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Eric Wepsic
*Rose-Hulman Institute of Technology*

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Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute, IN 47803

FAX(812) 877-3198 Phone: (812) 877-8391

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Let

$$Pr_3(G) = \frac{|\{(x,y,z) : xyz \in \{xzy, yxz, yzx, zxy, zyx\}|}{|G|^3}.$$

This number is the proportion of triples of elements of $G$ which are 3–rewritable. Ellenberg [1] has shown that

$$Pr_3(G) \in (0, 17/18] \cup \{1\},$$

where $Pr_3(G) = 17/18$ iff $G/Z \cong S_3$. The proof, which is long and difficult, involves the formula

$$Pr_3(G) = \frac{5k - 8s + 4t}{|G|}, \tag{1}$$

with

- $k$ the number of conjugacy classes,
- $s$ the sum of reciprocals of conjugacy class sizes,
- $t$ the number of mutually commuting triples divided by $|G|^2$, that is,

$$t = \frac{|\{(x,y,z) \in G^3 : xy = yx,\ xz = zx,\ yz = zy\}|}{|G|^2}.$$

Here, we investigate some simpler variations on this theme, namely, we find an upper bound for $Pr_3$ for simple groups, and find a bound for $Pr_{3C}$, the core set rewritability, which is a different measure of 3–rewritability.

**Lemma 1** *For any group, $s \geq t$.*

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Proof. Let $K(G)$ denote the set of conjugacy classes of $G$, $k(g)$ represent the conjugacy class containing $g$, and $c_g = |k(g)|$. Finally, denote by $\mathbf{C}_G(g)$ the centralizer of $g$ in $G$. Then

$$\begin{aligned} s &= \sum_{k(g)\in K(G)} \frac{1}{c_g} = \sum_{g\in G} \frac{1}{c_g^2} = \sum_{g\in G} \frac{\mathbf{C}_G(g)^2}{|G|^2} \\ &= \frac{1}{|G|^2} \sum_{g\in G} \mathbf{C}_G(g)^2. \end{aligned}$$

We claim that this sum overcounts the number of mutually commuting triples. As one can see, it actually counts the number of triples $(g, x, y)$ so that $x, y \in \mathbf{C}_G(g)$: that is, $g$ commutes with both $x$ and $y$. This is a superset of the set of mutually commuting triples, so $s \geq t$. $\heartsuit$

**Theorem 1** *For finite simple groups $G$,*

$$Pr_3(G) \in (0, 27/100] \cup \{1\},$$

*where $Pr_3(G) = 27/100$ if and only if $G \cong A_5$.*

Proof. (modelled after [4]) Assume $Pr_3(G) \geq 27/100$. Since $s \geq t$,

$$Pr_3(G) = \frac{5k - 8s + 4t}{|G|} \leq \frac{5k - 4s}{|G|} < \frac{5k}{|G|},$$

so we have $k \geq 27|G|/500$. Let $\chi_1, \ldots, \chi_k$ denote the irreducible characters of $G$, and $d_1 \leq d_2 \leq \ldots \leq d_k$ the dimensions of the associated representations. Certainly, $d_1 = 1$, since $\chi_1$ is the trivial character. Because $G$ is simple, its irreducible representations are faithful, and, in particular, $\chi_2$ is faithful. We now have two cases based on $d_2$.

If $d_2 \leq 4$, then $G$ is isomorphic to a group of $2 \times 2$, $3 \times 3$, or $4 \times 4$ matrices over $\mathbf{C}$. The only such simple groups are $A_5$ and $PSL_2(\mathbf{F}_7)$ [5]. But $Pr_3(PSL_2(\mathbf{F}_7)) \leq 5k/168 < 30/168 < 27/100$, so we must have $G \cong A_5$.

If, on the other hand, $d_2 \geq 5$, then

$$|G| = d_1^2 + \ldots + d_k^2 \geq 1 + 25(k - 1).$$

Hence $|G| \leq 480/7 < 69$, so $G \cong A_5$.$\heartsuit$

It turns out that a group is 3-rewritable just in case it is rewritable with respect to certain subsets of $S_3$. We prove this below.

**Proposition 1** *The following are equivalent:*

1. $xyz \in \{xzy, yxz, yzx, zxy, zyx\}$ *for all* $(x, y, z) \in G^3$;
2. $xyz \in \{xzy, yzx, zyx\}$ *for all* $(x, y, z) \in G^3$;
3. $xyz \in \{yxz, zxy, zyx\}$ *for all* $(x, y, z) \in G^3$;

Proof. It suffices to show that $1 \Rightarrow 2$: the other parts are symmetric or trivial. Assume that $G$ is 3-rewritable. Then [2] the derived group $G'$ has $|G'| = 2$, and $G' \subseteq Z(G)$. Let $\alpha = [x, y] = x^{-1}y^{-1}xy$, $\beta = [x, z]$, $\gamma = [y, z]$. Then $xzy = \gamma xyz$, $yzx = \alpha\beta xyz$. $zyx = \alpha\beta\gamma xyz$, One of $\alpha\beta$, $\gamma$, $\alpha\beta\gamma$ must be the identity, since all of them are in $G'$, which has only one non-identity element. Therefore, one of $zyx, xyz, yzx$ is equal to $xyz$, as desired.♡

The set $\{xzy, yzx, zyx\}$ of permutations is called a **core set** for 3-rewritability. Let

$$Pr_{3C}(G) = \frac{|\{(x, y, z) : xyz \in \{xzy, yzx, zyx\}|}{|G|^3}.$$

The inclusion-exclusion argument used in [1] to establish (1) yields

$$Pr_{3C}(G) = \frac{3k - 2s}{|G|}. \tag{2}$$

Given this formula, bounding $Pr_{3C}$ is no harder than bounding the number of large conjugacy classes from below.

**Theorem 2** *For finite groups* $G$,

$$Pr_{3C}(G) \in (0, 8/9] \cup \{1\}.$$

Proof. Let $c_g$ denote the size of the conjugacy class of $G$ containing $g$. Then, by (2),

$$Pr_{3C}(G) = \frac{3k - 2s}{|G|} = \frac{1}{|G|}\sum_{g \in G} \frac{3c_g - 2}{c_g^2}.$$

Set $f(n) = (3n - 2)/n^2$: then $Pr_{3C}(G)$ is just $E(f(c_g))$, the average value of $f \circ c$ over all $g \in G$. We make a table of values of $f$ (Figure 1), and note that $f$ decreases monotonically over the integers. If $c_g = 1$ or $2$ for all $g$, then $G$ is 3-rewritable [2]. If $c_g = 3$ for some $g$, then $c_g \geq 3$ for at least $1/2$ of the elements of $G$ [1]. Hence $E(f) \leq 1/2(1) + 1/2(7/9) = 8/9$ in this

| $n$ | 1 | 2 | 3 | 4 | 5 | ... | $\infty$ |
|---|---|---|---|---|---|---|---|
| $f(n)$ | 1 | 1 | 8/9 | 5/8 | 13/25 | ... | 0 |

Figure 1: Values for $f(n)$

case. If $c_g \geq 4$ for some $g$, then $c_g \geq 4$ for at least 3/8 of the elements of $G$ [3]. Hence $E(f) \leq 5/8(1) + 3/8(5/8) = 55/64 < 8/9$ in this case.♡

Eric Wepsic
Harvard University
Cambridge, MA 02138

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