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Abstract. Many theorems in the complex plane have analogues in the dual ($x + jy$, $j^2 = 0$) and the double ($x + ky$, $k^2 = 1$) planes. In this paper, we prove that Schwarz reflection principle holds in the dual and the double planes. We also show that in these two planes the domain of an analytic function can usually be extended analytically to a larger region. In addition, we find that a certain class of regions can be mapped conformally to the upper half plane, which is analogous to the Riemann mapping theorem.

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1 Introduction

Analytic functions of a complex variable have a wide range of application in mathematics and in the physical sciences. In mathematics, they have important applications to the fields of algebraic geometry, number theory, and applied mathematics. In fact, one of the earliest uses was to give a proof for the Fundamental Theorem of Algebra. In the sciences, analytic functions are useful to the study of electrostatics, hydrodynamics, and electrical engineering.

In this paper, we continue a project of finding analogues of theorems from complex variables in the study of analytic functions of a dual variable or double variable. Dual numbers $\mathbb{D} = \{z = x + jy : x, y \in \mathbb{R}, j^2 = 0\}$ first appeared in work by the German mathematician Eduard Study (1862-1930) [7, 16]. They have been used in the analysis of robotic controls, in spatial mechanisms, and in virtual reality. Double numbers $\mathbb{P} = \{z = x + ky : x, y \in \mathbb{R}, k^2 = 1\}$ first appeared in work by the English mathematician William Clifford (1845-1879) [7, 3]. They are closely related to Minkowski geometry and have been applied recently in the study of nonlinear dynamics [13].

The study of analytic functions of a dual or double variable began with work by Deakin who formulated the Cauchy-Riemann equations and considered Taylor expansions [4]. He claimed that it is impossible to obtain an analogue of Cauchy's integral formula for the dual or double plane due to the presence of zero divisors. Nevertheless, we are beginning to see that the representation he gives for analytic functions might be considered as a suitable replacement.

DenHartigh and Flim continued the study of analytic functions and proved that Liouville's theorem regarding bounded, entire functions fails in both the dual and double planes [5]. They show that there are essentially two formulations of analyticity in the dual and double planes—Cauchy-Riemann analyticity and power series analyticity, with the latter being stronger. In each case, DenHartigh and Flim construct examples of bounded, entire functions that are non-constant.

In this manuscript we show there are other ways in which analytic functions of a dual or double variable are different from analytic functions of a complex variable. There also are ways in which they are similar.

We begin our study by identifying a convexity condition under which analytic functions on a region can be represented globally using a pair of real valued functions. This is what we call the Deakin representation and is essential to most of our work. Using the Deakin representation, we prove first that the analogue of the Schwarz reflection principle remains true in the dual plane and double plane. We then show that analytic functions of dual or double variable map vertical or diagonal line segments to vertical or diagonal lines segments, respectively. There is no corresponding result for analytic functions of a complex variable. The work on this probem was motivated by a study of domain colorings.

In the subsequent section, we show that regions without the convexity condition do not allow for (global) Deakin representations, and we identify conditions under which analytic functions always can be extended to regions larger than their original domain. This, too, is different from the case of the complex plane, where $f(z) = 1/(z-a)$ cannot be extended analytically past a where a is any boundary point.

In our final section, we describe the relationship between analyticity and conformal maps for functions of a dual or double variable. In both cases, a function is conformal provided it is analytic and its derivative is nowhere equal to a zero divisor. We conclude our study by proving an analogue of the Riemann mapping theorem for the dual plane.

Previous work in the subject concerns the algebra and geometry of the dual and double planes. For this we mention the classic text by Yaglom [17] as well as the recent article by Harkin and Harkin that includes a treatment of roots of unity [7]. Besides this, there has been work done by a number of authors on a special class of analytic functions, namely the linear fractional transformations, and how they relate to the geometry of the Euclidean plane [2, 6, 8, 11].

2 Background

Collectively, we refer to the complex, dual, and double numbers as the generalized complex numbers, and we denote them by $z = x + \ell y$ where $\ell = i, j, k$. Addition and multiplication are defined by

$$(x_1 + \ell y_1) + (x_2 + \ell y_2) = x_1 + x_2 + \ell(y_1 + y_2)$$
$$(x_1 + \ell y_1) \cdot (x_2 + \ell y_2) = x_1 x_2 + \ell^2 y_1 y_2 + \ell(x_1 y_2 + x_2 y_1).$$

With these definitions, multiplicative inverses are given by

$$\frac{1}{x + \ell y} = \frac{x - \ell y}{x^2 - \ell^2 y^2}$$

provided $x^2 \neq \ell^2 y^2$. Notice that dual and double numbers have zero divisors $\{x + jy : x = 0\}$ and $\{x + ky : x = \pm y\}$, respectively. So although the complex numbers form a field, the dual and the double numbers only form commutative rings.

Geometrically, we associate a generalized complex number $z = x + \ell y$ with the point (x, y) in the Cartesian plane. This gives three planes: the complex plane, the dual plane, and the double plane, respectively. The geometry of these planes is reflected in the definitions of modulus and argument. The modulus of $x + \ell y$ is

$$|x + \ell y| = \sqrt{x^2 - \ell^2 y}$$

and the argument is

$$\arg(x + \ell y) = \begin{cases} \arctan(y/x), & \ell = i \\ y/x, & \ell = j \\ \operatorname{arctanh}(y/x), & \ell = k \quad (|y| < |x|) \end{cases}$$

In the case of $|y| > |x|$ for a double number, $\arg(x + ky) = \operatorname{arctanh}(x/y)$. The key properties of modulus and argument are that multiplying generalized complex numbers involves multiplying the modulus and adding the argument. In particular,

$$|z_1 z_2| = |z_1| \cdot |z_2|$$
$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

for $z_1 = x_1 + \ell y_1$, $z_2 = x_2 + \ell y_2$. The latter identity is important for understanding the definition of a conformal map (i.e., angle preserving maps) in the final section. In each of the three cases, the identities can be verified using algebra and an appropriate trigonometric or hyperbolic identity.

It is not essential to our presentation, but we mention that unlike the case for the complex plane, there is not a unique polar representation for a dual or double number. In general, a given modulus and argument determines two dual numbers that differ by a factor of -1. In a similar way, a given modulus and argument determines four distinct double numbers. The interested reader is encouraged to see Harkin and Harkin [7] for additional properties of the these numbers—including roots of unity—that are not essential to this paper.

A helpful way to visualize the three planes is through domain coloring. Each point is assigned a color whose argument determines the hue and whose modulus determines the brightness. The correspondence for the three cases is illustrated in Figure 1. Notice that

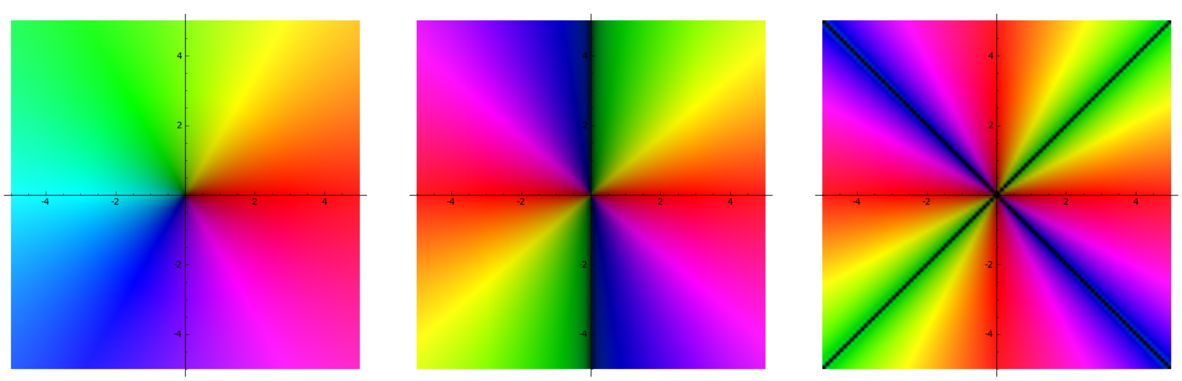

Figure 1: Identity maps of the complex, the dual, and the double planes.

the argument of complex numbers ranges between $-\pi$ and π, while that of dual and double numbers ranges between $-\infty$ and ∞. The generalized complex functions studied in later sections can be visualized using the method of Frank Farris as presented by DenHartigh and Flim. In particular, a drawing is made by everywhere recording the function's value using the color that corresponds with that values hue and brightness. The interested reader is encouraged to generate such pictures of their own or to see examples presented in [5].

3 Analyticity

In this section we explain analyticity and its different formulations in the generalized complex plane. We also examine representations of analytic functions in the dual and double planes that may be considered as analogues for the Cauchy integral formula in the complex plane. To illustrate the effectiveness of these representations, we use them to prove the analogues of the Schwarz reflection principle in the dual and double planes.

3.1 Formulations of analyticity

We consider a function of a generalized complex variable, $f(x+\ell y) = u+\ell v$ and $z = x+\ell y$. Let D be an open, connected region in the appropriate plane. DenHartigh and Flim [5] considered the following formulations of analyticity for a function defined on D:

1. *Differentiability*: For all $z \in D$, the limit $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists.
2. *Cauchy-Riemann Equations*: Partial derivatives of f satisfy $u_x = v_y$ and $u_y = \ell^2 v_x$ in D.
3. *Power Series Representation*: For each $z_0 \in D$, there exists $r > 0$ such that $f(z) = \sum_0^\infty c_n(z-z_0)^n$ for all $z \in D_r(z_0)$.

It is standard that in the complex plane, each of the three formulations implies the other two. See Mathews and Howell [10], for instance, for more detail.

In the dual and double planes, however, the formulations are not equivalent. In particular, the first formulation is stronger. So a function that is analytic according to the power series formulation must also be analytic according to the Cauchy-Riemann and differentiability formulations. The converse of this statement is not true. In particular, $f(z) = z^{10/3} = x^{10/3} + j(y\frac{10}{3}x^{7/3})$ serves as a counterexample for the dual plane. Similarly,

$$f(z) = z^{10/3} = \frac{(x+y)^{10/3}}{2} + \frac{(x-y)^{10/3}}{2} + k\Big[\frac{(x+y)^{10/3}}{2} - \frac{(x-y)^{10/3}}{2}\Big]$$

serves as a counterexample for the double plane.

Meanwhile, the Cauchy-Riemann and differentiability formulations are equivalent. For this latter claim it is assumed that the derivatives u_x, u_y, u_{xx}, v_x, v_y exist in the dual plane, and the derivatives u_x, u_y, v_x, v_y exist in the double plane. These results are proved by DenHartigh and Flim in [5, §4].

In what follows we assume that the derivatives mentioned in the last paragraph exist, so that the Cauchy-Riemann and differentiability formulations are equivalent.

A crucial ingredient that is used to prove the equivalence of the two formulations is the local representation of an analytic function that was observed earlier by Deakin [4] and which we are beginning to see as a natural analogue for the Cauchy integral formula. We discuss this representation in §§3.2 and show that for a large class of regions the representation is global.

3.2 Convex Regions and Global Deakin Representations

Here we introduce types of convexity for regions in the dual and the double planes to describe the situation where functions can be extended analytically.

Definition 3.1 (Vertically Linearly Convex)**.** In the dual plane, a region is vertically linearly convex (VLC) provided its intersection with every vertical line that intersects the region is connected. Examples are shown in Figure 2.

Figure 2: Examples of VLC and non-VLC regions

Our first result says that analytic functions of a dual variable have a unique global representation in the manner of Deakin provided the region is vertically linearly convex.

Theorem 3.2. *In the dual plane, let $f = u + jv$ satisfy the Cauchy-Riemann equations on a VLC region D. Then there exist unique real-valued functions α, β for which*

$$f(x + jy) = \alpha(x) + j(y\alpha'(x) + \beta(x)) \tag{1}$$

on D. The right-hand side of (1) *defines the unique analytic extension of f to the smallest vertical strip that contains D.*

Proof. Let $\{(x, y) : a < x < b\}$ be the smallest vertical strip that contains D and take $x_0 \in (a, b)$. The intersection of D with the line $x = x_0$ is a connected vertical line segment. Since $u_y = 0$ it follows that $u(x, y)$ is constant on the intersection. By considering all $x_0 \in (a, b)$, we see that $u(x, y) = \alpha(x)$ for some α defined on (a, b). Meanwhile, $v_y = u_x$. Taking again the intersection of D with the line $x = x_0$ we see that $v(x_0, y) = y\alpha'(x_0) + \beta(x_0)$ with $\beta(x_0)$ the constant of integration. So considering all $x_0 \in (a, b)$, we see that $v(x, y) = y\alpha'(x) + \beta(x)$ for some function β on (a, b). The fact that α is twice differentiable and β is differentiable follows from the existence of partial derivatives for u and v. The second claim of the theorem follows easily from the first claim. In particular, the right-hand side of (1) satisfies the Cauchy-Riemann equations. By the argument already given, any analytic function in the vertical strip has a unique representation as in (1). Clearly the representation must involve the same α and β. □

Definition 3.3 (Diagonally Linearly Convex)**.** In the double plane, a region is diagonally linearly convex (DLC) provided its intersection with every line of slope ± 1 that intersects the region is connected. Examples are shown in Figure 3.

Figure 3: Examples of DLC and non-DLC regions

Our next result says that analytic functions of a double variable, too, have a unique global representation in the manner of Deakin provided the region is diagonally linearly convex.

Theorem 3.4. *In the double plane, let $f = u + kv$ satisfy the Cauchy-Riemann equations on a DLC region D. Then there exist unique real valued functions, α and β for which*

$$f(x + ky) = \alpha(x + y) - \beta(x - y) + k(\alpha(x + y) + \beta(x - y)) \tag{2}$$

on D. The right-hand side of (2) *defines the unique analytic extension of f to the smallest rectangle with sides of slope ± 1 that contains D.*

Proof. We start by defining

$$\begin{aligned} a &= \inf\{x + y : z = x + ky \in D\}, \quad b = \sup\{x + y : z = x + ky \in D\}, \\ c &= \inf\{x - y : z = x - ky \in D\}, \quad d = \sup\{x - y : z = x + ky \in D\} \end{aligned}$$

so that the smallest rectangle with sides of slope ± 1 that contains D is $\{(x, y) : a < x + y < b,\ c < x - y < d\}$. This is illustrated in Figure 4.

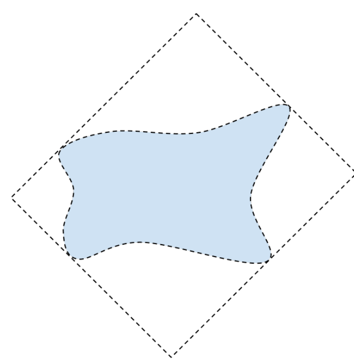

Figure 4: DLC Extension in the double plane

Take $x_0 \in (a, b)$. The intersection of D with line $x + y = x_0$ is connected. The derivative of $u + v$ in the direction of this line is zero since $(u + v)_x - (u + y)_y = 0$ by Cauchy-Riemann equations. It follows that $u + v$ is constant on the intersection. By considering all $x_0 \in (a, b)$ we see that $u(x, y) + v(x, y) = 2\alpha(x + y)$ for some α defined on (a, b). By considering the intersection of D with line $x - y = x_0$, one can show in the same manner that $u(x,y) - v(x,y) = -2\beta(x-y)$ for some β defined on (c,d). Solving for $u(x,y)$ and $v(x,y)$ gives (2). Once again, the second claim of the theorem follows easily from the first claim, which can be shown in a way similar to the proof for VLC extension. $\square$

We call the representations given in (1) and (2) *Deakin representations* for analytic functions in the dual and the double planes. For non-VLC and non-DLC regions, there do not always exist α and β such that (1) and (2) hold across the whole region. An example of this is provided in §4.

3.3 Schwarz Reflection Principle

To show the usefulness of Deakin representations, we use them to prove analogues for the Schwarz reflection principle in the dual and the double planes. The goal of the theorem is to extend the domain of an analytic function to include its reflection across the real axis. In the complex plane, the proof requires the Cauchy integral formula (Lang [9, XII.1]). In the dual and the double planes, we instead use the Deakin representations in the dual and the double planes.

Theorem 3.5 (Schwarz Reflection Principle - Dual and Double Planes)**.** *Let D be a simply connected open region and R the real axis such that $D \cap R = \emptyset$ and $\overline{D} \cap R = I \neq \emptyset$. Let $D^* = \{z : \overline{z} \in D\}$. Suppose that f is Cauchy-Riemann analytic in D, and extends continuous and is real-valued on I. If F is defined by*

$$F(z) = \begin{cases} f(z) & z \in D \cup I \\ \overline{f(\overline{z})} & z \in D^* \end{cases},$$

then F is Cauchy-Riemann analytic in $D \cup I \cup D^*$ (Figure 5).

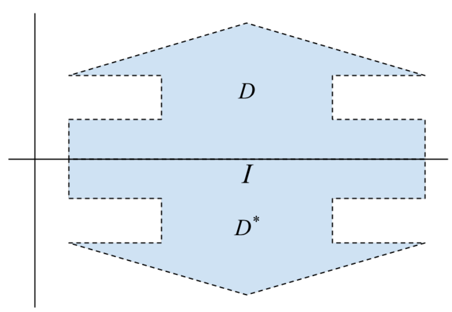

Figure 5: Schwarz reflection principle

Proof. Consider first the situation in the dual plane. Let $z = x + jy$ and $f(x+jy) = u(x,y) + jv(x,y)$. Denote the extended function by $F(x+jy) = U(x,y) + jV(x,y)$. For $z \in D^*$ we then have $F(x+jy) = u(x,-y) - jv(x,-y)$. Using the chain rule we find that

$V_y = -v_y(-1) = u_x = U_x$ and $U_y = u_y(-1) = 0$. So F satisfies Cauchy-Riemann equations in D^*.

We have left to check that F satisfies the Cauchy-Riemann equations in a neighborhood of I. For this, suppose that Δ is an open disk that intersects I and is contained in $D \cup I \cup D^*$. Then $D \cap \Delta$ is VLC and there exists a Deakin representation for f on $\Delta \cap D$. In particular,

$$F(x + jy) = f(x + jy) = \alpha(x) + j(y\alpha'(x) + \beta(x))$$

for $x + jy \in D \cap \Delta$. Next, since f extends continuously to I and takes real values on I, it is necessary that $\beta(x) = 0$ for $x + jy \in \Delta$. Then for $z \in D^* \cap \Delta$,

$$F(x + jy) = \overline{f(\overline{z})} = \alpha(x) - j(-y\alpha'(x)).$$

It follows that $F(z) = \alpha(x) + jy\alpha'(x)$ for all $z \in \Delta$. So F satisfies the Cauchy-Riemann equations in $D \cup I \cup D^*$.

The situation in the double plane is much the same. As before, one uses the chain rule to show that F satisfies the Cauchy-Riemann equations in D^*. To show that F satisfies the Cauchy-Riemann equations near I, one begins with a Deakin representation for f in the DLC region $D \cap \Delta$. That is,

$$F(x + ky) = f(x + ky) = \alpha(x + y) - \beta(x - y) + k(\alpha(x + y) + \beta(x - y))$$

for $z \in D \cap \Delta$. Since f extends continuously to I and takes real values on I, it is necessary that $\beta = -\alpha$ for $x + ky \in \Delta$. But then for $z \in D^* \cap \Delta$,

$$F(x + ky) = \overline{f(\overline{z})} = \alpha(x - y) + \alpha(x + y) - k(\alpha(x - y) - \alpha(x + y)).$$

It follows that $F(z) = \alpha(x + y) + \alpha(x - y) + k(\alpha(x + y) - \alpha(x - y))$ for all $z \in \Delta$. So F satisfies Cauchy-Riemann equations in $D \cup I \cup D^*$. $\square$

The Schwarz reflection principle remains true if we change Cauchy-Riemann analytic to power series analytic in both the hypothesis and the conclusion. The proof requires the additional step of showing that F is power series analytic on D^* and I. To do this requires Theorem 3.8 that is proved in §§3.5.

The Schwarz reflection principle is an example of analytic extension. We return to this more general topic in §4.

3.4 Further Consequences of the Deakin Representation

Studies of domain colorings for analytic functions of a dual variable suggest that they map vertical lines to vertical lines. For example, a coloring for $f(z) = \exp z$ is shown in Figure 6. (The exponential and sine functions are defined using power series.) Notice that points on the same vertical line all appear to have the same brightness. Since the brightness records

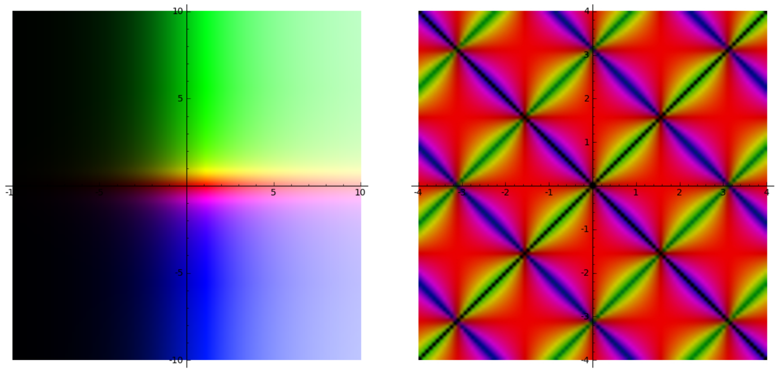

Figure 6: $\exp z$ in the dual plane and $\sin z$ in the double plane.

the modulus and since the modulus of a dual number is $|z| = |x|$, it follows that vertical lines are mapped to vertical lines. A statement and proof for the precise result is given below.

The corresponding result for the double plane is that analytic functions map diagonal lines to diagonal lines. This is harder to see in the colorings, but a coloring for $f(z) = \sin z$ that also is shown in Figure 6 reveals at least that the lines $y = \pm x + c$ for certain values of c are mapped to the lines $\pm x$. Since the modulus of a double number is the more complicated expression $|z| = |x^2 - y^2|^{1/2}$ it is harder to see the result more generally. Again, a statement and proof for the more precise result is given below.

Theorem 3.6. *Let $f = u + jv$ satisfy the Cauchy-Riemann equations on a region $D \subset \mathbb{D}$. Then f maps vertical line segments to (subsets of) vertical line segments.*

Proof. For a vertical line segment L contained in D, one always can find an open set in D that is VLC and that contains L. By Theorem 3.2 there is a real function $\alpha = \alpha(x)$ such that $u(x, y) = \alpha(x)$ on this open set. It follows that points on L (that have the same x-coordinate) are mapped to points of a vertical line segment (that have the same u-coordinate). $\square$

Theorem 3.7. *Let $f = u + kv$ satisfy the Cauchy-Riemann equations on a region $D \subset \mathbb{P}$. Then f maps line segments with slope ± 1 to (subsets of) line segments with slope ± 1, respectively.*

Proof. Consider a line segment L contained in D that has slope ± 1. One always can find an open set in D that is DLC and that contains L. By Theorem 3.4 there are real functions α and β such that

$$u(x, y) - v(x, y) = -2\beta(x - y)$$
$$u(x, y) + v(x, y) = +2\alpha(x + y)$$

on this open set. From the first of these equation we see that if L has slope $+1$, so that $x - y$ is constant, then L is mapped to a line segment with slope $+1$ where $u - v$ is constant.

Likewise, from the second of these equation we see that if L has slope -1, so that $x+y$ is constant, then L is mapped to a line segment with slope -1 where $u+v$ is constant. $\square$

It would be interesting to formulate converse statements to the two results. For instance, what additional conditions are needed in order for a function that maps vertical lines to vertical lines to be analytic? We leave this for question for later research.

3.5 Power Series Analyticity

We can say more about the Deakin representation of a power series analytic function. If f is power series analytic, then not only does it have a Deakin representation, but the functions α, β that are used in representing f also can be expressed using power series. The converse is also true.

Theorem 3.8. *A function f is power series analytic on an open, connected region D if and only if functions α and β that are used in its Deakin representation can be expressed using power series.*

Proof. Assume f has a power series representation centered at (x_0, y_0). Let $\Delta x = x - x_0$ and $\Delta y = y - y_0$. In the case of the dual plane,

$$\begin{aligned} f(x+jy) &= \sum_0^\infty (a_n + jb_n)(\Delta x + j\Delta y)^n \\ &= \sum_0^\infty (a_n + jb_n)\big[(\Delta x)^n + j(\Delta y)n(\Delta x)^{n-1}\big] \\ &= \sum_0^\infty a_n(\Delta x)^n + j\big[yna_n(\Delta x)^{n-1} - y_0na_n(\Delta x)^{n-1} + b_n(\Delta x)^n\big] \\ &= \sum_0^\infty a_n(\Delta x)^n + j\big[yna_n(\Delta x)^{n-1} + d_n(\Delta x)^n\big] \end{aligned}$$

where $d_n = -y_0(n+1)a_{n+1} + b_n$. Thus $f(z) = \alpha(x) + j[y\alpha'(x) + \beta(x)]$ where $\alpha(x) = \sum_0^\infty a_n(x-x_0)^n$ and $\beta(x) = \sum_0^\infty d_n(x-x_0)^n$.

Likewise, in the case of the double plane,

$$\begin{aligned} f(z) &= \sum_0^\infty (a_n + kb_n)(\Delta x + k\Delta y)^n \\ &= \sum_0^\infty (a_n + kb_n)\left[\frac{(\Delta x + \Delta y)^n + (\Delta x - \Delta y)^n}{2} + k\frac{(\Delta x + \Delta y)^n - (\Delta x - \Delta y)^n}{2}\right] \\ &= \sum_0^\infty c_n(\Delta x + \Delta y)^n + d_n(\Delta x - \Delta y)^n + k[c_n(\Delta x + \Delta y)^n - d_n(\Delta x - \Delta y)^n] \end{aligned}$$

where $c_n = \frac{1}{2}(a_n + b_n)$ and $d_n = \frac{1}{2}(a_n - b_n)$. Thus, $f(z) = \alpha(x+y) - \beta(x-y) + k[\alpha(x+y) + \beta(x-y)]$ where $\alpha(x+y) = \sum_0^\infty c_n[x+y-(x_0+y_0)]^n$ and $\beta(x-y) = -\sum_0^\infty d_n[x-y-(x_0-y_0)]^n$

All the steps of the above proof can be reversed and so the converse statement is true, too. □

Deakin [4] has given a proof for the double case, but for completeness we include the proof here, too. Notice that power series analytic functions have unique Deakin representations regardless of the shape of the domain. We examine this in more detail in §4.

4 Analytic Extension

In this section we consider the general problem of analytic extension. In particular, for a given region D_1 does there exist a larger region $D_2 \supsetneq D_1$ so that functions that are analytic on D_1 always can be extended analytically to D_2? For this one must show that for every function f_1 that is analytic on D_1, there is a function f_2 that is analytic on D_2 such that $f_1 = f_2$ on D_1.

By considering functions with isolated singularities, i.e., $f(z) = 1/(z-a)$ for various $a \in \mathbb{C}$, there is no such analytic extension in the complex plane. In particular, if $a \in D_2 \backslash D_1$, then $f(z) = 1/(z-a)$ is analytic on D_1 but cannot be extended analytically to D_2. Theorem 3.2, however, showed that for a large class of regions in the dual plane there is analytic extension to a vertical strip.

Here, we continue with the study of analytic extension in the dual plane. We begin by giving an example to demonstrate that the VLC condition is necessary if there is to be extension to a vertical strip. We then introduce a class of non-VLC regions for which there is analytic extension above and below the region except for finitely many lines of discontinuity. These results apply to the case of Cauchy-Riemann analyticity. We conclude by showing that for power series analyticity, there always is analytic extension to the vertical strip.

Example 4.1. Let $D \subset \mathbb{D}$ be the horseshoe region shown in Figure 7 and let f be defined by

$$f(z) = \begin{cases} +z^3 & \text{if } x < 0, y > 0 \\ -z^3 & \text{if } x < 0, y < 0 \\ 0 & \text{if } x \geq 0 \end{cases}.$$

It is clear from the figure that D is not a VLC region. Nevertheless, f is differentiable on D and

$$f'(z) = \begin{cases} +3z^2 & \text{if } x < 0, y > 0 \\ -3z^2 & \text{if } x < 0, y < 0 \\ 0 & \text{if } x \geq 0 \end{cases}.$$

(The fact that f is differentiable at the points in D on the imaginary axis uses the fact that $\pm 3z^2$ evaluates to zero on those intervals.)

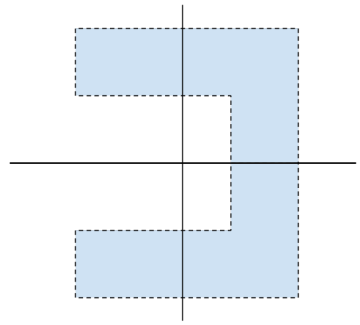

Figure 7: A non-VLC region for which not all analytic functions can be extended to a vertical strip

To see that f cannot be extended to a vertical strip containing D we consider the Deakin representation. Notice that

$$f(x+jy) = \begin{cases} \alpha_+(x) + jy\alpha_+'(x) & \text{for } y \geq 0 \\ \alpha_-(x) + jy\alpha_-'(x) & \text{for } y < 0 \end{cases}$$

where

$$\alpha_\pm(x) = \begin{cases} \pm x^3 & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}.$$

Since $\alpha_+(x) \neq \alpha_-(x)$ for $x < 0$ it follows there is no global Deakin representation for f on D. Theorem 3.2 shows that f cannot be extended analytically to any vertical strip containing D.

To understand further the behavior for Cauchy-Riemann analytic functions, we restrict to a special class of regions. Let $D \subset \mathbb{D}$ be a region that is not VLC. We construct from D a set D^{top} that is the union of the top interval from each intersection of D with a vertical line. (Notice that $D = D^{top}$ precisely when D is VLC.) We then say that D is *finitely non-VLC* provided that D^{top} has finitely many components. Notice that the region in Figure 7 is finitely non-VLC. With this condition on D, our next result says that Cauchy-Riemann analytic functions always can be extended analytically to a larger region.

Theorem 4.2. *Let D be a finitely non-VLC region in the dual plane. Then any analytic function defined on D can be extended analytically to the unbounded region above D with at most finitely many vertical lines of discontinuity.*

Proof. Let f be analytic on a finitely non-VLC region D. By definition, the set D^{top} has finitely many components. In general, D^{top} might not be open and it might not be connected. Figure 8 gives an example of the construction when D^{top} is neither open nor connected. Nevertheless, D^{top} must be a union of finitely many components whose interior regions are

Figure 8: A finitely non-VLC region D and its subset D^{top} of top intervals.

vertically linearly convex. Moreover, the interior regions belong to nonintersecting open vertical strips.

We next apply Theorem 3.2 to f restricted to each of the interior regions and obtain an analytic extension to the part of the vertical strip that lies above the corresponding interior region. (We ignore the part below the interior region because that part may include points of D that are not in D^{top}.) This defines the analytic extension of f to the region above D with the exception of finitely many vertical lines that correspond with the components of D^{top}. □

Notice that it is not necessary for the extension to have a vertical line of discontinuity. In fact, the proof shows that whenever D^{top} is connected, as it is in Example 4.1, there are no lines of discontinuity.

In addition, there always exist some analytic functions for which the extension is continuous. Our next result shows that functions that are power series analytic always extend continuously. In fact, for general regions, they can be extended analytically to the smallest vertical strip that contains the region.

Theorem 4.3. *If f is power series analytic on a region $D \subset \mathbb{D}$, then f has a global Deakin representation on D. In particular, f has a unique extension to the smallest vertical strip containing D that is power series analytic.*

Proof. Let $\{(x, y) : x_0 < x < x_1\}$ be the smallest vertical strip that contains D. It is enough to show that there exist unique real power series functions α, β defined on (x_0, x_1) for which

$$f(x + jy) = \alpha(x) + j(y\alpha'(x) + \beta(x)) \tag{3}$$

on D. Then the right-hand side of (3) defines the power series analytic extension on the vertical strip.

To obtain α and β, let $R = (a, b) \times (c, d)$ be any rectangle contained in D. Since f is power series analytic on R, there exist unique real power series functions α, β defined on (a, b) such that

$$f(x + jy) = \alpha(x) + j(y\alpha'(x) + \beta(x))$$

on R. This determines the functions α, β on a subinterval $(a, b) \subset (x_0, x_1)$.

To see that the definition is unique and does not depend on the choice of R, consider two overlapping rectangles

$$R_1 = (a_1, b_1) \times (c_1, d_1)$$
$$R_2 = (a_2, b_2) \times (c_2, d_2).$$

These determine real power series functions α_1, β_1 and α_2, β_2 on subintervals (a_1, b_1) and (a_2, b_2), respectively, as in the last paragraph. We will show that these functions piece together to give real power series functions on the union of the two intervals. For this we consider the intersection $R_3 = R_1 \cap R_2$ that is itself a rectangle $R_3 = (a_3, b_3) \times (c_3, d_3)$ where

$$a_3 = \max(a_1, a_2),\ b_3 = \min(b_1, b_2),\ c_3 = \max(c_1, c_2),\ d_3 = \min(d_1, d_2).$$

This rectangle determines real power series functions α_3, β_3 on $(a_3, b_3) = (a_1, b_1) \cap (a_2, b_2)$ that must be the restrictions of α_1, β_1 and α_2, β_2 from their respective intervals. It is immediate that these functions piece together to give real power series functions on $(a_1, b_1) \cup (a_2, b_2)$. More importantly, they are the only possible real power series extensions from the respective intervals. (See Rudin [15, p.177].)

By considering all possible rectangles in D one observes that there are unique real power series functions α, β defined on (x_0, x_1) for which (3) holds. $\square$

The results of this section have natural analogues for the case of the double plane. For example, one can rotate the region in Example 4.1 by 45 degrees counterclockwise and define f according to

$$f(x + ky) = \begin{cases} \alpha_+(x + y) + k\alpha_+(x + y) & \text{for } x - y \geq 0 \\ \alpha_-(x + y) + k\alpha_-(x + y) & \text{for } x - y < 0 \end{cases}$$

where $\alpha_\pm$ also were defined in Example 4.1. This gives a Cauchy-Riemann analytic function that cannot be extended analytically to the smallest rectangle containing the region. Similarly, one can define a condition called *finitely non-DLC* such that analytic functions can be extended outward to the smallest rectangle containing the region with the exception of finitely many lines of discontinuity. Finally, power series analytic functions always can be extended to the smallest rectangle with sides of slope ± 1 that contains the region. The proofs are similar to the proofs already given.

5 Conformal mapping

In this section we study conformal maps in the generalized complex plane. We show first that a map is conformal provided that it can be represented by an analytic function with derivative that is not a zero divisor or zero. This is analgous to the situation in the complex plane. Subsequently, our main result is an analogue of the Riemann mapping theorem for the dual plane.

5.1 Conformal Maps In the Generalized Complex Plane

By a conformal map we mean an injective map of a region in the complex, dual, or double plane that preserves angles of intersection. In all three cases, angles are measured as the difference in argument between tangent vectors at a point of intersection of two curves. This means that in the complex plane the angles are Euclidean. In the dual plane the angle is a difference of slope. In the double plane the angles are hyperbolic. The precise definition of argument was given in §2.

It is standard in complex analysis that conformal maps correspond with analytic functions whose derivatives are nonzero. To be precise, an analytic function $w = f(z)$ with nonzero derivative gives a conformal map near any point of its domain. (It may be necessary to restrict to a small neighborhood of the point in order for the map to be injective.) We begin by showing that the same result holds for analytic functions of a dual or double variable provided the derivative is not a zero divisor.

The proof for the double case is nearly identical to the proof for the complex case as found in Ahlfors [1, p.74]. To handle both we introduce operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{\ell}\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{\ell}\frac{\partial}{\partial y}\right)$$

for $\ell = i, k$. Then f satisfies the Cauchy-Riemann equations precisely when $\partial f/\partial\overline{z} = 0$, and in this case, $f'(z) = \partial f/\partial z$.

Furthermore, if $z(t)$ is a curve in the plane, then its image under a map $w = f(z)$ is the curve $w(t) = f(z(t))$, and the tangent vector at $w(t_0)$ is given by

$$w'(t_0) = \frac{\partial f}{\partial z}\, z'(t_0) + \frac{\partial f}{\partial \overline{z}}\, \overline{z'(t_0)}. \tag{4}$$

For angles to be preserved we need $\arg[w'(t_0)/z'(t_0)]$ to be independent of $\arg z'(t_0)$. That is,

$$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}}\, \frac{\overline{z'(t_0)}}{z'(t_0)}$$

must have constant argument independent of $\arg z'(t_0)$. This requires $\partial f/\partial\overline{z} = 0$ and therefore f satisfies the Cauchy-Riemann equations.

It remains to be seen that $f'(z)$ is not a zero divisor or zero. But this is apparent from (4). For if $f'(z)$ is a zero divisor then f maps all tangent vectors to vectors that are zero divisors. Likewise, if $f'(z) = 0$ then f maps all tangent vectors to zero. In neither case are angles preserved.

For the dual case, let $z(t) = x(t) + jy(t)$ be a curve in the plane and consider the map $w = f(z)$. By the chain rule, the tangent vector $z'(t_0) = x'(t_0) + jy'(t_0)$ at $z(t_0)$ is mapped to tangent vector

$$w'(t_0) = \left(\frac{\partial u}{\partial x}\, x'(t_0) + \frac{\partial u}{\partial y}\, y'(t_0)\right) + j\left(\frac{\partial v}{\partial x}\, x'(t_0) + \frac{\partial v}{\partial y}\, y'(t_0)\right)$$

at $w(t_0)$. For angles to be preserved at $z(t_0)$ it is necessary that the slope of these tangent vectors differ by a constant that is independent of $z'(t_0)$. So

$$\frac{v_x\, x'(t_0) + v_y\, y'(t_0)}{u_x\, x'(t_0) + u_y\, y'(t_0)} - \frac{y'(t_0)}{x'(t_0)}$$

is constant, independent of $x'(t_0)$ and $y'(t_0)$. It is easy to verify algebraically that this condition requires $u_y = 0$ and $u_x = v_y$, so f satisfies the Cauchy-Riemann equations. For the same reason as before it is necessary that the derivative is not a zero divisor or zero.

5.2 Riemann Mapping Theorem For the Dual Plane

Recall that for the complex plane, the Riemann mapping theorem says that any simply connected region that is not the entire plane, can be mapped conformally to the upper half plane (or unit disc) in an essentially unique way. In determining an analogue of this statement for the dual plane, it becomes apparent that one must restrict to a different class of regions. In particular, by considering the behavior of analytic functions on vertical line segments, it is apparent that a conformal image of the upper half-plane must be unbounded above or below. We have the following.

Theorem 5.1 (Riemann Mapping Theorem - Dual Plane)**.** *Every conformal image of the upper half plane is a region that is bounded above or below by the graph of a differentiable function on an open interval. In addition, any such region is the image of the upper half plane under a conformal map.*

Proof. Let $F(z)$ be a conformal mapping on the upper half-plane. Then $F(x+jy) = \alpha(x) + j(y\alpha'(x)+\beta(x))$ for differentiable functions α, β with $\alpha' \neq 0$. It is necessary that α' is always positive or always negative. (See Ross [14, p.217].) Let's suppose that α' is always positive. Then α is invertible and takes values in the open interval (a, b) where $a = \lim_{x\to-\infty} \alpha(x)$ and $b = \lim_{x\to+\infty} \alpha(x)$. (It is possible that the interval is infinite; i.e., if $a = -\infty$ or $b = +\infty$.) We will show that the image of the upper half plane $H^+ = \{(x, y) : x \in \mathbb{R},\, y > 0\}$ under F is exactly the set $R^+ = \{(u, v) : u \in (a, b),\, v > (\beta \circ \alpha^{-1})(u)\}$ where $\beta \circ \alpha^{-1}$ is differentiable. Using coordinate functions $u(x, y) = \alpha(x)$, $v(x, y) = y\alpha'(x)+\beta(x)$ this is easy. For if $(x, y) \in H^+$, then $u = \alpha(x) \in (a, b)$ and $v = y\alpha'(x)+\beta(x) > \beta(x) = (\beta\circ\alpha^{-1})(u)$ so that $(u, v) \in R^+$. Likewise, if $(u, v) \in R^+$, then $x = \alpha^{-1}(u)$ and $y = (v - (\beta \circ \alpha^{-1})(u))/\alpha'(\alpha^{-1}(u)) > 0$. So the first claim is proved in the case α' is positive. The case α' is negative is handled similarly and results in a region bounded above by the graph of the differentiable function $\beta \circ \alpha^{-1}$.

For the second claim, let f be differentiable on the open interval $I = (a, b)$ and let $R^\pm = \{(x, y) : x \in I, y \gtrless f(x)\}$. We must show there is a conformal map $F^\pm(z)$ such that

$F^{\pm}(H^{+}) = R^{\pm}$. We will show that if

$$G_1(z) = \begin{cases} \frac{a+b}{2} + \frac{a-b}{\pi}\tan^{-1} z & a > -\infty,\ b < +\infty \\ b - e^{-z} & a = -\infty,\ b < +\infty \\ a + e^{z} & a > -\infty,\ b = +\infty \\ z & a = -\infty,\ b = +\infty \end{cases}$$

and $G_2(z) = x + j(y + f(x))$, then $F^{\pm}(z) = (G_2 \circ G_1)(\pm z)$ has the desired properties. As written, $F^{\pm}$ is a composition of three functions. It can be readily checked that each of them is analytic on its domain and has derivative that is nowhere a zero divisor. So $F^{\pm}$ is conformal. It remains to be seen that $F^{\pm}(H^{+}) = R^{\pm}$. This is straightforward. As can be checked in all four cases, the image of H^{+} under $G_1(+z)$ is exactly the region above the real interval $I = (a, b)$. In order for $F^{+}(H^{+}) = R^{+}$, it then is enough to observe that under the map $G_2(z)$, i.e., under $u = x$ and $v = y + f(x)$, the set $a < x < b$, $y > 0$ corresponds exactly with the set $a < u < b$, $v > f(u)$. That $F^{-}(H^{+}) = R^{-}$ is handled similarly. The only difference is that the image of H^{+} under $G_1(-z)$ is exactly the region *below* the real interval $I = (a, b)$. $\square$

To conclude, we mention that the map from the upper half-plane to a given region that is bounded above or below by a differentiable function is not unique. In fact, there are many such maps because there are many conformal maps from the upper half-plane to itself. Such self-maps of the upper half-plane are called automorphisms. All automorphisms of the upper half-plane can be written as $f(z) = \alpha(x) + jy\alpha'(x)$ where $\alpha(x)$ is a real function with positive derivative and range $\mathbb{R}$. This is an application of the Deakin representation for the upper half-plane.

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