TrnGp(5 , 5) , 120	58
4	5	1	49	TrnGp(5 , 5) , 120	15
4	5	1	51	TrnGp(5 , 5) , 120	86
4	5	1	56	TrnGp(5 , 5) , 120	48
4	5	1	61	TrnGp(5 , 5) , 120	34
4	5	2	69	TrnGp(5 , 5) , 120	25
4	5	2	4	TrnGp(5 , 2) , 10	4
4	5	2	71	TrnGp(5 , 5) , 120	52
4	5	2	9	TrnGp(5 , 3) , 20	10
4	5	2	76	TrnGp(5 , 5) , 120	32
4	5	2	11	TrnGp(5 , 3) , 20	18
4	5	2	81	TrnGp(5 , 5) , 120	12
4	5	2	19	TrnGp(5 , 4) , 60	15
4	5	2	24	TrnGp(5 , 4) , 60	24
4	5	2	29	TrnGp(5 , 4) , 60	34
4	5	2	49	TrnGp(5 , 5) , 120	35
4	5	2	52	TrnGp(5 , 5) , 120	24
4	5	2	54	TrnGp(5 , 5) , 120	30
4	5	2	57	TrnGp(5 , 5) , 120	14
4	5	2	59	TrnGp(5 , 5) , 120	80

No. Branch Pts.	Degree	Genus	G.C. Genus	Monodromy Group, Order	No. Classes
4	5	2	61	TrnGp(5 , 5) , 120	22
4	5	2	64	TrnGp(5 , 5) , 120	40
4	5	2	66	TrnGp(5 , 5) , 120	72
4	5	3	67	TrnGp(5 , 5) , 120	64
4	5	3	72	TrnGp(5 , 5) , 120	48
4	5	3	77	TrnGp(5 , 5) , 120	34
4	5	3	12	TrnGp(5 , 3) , 20	8
4	5	3	28	TrnGp(5 , 4) , 60	28
4	5	3	33	TrnGp(5 , 4) , 60	36
4	5	4	4	TrnGp(5 , 1) , 5	6
4	5	4	37	TrnGp(5 , 4) , 60	20
5	3	1	3	TrnGp(3 , 2) , 6	9
5	3	2	4	TrnGp(3 , 2) , 6	4
5	3	3	3	TrnGp(3 , 1) , 3	1
5	4	0	7	TrnGp(4 , 5) , 24	12
5	4	0	9	TrnGp(4 , 5) , 24	27
5	4	1	3	TrnGp(4 , 3) , 8	6
5	4	1	4	TrnGp(4 , 3) , 8	4
5	4	1	9	TrnGp(4 , 5) , 24	9
5	4	1	10	TrnGp(4 , 5) , 24	12
5	4	1	11	TrnGp(4 , 5) , 24	24
5	4	1	12	TrnGp(4 , 5) , 24	36
5	4	1	13	TrnGp(4 , 5) , 24	60
5	4	2	2	TrnGp(4 , 2) , 4	2
5	4	2	4	TrnGp(4 , 3) , 8	7
5	4	2	5	TrnGp(4 , 3) , 8	2
5	4	2	6	TrnGp(4 , 4) , 12	9
5	4	2	7	TrnGp(4 , 4) , 12	12
5	4	2	8	TrnGp(4 , 4) , 12	48
5	4	2	9	TrnGp(4 , 4) , 12	21
5	4	2	12	TrnGp(4 , 5) , 24	9
5	4	2	13	TrnGp(4 , 5) , 24	12
5	4	2	14	TrnGp(4 , 5) , 24	24
5	4	2	15	TrnGp(4 , 5) , 24	36
5	4	2	16	TrnGp(4 , 5) , 24	64
5	4	3	17	TrnGp(4 , 5) , 24	24
5	4	3	18	TrnGp(4 , 5) , 24	36
5	4	3	19	TrnGp(4 , 5) , 24	64
5	4	3	3	TrnGp(4 , 1) , 4	1
5	4	3	5	TrnGp(4 , 3) , 8	6
5	4	3	6	TrnGp(4 , 3) , 8	4
5	4	3	15	TrnGp(4 , 5) , 24	9
5	4	3	16	TrnGp(4 , 5) , 24	12
5	4	4	19	TrnGp(4 , 5) , 24	12
5	4	4	4	TrnGp(4 , 1) , 4	4
5	4	4	21	TrnGp(4 , 5) , 24	36
6	3	1	4	TrnGp(3 , 2) , 6	10
6	3	2	5	TrnGp(3 , 2) , 6	18
6	3	3	6	TrnGp(3 , 2) , 6	8
6	3	4	4	TrnGp(3 , 1) , 3	4

5.2. **Summary data.** The 'Some Unbraid' column gives the number of inequivalent representatives as identified by applying Unbraid to the representatives identified by GenClasses.

No. Branch Pts.	Degree	No. Classes	Some Unbraid
3	3	2	2
3	4	7	7
3	5	24	24
3	6	139	139
3	7	899	885
4	3	6	4
4	4	74	46
4	5	1364	775
5	3	14	5
5	4	644	225
6	3	40	11

APPENDIX A

Magma's descriptions of the transitive subgroups of each symmetric group are as follows:

```
TransitiveGroup( 3 , 1 ):
Permutation group acting on a set of cardinality 3
    (1, 2, 3)
C(3) = A(3) = 3

TransitiveGroup( 3 , 2 ):
Permutation group acting on a set of cardinality 3
    (1, 3)
    (2, 3)
S(3)

TransitiveGroup( 4 , 1 ):
Permutation group acting on a set of cardinality 4
    (1, 2, 3, 4)
C(4) = 4

TransitiveGroup( 4 , 2 ):
Permutation group acting on a set of cardinality 4
    (1, 4)(2, 3)
    (1, 2)(3, 4)
E(4) = 2[x]2

TransitiveGroup( 4 , 3 ):
Permutation group acting on a set of cardinality 4
    (1, 2, 3, 4)
    (1, 3)
D(4)

TransitiveGroup( 4 , 4 ):
Permutation group acting on a set of cardinality 4
    (1, 2, 4)
    (2, 3, 4)
A(4)

TransitiveGroup( 4 , 5 ):
Permutation group acting on a set of cardinality 4
    (1, 2, 3, 4)
    (1, 2)
S(4)

TransitiveGroup( 5 , 1 ):
Permutation group acting on a set of cardinality 5
    (1, 2, 3, 4, 5)
C(5) = 5

TransitiveGroup( 5 , 2 ):
Permutation group acting on a set of cardinality 5
    (1, 2, 3, 4, 5)
    (1, 4)(2, 3)
D(5) = 5:2

TransitiveGroup( 5 , 3 ):
Permutation group acting on a set of cardinality 5
    (1, 2, 3, 4, 5)
    (1, 2, 4, 3)
F(5) = 5:4

TransitiveGroup( 5 , 4 ):
Permutation group acting on a set of cardinality 5
    (3, 4, 5)
    (1, 2, 3)
A(5)

TransitiveGroup( 5 , 5 ):
Permutation group acting on a set of cardinality 5
    (1, 2, 3, 4, 5)
    (1, 2)
S(5)

TransitiveGroup( 6 , 1 ):
Permutation group acting on a set of cardinality 6
    (1, 2, 3, 4, 5, 6)
C(6) = 6 = 3[x]2

TransitiveGroup( 6 , 2 ):
Permutation group acting on a set of cardinality 6
    (1, 3, 5)(2, 4, 6)
    (1, 4)(2, 3)(5, 6)
D_6(6) = [3]2

TransitiveGroup( 6 , 3 ):
Permutation group acting on a set of cardinality 6
    (1, 2, 3, 4, 5, 6)
    (1, 4)(2, 3)(5, 6)
D(6) = S(3)[x]2

TransitiveGroup( 6 , 4 ):
Permutation group acting on a set of cardinality 6
    (1, 4)(2, 5)
    (1, 3, 5)(2, 4, 6)
A_4(6) = [2^2]3
```

```
TransitiveGroup( 6 , 5 ):
Permutation group acting on a set of cardinality 6
    (2, 4, 6)
    (1, 4)(2, 5)(3, 6)
F_18(6) = [3^2]2 = 3 wr 2

TransitiveGroup( 6 , 6 ):
Permutation group acting on a set of cardinality 6
    (3, 6)
    (1, 3, 5)(2, 4, 6)
2A_4(6) = [2^3]3 = 2 wr 3

TransitiveGroup( 6 , 7 ):
Permutation group acting on a set of cardinality 6
    (1, 4)(2, 5)
    (1, 3, 5)(2, 4, 6)
    (1, 5)(2, 4)
S_4(6d) = [2^2]S(3)

TransitiveGroup( 6 , 8 ):
Permutation group acting on a set of cardinality 6
    (1, 4)(2, 5)
    (1, 3, 5)(2, 4, 6)
    (1, 5)(2, 4)(3, 6)
S_4(6c) = 1/2[2^3]S(3)

TransitiveGroup( 6 , 9 ):
Permutation group acting on a set of cardinality 6
    (2, 4, 6)
    (1, 5)(2, 4)
    (1, 4)(2, 5)(3, 6)
F_18(6):2 = [1/2.S(3)^2]2

TransitiveGroup( 6 , 10 ):
Permutation group acting on a set of cardinality 6
    (2, 4, 6)
    (1, 5)(2, 4)
    (1, 4, 5, 2)(3, 6)
F_36(6) = 1/2[S(3)^2]2

TransitiveGroup( 6 , 11 ):
Permutation group acting on a set of cardinality 6
    (3, 6)
    (1, 3, 5)(2, 4, 6)
    (1, 5)(2, 4)
2S_4(6) = [2^3]S(3) = 2 wr S(3)

TransitiveGroup( 6 , 12 ):
Permutation group acting on a set of cardinality 6
    (1, 2, 3, 4, 6)
    (1, 4)(5, 6)
L(6) = PSL(2,5) = A_5(6)

TransitiveGroup( 6 , 13 ):
Permutation group acting on a set of cardinality 6
    (2, 4, 6)
    (2, 4)
    (1, 4)(2, 5)(3, 6)
F_36(6):2 = [S(3)^2]2 = S(3) wr 2

TransitiveGroup( 6 , 14 ):
Permutation group acting on a set of cardinality 6
    (1, 2, 3, 4, 6)
    (1, 2)(3, 4)(5, 6)
L(6):2 = PGL(2,5) = S_5(6)

TransitiveGroup( 6 , 15 ):
Permutation group acting on a set of cardinality 6
    (1, 2)(3, 4, 5, 6)
    (1, 2, 3)
A(6)

TransitiveGroup( 6 , 16 ):
Permutation group acting on a set of cardinality 6
    (1, 2, 3, 4, 5, 6)
    (1, 2)
S(6)
```