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Counting Maximal Chains in Weighted Voting Posets

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Abstract. Weighted voting is built around the idea that voters have differing amounts of influence in elections, with familiar examples ranging from company shareholder meetings to the United States Electoral College. We examine the idea that each voter has a uniquely determined *weight*, paying particular attention to how voters leverage this weight to get their way on a specific yes/no motion (for example, by forming *coalitions*). After some more background on weighted voting, we describe a natural partial order relation between these coalitions of voters. This ordering can be modeled by a partially ordered set (poset), which we call a *coalitions poset*. Using this poset, we derive another important poset via a natural ordering on collections of coalitions. Our results begin by detailing a method for counting the number of maximal chains in the derived poset. After employing this method to find the number of maximal chains in the derived poset with 5 voters, we extend our method for use in the coalitions poset. Finally, we conjecture a formula for the number of maximal chains in the coalitions poset with n voters.

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1 An Introduction to Weighted Voting Systems

The focus of this paper is the study of weighted voting systems. A weighted voting system is defined as a collection of n voters, $v_1, v_2, v_3, ..., v_n$, who vote on a yes or no motion. Each voter v_i has some given weight w_i. In order for a motion to pass, the sum of the weights of all voters voting for the motion must meet or exceed some fixed quota q. Otherwise, the motion is said to fail. We define a coalition as a subset of our n voters who all vote the same way on a motion. A coalition can contain any number of voters, from no voters to all voters in the system. The weight of a coalition is the sum of the weights of each of its voters. The interested reader may wish to peruse [1] and [4] as a supplement to our introduction of weighted voting.

Now that we have been acquainted with the basic concepts of weighted voting systems, let us introduce more mathematical definitions:

Definition 1.1. A *coalition* A is a collection of j voters from a system with n voters such that every member of A votes the same way on a motion. The coalition A has the following properties:

- $0 \leq j \leq n$
- $w_A = w_1 + w_2 + w_3 + \cdots + w_j$, where w_A is the weight of coalition A

The *grand coalition* is precisely the coalition containing all voters in the system. In other words, the grand coalition occurs when $j = n$. Similarly, the *empty coalition* contains no voters and occurs when $j = 0$.

Definition 1.2. A *weighted voting system* $(q : w_n, w_{n-1}, ..., w_2, w_1)$ is defined by the following characteristics:

- There are n voters, to whom we give the names $v_1, v_2, v_3, ..., v_n$.
- There is a weight $w_i \geq 0$ corresponding to each voter v_i.
- There is a quota q such that a motion will pass if, given the coalition A containing all voters voting in favor of the motion, we have $w_A \geq q$.

In most literature, voters are referred to by their subscripts. In other words, we would simply call voter v_n by the abbreviated name, voter n. As a convention, we enumerate our n voters by increasing weight. That is, voter 1 has the least weight, voter 2 has the second least weight,..., and voter n has the greatest weight. We do not require that the weights be strictly increasing. So, we have:

$$0 \leq w_1 \leq w_2 \leq w_3 \leq \cdots \leq w_n$$

Contrastingly, we list the voters comprising a coalition by decreasing weight. Consequently, the coalition containing voters 1, 3, and 4 would be written $\{4, 3, 1\} = \{431\}$ because voter 4 has the greatest weight, voter 3 has the second greatest weight, and voter 1 has the least weight. Similarly, the grand coalition containing all n voters would be written $\{n, n-1, n-2, ..., 2, 1\}$. We should note that this ordering is contrary to the majority of voting literature, which tends to list the weights of the voters in a coalition in increasing order. The convention of ordering weights in this new way came from a paper by Mason and Parsley [3]. The advantage to this ordering is that it makes it much easier to determine the rank of a coalition in a certain partially ordered set $M(n)$, which is central to the study of weighted voting theory and to this paper.

Essentially, weighted voting is predicated on the idea that not all voters are equal. Familiar real world applications of weighted voting range from shareholder meetings to the Electoral College of the United States. In shareholder meetings, some shareholders will have more weight than others by virtue of owning more stock in the company. For example, a shareholder's weight might be equal to the number of shares he owns. Similarly, some states have more weight than others in the Electoral College because they have a larger population. Let us examine an example of weighted voting in a real-world scenario.

Example 1.1. Alice, Bob, and Charlie are the sole shareholders of a small construction company. The distribution of shares is illustrated in the following table:

Shareholder	Number of Shares
Alice	49
Bob	46
Charlie	5

The company has recently fallen on hard times because of the decline in the housing market. Alice proposes expanding the business to other markets, reasoning that this will increase their revenue. Bob opposes the idea, explaining that this proposition will also lead to higher expenditures. Given the share distribution shown above, how can Alice get her motion to pass? We will set the quota q at 51 shares.

Because we can pick a quota (namely, $q = 51$) and assign weights to each voter i (simply take $w_i =$ the number of shares held by i), we have a weighted voting system. Using Definition 1.2, we notate this weighted voting system as $(51 : 49, 46, 5)$. If Alice is the only one to vote for the motion, she will form a coalition with weight $w = 49$. But $49 < q$, so Alice's motion would fail. On the other hand, if Charlie votes with Alice on the motion, they will form a coalition with weight $w = 49 + 5 = 54$. Since $54 > q$, Alice's motion would pass. Finally, if Alice is able to convince Bob to support her motion, they will form a coalition with weight $w = 49 + 46 = 95$. Since $95 > q$, Alice's motion would pass. So, we have found that Alice cannot singlehandedly force her motion to pass; she needs the help of either Bob or Charlie. Of course, Alice could also win by teaming up with both Bob and Charlie. However, this observation is trivial. Once Alice forms a coalition with one of the other two shareholders, they already have enough weight between themselves to force her motion to pass. Adding the third shareholder to the coalition would have no effect on the outcome. This idea is important and will come up again when we discuss the notion of minimal winning coalitions.

Example 1.1 is meant to be a representative example of weighted voting systems. So, we should note that while it was convenient for the weights of our voters to sum to 100, it was not necessary. Furthermore, we did not have to choose $q = 51$ as our quota; it could have just as easily been taken to be $q = 30$ or even $q = 100$. However, to avoid the situation in which two opposing coalitions are both winning, we will disallow quotas set at less than or equal 50 percent of the combined weight of all voters. The reader may also notice that the weights of each voter were different in Example 1.1. This was also not necessary. It is perfectly valid to have a weighted voting system in which the weights of certain (or even all) voters are equal. Consider the following example, which is a variation of Example 1.1:

Example 1.2. Alice, Bob, and Charlie are the sole shareholders of the construction company. Their shares are distributed according to the following table.

Shareholder	Number of Shares
Alice	33
Bob	33
Charlie	33

Alice again makes her proposition, and is opposed by Bob as in Example 1.1. In this scenario, how can Alice get her motion to pass? The quota q is set at 50 shares.

Analyzing the system as we did before, we find that Alice needs the help of either Bob or Charlie to get her motion to pass. The reader may notice that this result is identical to the result from Example 1.1. This similarity will be even more apparent once the notion of winning coalitions is introduced.

2 Winning Coalitions

Before we proceed any further, we will give a formal definition for an idea that we have been taking for granted, winning coalitions. Our intuitive notion of a winning coalition has served us well up to this point, but it will be useful to have an explicit definition. In a natural extension of this idea, we will also define losing coalitions.

Definition 2.1. Given a weighted voting system $(q : w_n, w_{n-1}, ..., w_2, w_1)$, define the coalition A as a collection of all voters who vote the same way on some motion.

- A is a *winning coalition* if and only if $w_A \geq q$
- A is a *losing coalition* if and only if $w_A < q$

By Definition 1.1, the weight of any given coalition A must be unique. In other words, no one coalition can have two weights at once. Therefore, the weight of coalition A must satisfy either $w_A \geq q$ or $w_A < q$, but not both. This tells us that a coalition cannot be both winning and losing; however, it must be one of the two. To help us see this more explicitly, we will find all of the winning and losing coalitions from Example 1.1. Recall that the quota was set at $q = 51$.

Coalition	Weight of Coalition	Winning/Losing
$\{\emptyset\}$	0	Losing
{Alice}	49	Losing
{Bob}	46	Losing
{Charlie}	5	Losing
{Alice, Bob}	$49 + 46 = 95$	Winning
{Alice, Charlie}	$49 + 5 = 54$	Winning
{Bob, Charlie}	$46 + 5 = 51$	Winning
{Alice, Bob, Charlie}	$49 + 46 + 5 = 100$	Winning

Next, we will compute all of the winning and losing coalitions from Example 1.2. Upon doing so, the similarity to Example 1.1 will be even more obvious.

Coalition	Weight of Coalition	Winning/Losing
$\{\emptyset\}$	0	Losing
{Alice}	33	Losing
{Bob}	33	Losing
{Charlie}	33	Losing
{Alice, Bob}	$33 + 33 = 66$	Winning
{Alice, Charlie}	$33 + 33 = 66$	Winning
{Bob, Charlie}	$33 + 33 = 66$	Winning
{Alice, Bob, Charlie}	$33 + 33 + 33 = 99$	Winning

Notice that the winning coalitions in Example 1.1 and Example 1.2 are the same. Furthermore, the losing coalitions are also the same. We would like to show that the second observation follows from the first. Because we set our quota at greater than 50 percent of the combined weight of all voters, the set of losing coalitions L is the complement of the set of winning coalitions W. That is, $W^c = L$. In order to generalize our argument, we will assume that two arbitrarily chosen weighted voting systems have the same winning coalitions (as was the case with Example 1.1 and Example 1.2). Restating the assumption, the set of winning coalitions of system 1 must equal the set of winning coalitions of system 2.

$$
\begin{array}{ll}
W_1 = W_2 & \text{(Assumption)} \\
(W_1)^c = (W_2)^c & \text{(Taking the complement preserves equality)} \\
L_1 = L_2 & \text{(Since } W^c = L\text{)}
\end{array}
$$

We have now seen that all information regarding a voting system's losing coalitions can be obtained from information about the winning coalitions. This is an important observation because it tells us that voting systems are characterized by their winning coalitions. In fact, voting systems are actually characterized by their *minimal* winning coalitions, but we will get to that in the next section. Essentially, if we know the winning coalitions of a weighted voting system, then the losing coalitions are the leftover coalitions (assuming q is set at greater than 50 percent of the combined weight of all voters).

3 Minimal Winning Coalitions

Now that we have been introduced to winning coalitions, let us try to gain a deeper understanding via an example. Namely, consider all four of the winning coalitions in Example 1.1. By definition of a winning coalition A, it must be true that $w_A \geq q$, or in this case $w_A \geq 51$. Indeed, one of these coalitions has a weight equal to the quota, while the other three have more weight than is needed to win ($w_A > q$). A good question to ask would be if we could remove a voter from one of these three coalitions and still have a winning coalition. If we could do so, then that voter would not be critical to his coalition winning. This leads us to another definition:

Definition 3.1. A *critical voter* in a coalition is a voter whose removal would cause a winning coalition to become a losing coalition.

We can use Definition 3.1 to define minimal winning coalitions:

Definition 3.2. A *minimal winning coalition* is a winning coalition in which every voter is a critical voter.

Because every voter in a minimal winning coalition is critical, removing any one would cause the coalition to become a losing coalition. To get a better understanding of this idea, we will identify the minimal winning coalitions in Example 1.1. Recall that the quota was set at 51 shares. We begin by identifying all of the winning coalitions. During our discussion of winning coalitions, we found that there were exactly four. Now, we select one of these four coalitions to start with. We choose {Alice, Bob}. For each voter in the coalition (in this case, Alice and Bob), we must determine if that voter is critical. We will start with Alice. Our first step is to remove Alice from the coalition {Alice, Bob}. We are left with the coalition {Bob}. The weight w of this coalition is equal to Bob's weight, which is 46. But $46 < q$. Alice's removal has caused the winning coalition to become a losing coalition. Therefore, Alice is a critical voter in the coalition {Alice, Bob}. Similarly, removing Bob from the coalition leaves only {Alice}. But $49 < q$. Bob's removal has also caused the coalition to become a losing coalition. Therefore, Bob must be a critical voter in the coalition {Alice, Bob}. Since every voter in the coalition {Alice, Bob} is critical, it must be a minimal winning coalition. The same process can be repeated for the other three winning coalitions. Doing so, we find:

Coalition	Weight of Coalition	Minimal/Not Minimal
{Alice, Bob}	$49 + 46 = 95$	Minimal
{Alice, Charlie}	$49 + 5 = 54$	Minimal
{Bob, Charlie}	$46 + 5 = 51$	Minimal
{Alice, Bob, Charlie}	$49 + 46 + 5 = 100$	Not Minimal

We can see that there are no critical voters in the coalition {Alice, Bob, Charlie}. The removal of any one voter still leaves the coalition with a weight of at least 51. In a way, this winning coalition is less important than the other three winning coalitions. Imagine that we only knew the minimal winning coalitions for this weighted voting system. We could use this knowledge to find all of the non-minimal winning coalitions. In this case, we would know that {Alice, Bob} is winning. Adding Charlie to this coalition could only increase its weight, so {Alice, Bob, Charlie} must also be winning. So, as we hinted at in Section 2, weighted voting systems are completely characterized by their minimal winning coalitions. This leads us to the following definition.

Definition 3.3. Two distinct weighted voting systems are *isomorphic* if and only if they have the same minimal winning coalitions.

When two distinct weighted voting systems are isomorphic, it means that they are essentially the same. While the two isomorphic weighted voting systems may have different quotas and different weights assigned to their voters, the structure and the power of each voter is the same in each system. We talked about the similarities between Examples 1.1 and 1.2. Now, with this new idea in mind, we might guess that these two weighted voting systems are isomorphic. Indeed, they both have precisely the same minimal winning coalitions, so they must be isomorphic. In fact, we have a name for this type of weighted voting system: majority rule. As we can see from our minimal winning coalitions, any two voters can team up to win.

Definition 3.4. A weighted voting system with n voters is called a *majority rule* system if and only if any coalition containing greater than 50 percent of the n voters is a winning coalition.

In the case of Example 1.1 and Example 1.2, $n = 3$. Since $\frac{3}{2} = 1.5$, any coalition with two or more voters must be winning. Indeed, this was exactly what we observed. In more general terms, when n is odd, any coalitions with $\frac{n+1}{2}$ or more voters must be winning. Likewise, when n is even, any coalitions with $\frac{n}{2} + 1$ or more voters must be winning.

We previously noted that the choice of quota $q = 51$ in Example 1.1 was arbitrary. In the following example, we will change the quota to $q = 52$. The reader might not expect this to make a significant difference on our example. However, we will see that it changes the entire structure of the system.

Example 3.1. Since we are raising our quota, all losing coalitions from Example 1.1 will stay losing. So, we only need to consider the winning coalitions. The only difference between this example and Example 1.1 is that the coalition {Bob, Charlie} no longer has enough weight to win with a quota of 52. So, our only minimal winning coalitions are {Alice, Bob} and {Alice, Charlie}.

Coalition	Weight of Coalition	Minimal/Not Minimal
{Alice, Bob}	$49 + 46 = 95$	Minimal
{Alice, Charlie}	$49 + 5 = 54$	Minimal
{Bob, Charlie}	$46 + 5 = 51$	Losing
{Alice, Bob, Charlie}	$49 + 46 + 5 = 100$	Not Minimal

We can see that Alice is in every winning coalition. In other words, any coalition that Alice is not a part of is losing. So, in this weighted voting system, Alice can force a motion to fail by voting against it. We define this type of weighted voting system by saying that Alice has veto power.

Definition 3.5. A voter in a weighted voting system has *veto power* if and only if that voter appears in every (minimal) winning coalition.

Suppose that we change the quota again, this time to $q = 55$. In the following example, we will examine the effect that this increase in quota has on our system.

Example 3.2. As in Example 3.1, we are raising our quota. So, all coalitions that were losing at lower quotas will still be losing here. The only difference between this example and Example 3.1 is that the coalition {Alice, Charlie} no longer has enough weight to win with a quota of $q = 55$. Thus, our only minimal winning coalition is {Alice, Bob}.

Coalition	Weight of Coalition	Minimal/Not Minimal
{Alice, Bob}	$49 + 46 = 95$	Minimal
{Alice, Charlie}	$49 + 5 = 54$	Losing
{Bob, Charlie}	$46 + 5 = 51$	Losing
{Alice, Bob, Charlie}	$49 + 46 + 5 = 100$	Not Minimal

We can see that Charlie is not in any minimal winning coalitions. The only way he can win is to form a coalition with both Alice and Bob. However, Alice and Bob can win without Charlie. So, Charlie has no power in this system. Formally, we call him a dummy voter.

Definition 3.6. A voter in a weighted voting system is called a *dummy* if and only if that voter does not appear in any minimal winning coalitions.

Finally, we will consider the effect of changing the quota to $q = 96$.

Example 3.3. Again, since we are raising the quota, all coalitions that were losing at lower quotas will still be losing here. The only difference between this example and Example 3.2 is that the coalition {Alice, Bob} no longer has enough weight to win with a quota of $q = 96$. Thus, our only minimal winning coalition is {Alice, Bob, Charlie}.

Coalition	Weight of Coalition	Minimal/Not Minimal
{Alice, Bob}	$49 + 46 = 95$	Losing
{Alice, Charlie}	$49 + 5 = 54$	Losing
{Bob, Charlie}	$46 + 5 = 51$	Losing
{Alice, Bob, Charlie}	$49 + 46 + 5 = 100$	Minimal

We can see that all voters must vote together in order to win. We call this type of system a consensus system.

Definition 3.7. A weighted voting system with n voters is called a *consensus* system if and only if the only winning coalition is the grand coalition containing all n voters.

4 Partial Order Relations

The results of this paper are centered around partially ordered sets (also known as posets). Because of their importance in our research, will we take a step back from voting theory to explain the mathematics of partially ordered sets. We will begin with partial order relations, the building blocks of posets.

Definition 4.1. A binary relation R on a set S is a *partial order relation* if and only if R satisfies the following conditions:

1. R is reflexive: For all $x \in S,\ x\ R\ x$
2. R is antisymmetric: For all $x, y \in S,\ x\ R\ y$ and $y\ R\ x$ implies $x = y$
3. R is transitive: For all $x, y, z \in S,\ x\ R\ y$ and $y\ R\ z$ implies $x\ R\ z$

Consider the "subset" relation on the power set of a set S. For readers unfamiliar with this terminology, the power set $P(S)$ is simply the set of all subsets of S. We define the subset relation as follows. Given two subsets U and V in the power set $P(S)$, $U\ R\ V$ if and only if $U \subseteq V$. In Figure 1, we can see a visual representation of $U\ R\ V$.

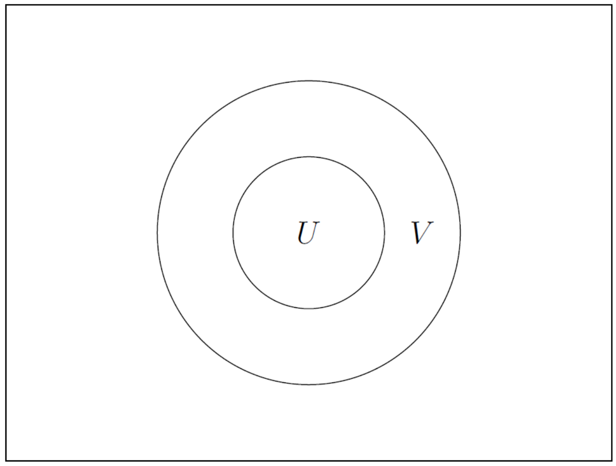

Figure 1: $U\ R\ V$

Example 4.1. Is the subset relation $\subseteq$ a partial order relation? We must check that $\subseteq$ is reflexive, antisymmetric, and transitive.

1. Take an arbitrary subset $U \in P(S)$. Every set must contain itself, so $U \subseteq U$. This shows that $\subseteq$ is reflexive.

2. Take two arbitrary subsets $U, V \in P(S)$. Assume that $U \subseteq V$ and $V \subseteq U$. Then, by the definition of set equality, $U = V$. This shows that $\subseteq$ is antisymmetric.

3. Take three arbitrary subsets $U, V, W \in P(S)$. Assume that $U \subseteq V$ and $V \subseteq W$. Then, for any element $x \in U$, it follows by definition of a subset that $x \in V$. Using the same argument, $x \in V$ implies $x \in W$. So, $x \in U$ implies $x \in W$. By definition of a subset, $U \subseteq W$. This shows that $\subseteq$ is transitive.

So, we have shown that $\subseteq$ is indeed a partial order relation. As a convention, partial order relations are commonly denoted by $\preceq$.

5 Partially Ordered Sets

Now that we have introduced partial order relations, we have the necessary tools to understand partially ordered sets.

Definition 5.1. Consider a relation R on a set S. S is a *partially ordered set* if and only if R is a partial order relation on S.

Given any poset S, we can draw a *poset diagram* (often called a Hasse diagram). This diagram is useful for visualizing the structure of the poset. We draw the diagram by connecting related elements of the poset with either a line or a series of lines. Given any two distinct elements $a, b \in S$, we connect a and b with a single line if and only if $a \preceq b$ and $\forall c \in S$ such that $a \preceq c \preceq b$, either $a = c$ or $c = b$. This idea leads to the following definition.

Definition 5.2. Let a and b be arbitrary elements of some poset S. We say that a is *covered* by b if and only if $a \preceq b$ and a and b are connected by a single line in the poset diagram.

As an example, we will return to the subset relation on the power set $P(S)$.

Example 5.1. Since $\subseteq$ is a partial order relation on $P(S)$, it follows that $P(S)$ is a partially ordered set. Thus, we may draw a poset diagram of $P(S)$. First, we will make $P(S)$ more concrete by letting $S = \{a, b, c\}$. There are 2^3 subsets of S, and these eight subsets comprise the power set $P(S)$. We start by placing the empty set at the bottom of our diagram since it is a subset of everything (and therefore related to all subsets of S). Specifically, $\emptyset \subseteq \{a\}$. Thus, $\emptyset$ and $\{a\}$ must be connected with either a line or a series of lines. Moreover, $\forall x \in P(S)$ with $\emptyset \preceq x \preceq \{a\}$, $\emptyset = x$ or $x = \{a\}$. Therefore, $\emptyset$ and $\{a\}$ are connected with a single line in the poset diagram. So, we say that $\{a\}$ covers $\emptyset$. In the same way, we can show that both $\{b\}$ and $\{c\}$ cover the empty set. Similarly, we find that both $\{a, b\}$ and $\{a, c\}$ cover $\{a\}$. Now, we would like to determine how the empty set is connected to $\{a, b\}$ and $\{a, c\}$. Since $\emptyset \subseteq \{a, b\}$, it must be true that $\emptyset$ is connected to $\{a, b\}$ with either a line or a series of lines. Notice that $\emptyset \preceq \{a\} \preceq \{a, b\}$. Thus, $\emptyset$ must be connected to $\{a, b\}$ with a series of lines. Using the same argument, we can see that $\emptyset$ is also connected to $\{a, c\}$ with a series of lines. Consequently, neither $\{a, b\}$ nor $\{a, c\}$ covers the empty set. We continue drawing lines in this manner until all related elements are connected. The end result is shown in Figure 2.

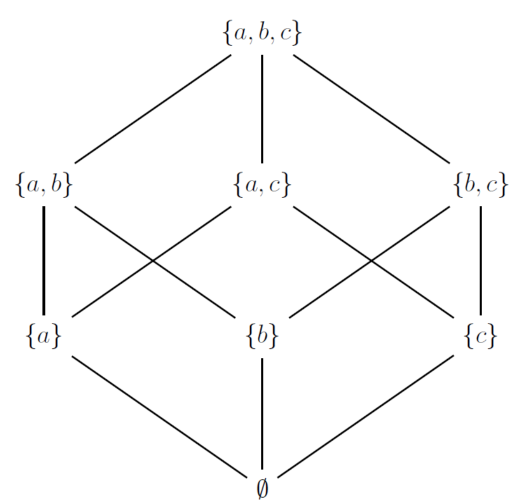

Figure 2: Poset for $P(\{a,b,c\})$

Observe that this poset diagram has a nice, tiered structure. More specifically, each element of $P(\{a, b, c\})$ lies on one of four distinct "levels". So, we could choose to "rank" each element of the poset diagram by the level on which it lies (starting with the empty set on level 0). Notice that $\{a\}$ covers $\emptyset$, and the difference in these elements' ranks is 1. Furthermore, notice that $\{a, b\}$ does not cover $\emptyset$, and the difference in these elements' ranks is 2. This is a defining characteristic of a ranked poset.

Definition 5.3. Consider a function that assigns a value $\rho(x)$ to each element x of a connected poset S. Given $a, b \in S$, we say that S is *ranked* if $\rho(b) = \rho(a) + 1$ only when b covers a.

In the case of our poset for $P(\{a, b, c\})$, we let the rank of a subset be equal to the number of elements it contains. So, $\{a, b, c\}$ has rank 3, and $\emptyset$ has rank 0. Again, we can think of the rank of an element as corresponding to the level of the poset diagram on which that element sits. Now, consider a ranked poset S and elements $a, b, c \in S$. If b covers a and c covers a, then $\rho(b) = \rho(c) = \rho(a) + 1$. So, elements b and c lie on the same level of the poset diagram (they both lie one level above a).

Looking back at our poset diagram for $P(\{a, b, c\})$, we notice that some elements are not related to each other. This observation leads us to another important definition.

Definition 5.4. Let $\preceq$ be a partial order relation on the set S. Two elements $x, y \in S$ are said to be *comparable* if and only if either $x \preceq y$ or $y \preceq x$. Otherwise, x and y are not related and are said to be *incomparable*.

Consider elements $\{a, b\}$ and $\{a, c\}$ from Example 5.1. Since $\{a, b\} \nsubseteq \{a, c\}$ and $\{a, c\} \nsubseteq \{a, b\}$, these two elements are incomparable. In general, two elements in a poset diagram are comparable if and only if we can follow lines up the diagram from one element to the other. This method works because of the transitive property of the partial order relation. Now, consider the case in which any two given elements of a set S are comparable under a relation R. Whenever this is true, we call R a *total order relation* on S. We will use this idea to define the totally ordered set.

Definition 5.5. Let S be a partially ordered set under a relation $\preceq$. We call S a *totally ordered set* if and only if $\preceq$ is a total order relation on S.

In a totally ordered set, every element is comparable. So, drawing a poset diagram would yield a single "chain" with no branches. In fact, this notion of a chain has a mathematical definition:

Definition 5.6. Let A be a partially ordered set under a relation $\preceq$. A subset B of A is called a *chain* if and only if every pair of elements in B are comparable.

Lemma 5.1. If B is a chain in A, then B must be a totally ordered set under the relation $\preceq$.

To get a better understanding of chains, we will go back to Example 5.1. Consider the subset $B = \{\emptyset, \{c\}, \{b, c\}, \{a, b, c\}\}$ of $P(\{a, b, c\})$. Every pair of elements in B are comparable since $\emptyset \subseteq \{c\} \subseteq \{b, c\} \subseteq \{a, b, c\}$. Thus, B is a chain. Look back at the poset diagram for $P(\{a, b, c\})$ and notice how B sits in it; the resemblance to a chain should be apparent.

Now, let us consider maximal chains, which are specific types of chains. Essentially, a maximal chain is a chain that spans the entire poset diagram. No elements can be added onto a maximal chain B without causing B to lose the property of being totally ordered. To formally define the maximal chain, we will need the idea of maximal and minimal elements.

Definition 5.7. The element a is a *maximal element* of A if and only if $a \preceq b$ implies $a = b$. The element a is a *minimal element* of A if and only if $b \preceq a$ implies $a = b$.

Now, we are ready to define the (finite) maximal chain.

Definition 5.8. A finite chain B is a *maximal chain* if and only if it contains both a minimal element and a maximal element.

Going back to Example 5.1, we propose that $B = \{\emptyset, \{c\}, \{b, c\}, \{a, b, c\}\}$ is one of several maximal chains. If this statement is true, then B should have a maximal element and a minimal element. Examining the poset diagram for $P(\{a, b, c\})$, we find that $\{a, b, c\}$ is a maximal element and $\emptyset$ is a minimal element. Since B contains both of these elements, it is indeed a maximal chain. Counting these maximal chains is a very interesting problem. In fact, our attempt to do so led to several new discoveries, which will be discussed shortly.

6 Posets Applied to Voting Theory

In this section, we will apply our knowledge of relations and posets to voting theory. More specifically, we will define a partial order relation on the set S of all coalitions in a weighted voting system with n voters. We will then use this partial order relation to make a poset diagram.

Definition 6.1. Let S be the set of all coalitions in a weighted voting system with n voters. Given any two coalitions A and B in S, we say that B covers A if and only if we can make B from A using exactly one of the following two techniques:

1. Adding voter 1 to the coalition A

2. Replacing voter a in A with voter b, provided $a < b$

We will say that coalition A is related to coalition B (written $A \preceq B$) if and only if we can make B from A using a sequence of covers as described in Definition 6.1. Furthermore, $A = B$ if and only if we can make B by leaving A unchanged (a sequence of zero covers). The following example will give the reader a clearer idea of how our partial order relation $\preceq$ works.

Example 6.1. Consider a weighted voting system with $n = 3$ voters. The possible coalitions of voters are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{21\}$, $\{31\}$, $\{32\}$, and $\{321\}$. Recall that voters in a coalition are listed in order of decreasing weight (i.e. voter 3 has the greatest weight). We say that $\{1\}$ covers $\emptyset$ because we can make $\{1\}$ from $\emptyset$ (use method 1 from Definition 6.1). Likewise, we say that $\{2\}$ covers $\{1\}$ because we can make $\{2\}$ from $\{1\}$ (use method 2 from Definition 6.1). Since $\{2\}$ can be made from $\emptyset$ using a sequence of covers (two, to be exact), we say that $\emptyset \preceq \{2\}$. Of course, it is also true that $\emptyset \preceq \{1\}$ and $\{1\} \preceq \{2\}$ (since each uses a sequence of one cover). We can continue this process to relate most of the elements of S. However, the elements $\{21\}$ and $\{3\}$ will be incomparable. We say that $\{21\} \not\preceq \{3\}$ because we cannot make $\{3\}$ from $\{21\}$ with a sequence of covers. Similarly, $\{3\} \not\preceq \{21\}$.

Now, we will show that $\preceq$ is a partial order relation on the set S of all coalitions. Recall that this requires us to show that $\preceq$ is reflexive, antisymmetric, and transitive.

1. Take an arbitrary coalition A in S. We can make A from A by simply leaving A unchanged (a sequence of zero covers). Therefore, $A = A$. More generally, $A \preceq A$. This shows that $\preceq$ is reflexive.

2. Take two arbitrary coalitions A and B in S. Assume that $A \preceq B$ and $B \preceq A$. Consider the scenario in which we make coalition B from coalition A by using a sequence of covers. In this case, it will be impossible to make coalition A from coalition B using a sequence of covers (our definition of cover only allows us to add rank to coalitions, not take it back away). Therefore, it must be true that we make coalition B by leaving coalition A unchanged, which tells us that $A = B$. This shows that $\preceq$ is antisymmetric.

3. Take three arbitrary coalitions A, B, and C in S. Assume that $A \preceq B$ and $B \preceq C$. So, we can make B out of A using a sequence of covers and C out of B using a sequence of covers. It follows that we can make C out of A using a sequence of covers (just chain the two sequences together). This shows that $\preceq$ is transitive.

Since $\preceq$ is a partial order relation on S, it must be true that S is a partially ordered set. We will call this set S the coalitions poset and denote it by $M(n)$, where n is the number of voters in the corresponding weighted voting system. We may better understand $M(n)$ by drawing a poset diagram of its elements, which are coalitions of voters. We begin by putting the empty coalition at the bottom of the diagram. Then, we connect the empty set to any coalitions that cover it. The only such coalition is made by adding voter 1 to the empty set. We continue this process until we have used all 2^n coalitions in our diagram. The resulting poset diagrams for $n = 3$ and $n = 4$ can be seen in Figure 3.

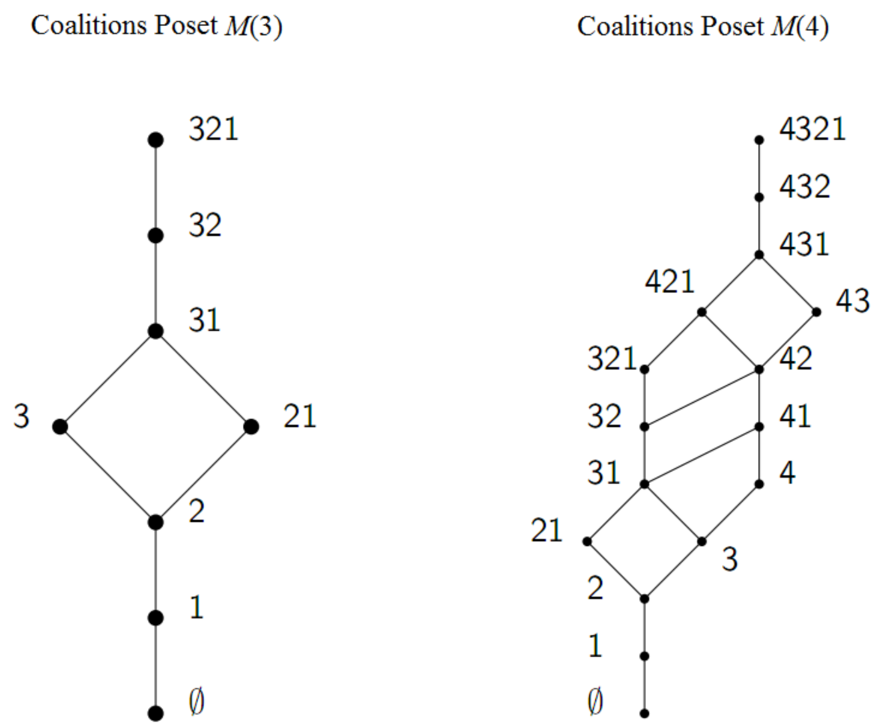

Figure 3: Coalitions Posets

Now that the reader is familiar with the coalitions poset, we will show how it can be used to derive another important poset. First, however, we must introduce the concept of filters.

Definition 6.2. A *filter* of $M(n)$ is completely generated by one or more lowest elements, which are called *generators*. Given a generator, a filter consists of all elements to which that generator is related.

To fully specify a filter, it is enough to simply name the generator(s). Once the generator(s) is (are) specified, the filter consists of all other coalitions that the generator(s) is (are) related to. In our poset diagram, this corresponds to all coalitions above (and including) our generator(s).

Example 6.2. Suppose that the filter a is generated by $\{31\}$ in $M(3)$. Then, the filter a consists of all coalitions that $\{31\}$ is related to. Examining the diagram of $M(3)$, we can see that $\{31\} \preceq \{32\} \preceq \{321\}$. So, $a = \{\{31\}, \{32\}, \{321\}\}$.

Recall from Section 2 that each weighted voting system is uniquely characterized by its set of winning coalitions. So, we can think of a weighted voting system as a specific filter a of $M(n)$, where a is precisely the set of winning coalitions for the given weighted voting system.

In other words, we will say that all coalitions in a specific filter are winning. Similarly, we say that any coalition not in the given filter is losing. An example will make this idea clearer.

Example 6.3. Consider the weighted voting system generated by $\{31\}$ in $M(3)$. In this voting system, we say that voter 3 has veto power (in our previous examples, voter 3 was Alice). For this scenario, the winning coalitions are $\{31\}$, $\{32\}$, and $\{321\}$. So, we can think of this weighted voting system as the filter $a = \{\{31\}, \{32\}, \{321\}\}$ of $M(3)$. In Figure 4, the filter a is circled. Pictorially, all coalitions inside the circle are winning coalitions, and all coalitions outside the circle are losing coalitions.

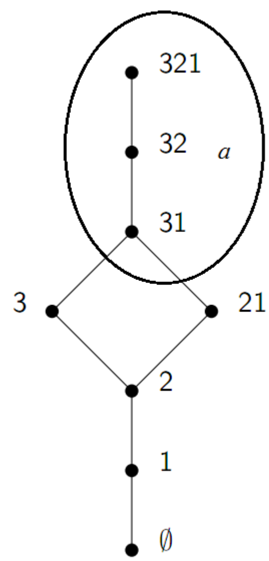

Figure 4: A Filter of $M(3)$

Now, we are ready to define the relation that gives us our new poset:

Definition 6.3. Let J be the set of all filters of $M(n)$. Given two filters a and b in J, we say that $b \preceq a$ if and only if $a \subseteq b$.

Example 6.4. Let $n = 4$. Suppose we have filters a and b such that $a = \{\{431\}, \{432\}, \{4321\}\}$ and $b = \{\{421\}, \{43\}, \{431\}, \{432\}, \{4321\}\}$. In Figure 5, it is clear that $a \subseteq b$. So, by Definition 6.3, $b \preceq a$.

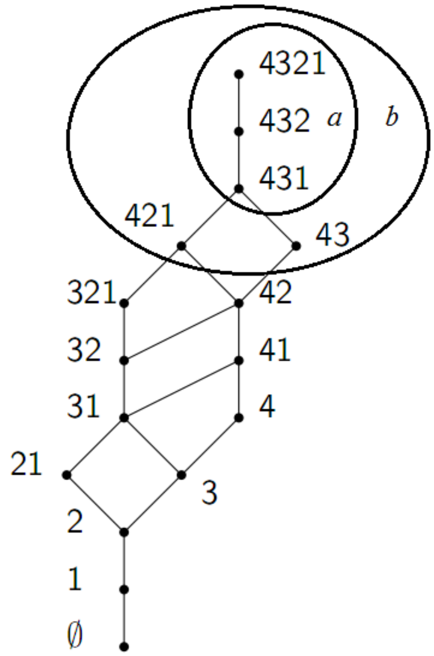

Figure 5: $b \preceq a$

This relation is essentially the reverse of the subset relation. Therefore, it must be a partial order relation on the set J of all filters of $M(n)$. So, J is a poset by Definition 5.1. Since each filter represents a unique set of winning coalitions, we call this poset the *poset of winning coalitions* and denote it $J(M(n))$ or J_n. When we are constructing J_n, we denote filters by their generator(s).

However, an important remark is in order. We are not interested in collections of winning coalitions (filters) in which the coalition A and its complement A^c are both winning. For example, let $A = \{43\}$. For $n = 4$ voters, we have $A^c = \{21\}$. Suppose both A and A^c are winning. If A and A^c are voting opposite ways on a motion, then it would be impossible to say whether the motion passes or fails. To avoid this problem, we will only consider the top half of J_n, which we will denote J_n^+. So that the reader may get a better idea of all of these posets, we will show $M(4)$, J_4, and J_4^+ in Figure 6.

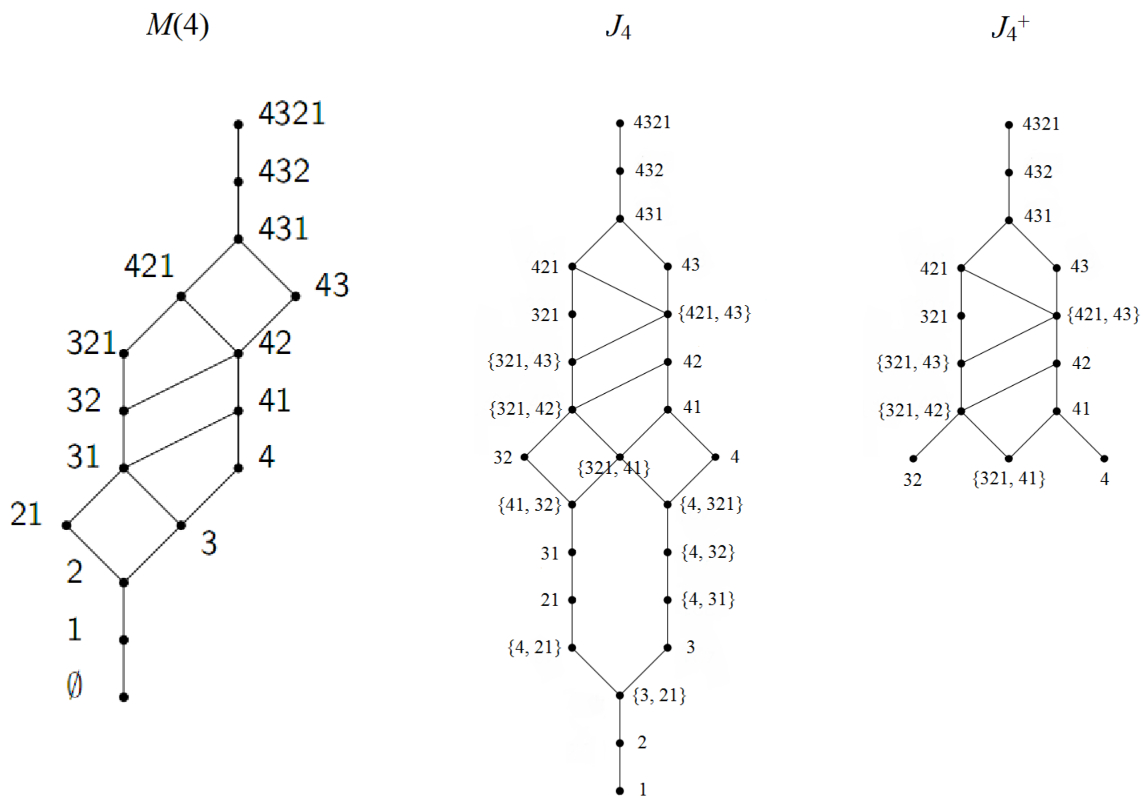

Figure 6: A Comparison of $M(4)$, J_4, and J_4^+

We should note that while any weighted voting system with n voters has a corresponding generator in J_n^+, the converse is not necessarily true. An excellent discussion on this topic can be found in a paper by Mason and Parsley [3]. In brief, they write that each generator in J_n^+ does correspond to a weighted voting system when $n \leq 5$. However, for $n \geq 6$, not all generators in J_n^+ correspond to weighted voting systems. For further details and examples, we refer the interested reader to Mason and Parsley [3].

7 Counting Maximal Chains in J_n^+

Looking back at the poset diagram for J_4^+, we can see that it is not very complicated. Suppose, for example, that we wanted to count the number of maximal chains in J_4^+. With relatively little effort, we can trace all of the different paths from the bottom to the top. Keeping track of each separate path, we find that there are 14 maximal chains. While this is not hard to compute for $n = 4$, consider J_5^+ as pictured in Figure 7.

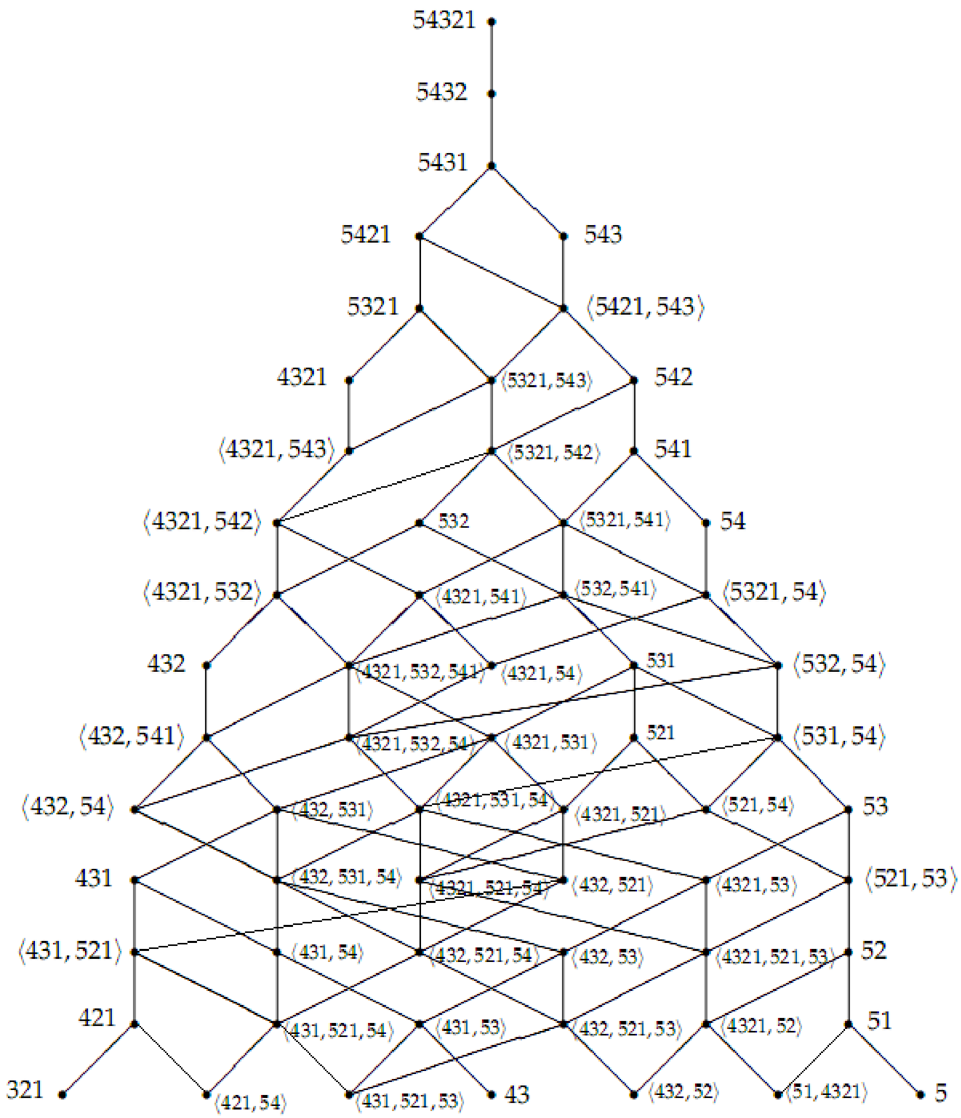

Figure 7: J_5^+

This poset is significantly more complicated, and it would be quite tedious to trace out each maximal chain by hand. In addition, the complexity of the poset increases the chance for error in a manual approach. We need a simple, mathematical way of counting these chains. Since it is much easier to start with simple cases and then generalize to more complicated cases, we will return to J_4^+. This time, we want to find a way to count the maximal chains without tracing out each path. However, this does not mean that there is nothing to be learned from our first approach. When we manually trace a path, we start at the bottom of the poset and work our way to the top. We travel along the same path until we meet a branch in the poset. These branches are the reason that there are multiple paths to the top. In the very simple case of a poset with no branches, there would only be one path to the top. Therefore, there would only be one maximal chain. So, we need only consider the branches in our poset diagram in order to count the number of maximal chains. This leads us to a definition.

Definition 7.1. A *decision hub* is an element in a poset that is covered by at least two other elements.

In terms of the poset diagram, a decision hub is a point that has at least two lines diverging from it and traveling upwards. In Figure 8, the four decision hubs of J_4^+ are highlighted.

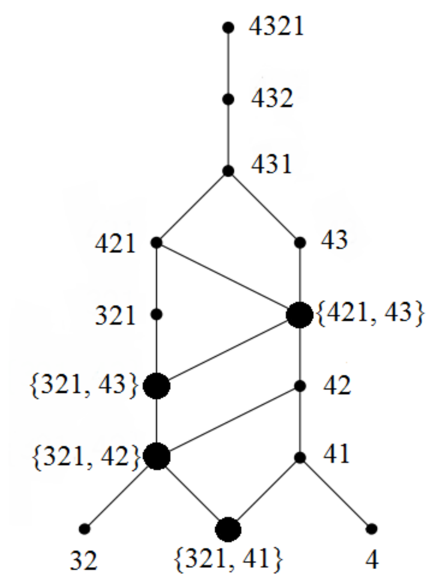

Figure 8: Decision Hubs in J_4^+

We will start at the top of the diagram and work our way down. The first decision hub we reach is $\{421, 43\}$. There are two paths diverging from this hub and traveling upwards. Neither path branches, so there are two ways to get to the top from this hub. So, we assign the value 2 to the hub. Moving down, the next decision hub we reach is $\{321, 43\}$. Again, there are two paths diverging from this hub. One path travels straight to the top without branching. The other path runs into the previous decision hub with value 2. From this point, the path has 2 ways to the top. So, the total number of paths to the top of the poset from $\{321, 43\}$ is $1 + 2 = 3$. We assign this value to the decision hub. We continue moving down, next reaching the decision hub $\{321, 42\}$. There are two paths diverging from this hub. Each path runs into a decision hub. One of the paths runs into a decision hub of value 3. The other path runs into a decision hub of value 2. Therefore, there are $3 + 2 = 5$ paths to the top of the poset from this hub. We assign this value to the hub. Finally, we move down to our last decision hub, $\{321, 41\}$, which is on the bottom level of the poset. There are two paths diverging from this hub. One path runs into a decision hub of value 5, while the other runs into a decision hub of value 2. Adding these values, we find that the total number of paths from $\{321, 41\}$ to the top of the poset is 7. We assign this value to the hub. Now, we just need to find the number of paths to the top of the poset starting from 32 and 4. Since neither of these points are decision hubs, there is only one path leaving each of them. Each of these paths runs into a decision hub. So, we assign the value of the first decision hub encountered to each point. The point 32 is assigned the value 5, and the point 4 is assigned the value 2. The number of maximal chains is just the sum of the values of the three minimal elements: $7 + 5 + 2 = 14$. Not surprisingly, this answer agrees with the answer we found manually. Figure 9 illustrates this method by including the values of the decision hubs and minimal elements.

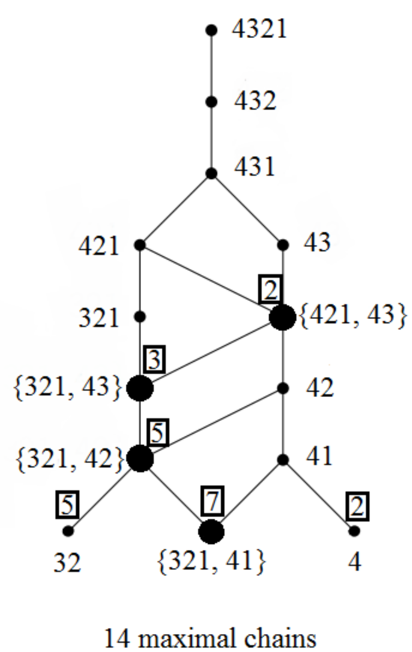

Figure 9: Decision Hubs with Values in J_4^+

We can implement this procedure in exactly the same way for larger values of n. However, there are two disadvantages to doing so. First, the procedure requires us to know the structure of J_n^+, which gets immensely complicated for $n \geq 6$. Second, values have to be computed for each decision hub. This means that our procedure will take a considerable amount of time for larger values of n. A computer program could drastically simplify this process, assuming that we already knew the structure of J_n^+. A program that could generate the poset and then calculate the number of maximal chains using our procedure would be ideal.

Since J_5^+ is not extremely complicated (especially in terms of decision hubs), we will use our procedure to calculate its number of maximal chains. First, however, we must state our procedure formally and prove it mathematically. We will need to use the fact that $M(n)$ and J_n^+ are both ranked posets. The interested reader may find the rank function for these posets in Mason and Parsley's paper [3].

Theorem 7.1. Given some poset S, take any non-maximal element a that is covered by each of the points $1, 2, ..., n$. Let h_a be the number of paths from point a to a maximal element in the poset diagram. Then, $h_a = h_1 + h_2 + \cdots + h_n$, where h_1 is the number of paths from point 1 to a maximal element, h_2 is the number of paths from point 2 to a maximal element,..., and h_n is the number of paths from point n to a maximal element.

Proof. This will be a proof by induction.

Base case: Our base case is to consider the point a that is covered by n distinct maximal elements. Since there are no elements above a maximal element, there is only 1 path from point a to a maximal element for each of the n distinct maximal elements. Therefore, there are $1+1+\cdots+1 = n$ paths from point a to a maximal element. Now, we check this against our equation. For each of the n maximal elements that a is directly connected to, there is exactly 1 path from the given element to a maximal element (namely, the stationary path). So, $h_1 = h_2 = \cdots = h_n = 1$. Then, $h_a = 1 + 1 + \cdots + 1 = n$. This agrees with the answer we found, so we have verified our base case.

Inductive step: We are given some point b that is covered by each of the points $1, 2, ..., m$. Assume that there are h_1 paths from point 1 to a maximal element, h_2 paths from point 2 to a maximal element,..., and h_m paths from point m to a maximal element. We want to prove show that there are $h_b = h_1 + h_2 + \cdots + h_m$ paths from point b to a maximal element. Let $P = \{p_1, p_2, ..., p_\ell\}$ be the chain that starts at point i (where $1 \le i \le m$) and then follows one of the h_i paths from point i to a maximal element. Likewise, let $Q = \{q_1, q_2, ..., q_\ell\}$ be the chain that starts at point j (where $1 \le j \le m$) and then follows one of the h_j paths from point j to a maximal element. Furthermore, specify that $i \ne j$. Since points i and j are on the same level of the poset (each is one level above b), it is impossible to start at i and go through j (and vice versa). Therefore, point i is a part of chain P, but it cannot be a part of chain Q. That is, $i \in P$ but $i \notin Q$. Thus, $P \ne Q$. As a result, $P \cup \{b\} \ne Q \cup \{b\}$. So, each of the h_i distinct chains starting at point i are distinct from each of the h_j distinct chains starting at point j. So, we must sum all of the distinct chains starting points $1, 2, ..., m$ in order to find the total number of chains. Therefore, $h_b = h_1 + h_2 + \cdots + h_m$, as desired.

□

Now that we have proven that our procedure works, we will calculate the number of maximal chains in J_5^+ and combine this with our knowledge of J_n^+ for $n < 5$. Figure 10 shows the values of the decision hubs and minimal elements.

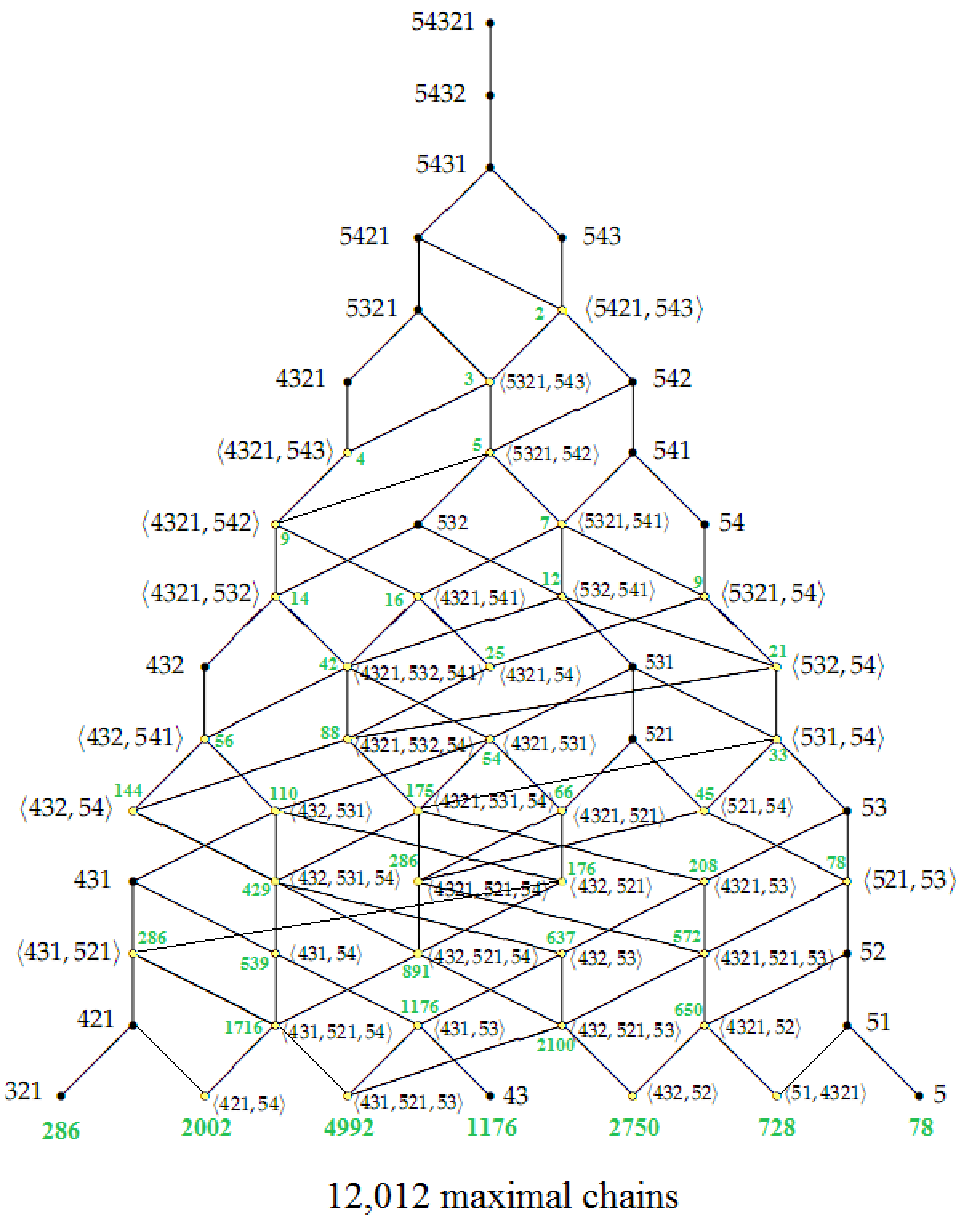

Figure 10: Decision Hubs with Values in J_5^+

Combining this with our previous knowledge, we obtain the sequence of values shown in the following table:

	J_1^+	J_2^+	J_3^+	J_4^+	J_5^+
Number of Maximal Chains	1	1	2	14	12,012

The On-Line Encyclopedia of Integer Sequences (OEIS) does not contain any matches for these values [2]. So, it is possible that we are dealing with an unknown sequence. We believe that the key to finding a formula for this sequence is understanding how J_n^+ lies in J_{n+1}^+. If we could understand this, then we could find a recurrence relation for the number of maximal chains in J_n^+. We could then use the recurrence relation to help us find and prove a formula for the number of maximal chains in J_n^+.

8 Counting Maximal Chains in $M(n)$

Although the procedure described in Section 7 was constructed for counting maximal chains in J_n^+, it might also be effectively applied to $M(n)$. Unlike in J_n^+, we can readily see how $M(n)$ lies in $M(n+1)$. In Figure 11, observe how 2 copies of $M(3)$ lie in $M(4)$.

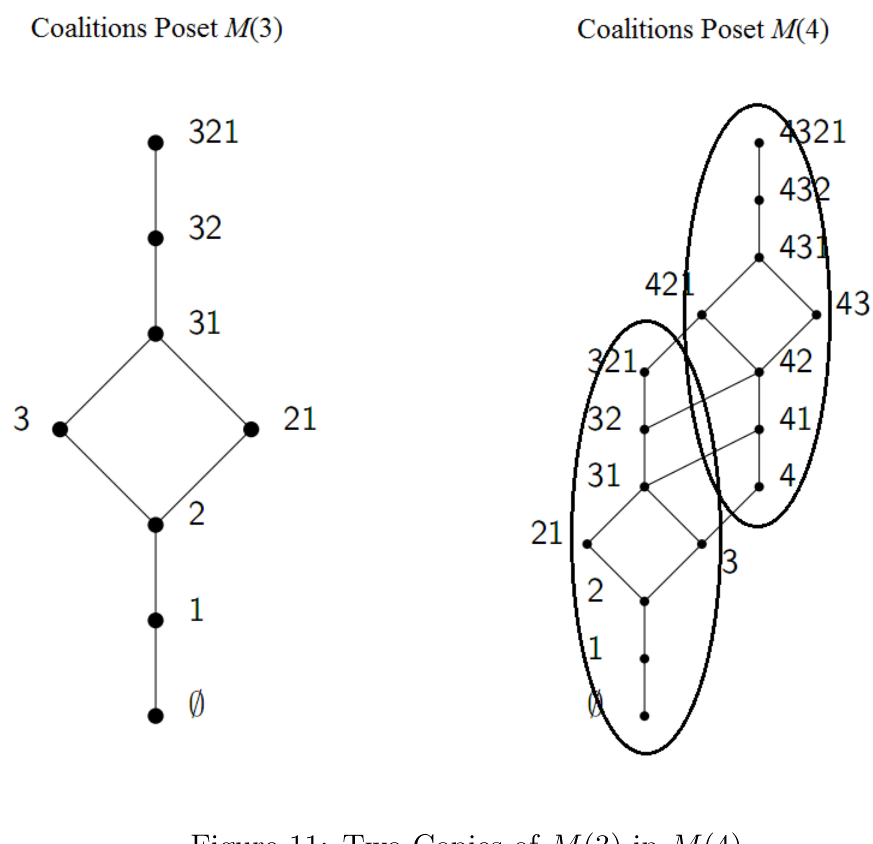

Figure 11: Two Copies of $M(3)$ in $M(4)$

Mason and Parsley discuss why this pattern holds for all values of n [3]. If we were to compare $M(3)$ and $M(5)$, we would notice that 4 copies of $M(3)$ lie inside of $M(5)$. Similarly, if we were to compare $M(3)$ and $M(6)$, we would notice that 8 copies of $M(3)$ lie inside of $M(6)$. In fact, Mason and Parsley generalize this idea to the following proposition:

Proposition 8.1. There are 2^{b-a} copies of $M(a)$ in $M(b)$ [3].

This "nice" behavior of $M(n)$ encouraged us that our procedure would yield a pattern and a formula for the number of maximal chains in $M(n)$. So, we calculated the number of maximal chains in $M(n)$ through $n = 6$. Our results are summarized in the table below:

	$M(1)$	$M(2)$	$M(3)$	$M(4)$	$M(5)$	$M(6)$
Number of Maximal Chains	1	1	2	12	286	33,592

These values match exactly one sequence from the OEIS [2].

Conjecture 8.1. The number of maximal chains $a(n)$ in the coalitions poset with n voters is given by:

$$a(n) = \frac{\binom{n+1}{2}!(1!2!3!...(n-1)!)}{(1!3!5!...(2n-1)!)} = (\frac{(n)(n+1)}{2})! \prod_{i=1}^{n} \frac{(i-1)!}{(2i-1)!}$$

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[3] Sarah Mason and Jason Parsley. A geometric and combinatorial view of weighted voting. September 2011. arXiv:1109.1082v1.

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