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**CYCLICIZERS, CENTRALIZERS  
and NORMALIZERS**

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# Cyclicizers, Centralizers, and Normalizers

David Patrick\* and Eric Wepsic\*

## Abstract

Our goal is to define the cyclicizer, which is analogous to the centralizer and normalizer, and to examine groups in which these subsets have certain special properties.

## 1 Definitions

Note: In this paper, all of the groups mentioned are finite. Also, for notational convenience,  $G^\times$  denotes  $G$  without the identity element.

We begin by restating a familiar definition.

**Definition:** The *centralizer* of an element  $x \in G$ , denoted  $C(x)$ , is defined by

$$C(x) = \{y \in G \mid xy = yx\}.$$

We will also define the normalizer of an element.

**Definition:** The *normalizer* of an element  $x \in G$ , denoted  $N(x)$ , is defined by

$$N(x) = \{y \in G \mid y^{-1}xy \in \langle x \rangle\}.$$

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Note that this is equivalent to  $N(\langle x \rangle)$  under the more customary definition of normalizer.

Finally, we define a new subset of  $G$ , analogous to the two defined above:

**Definition:** The *cyclicizer* of an element  $x \in G$ , denoted  $\text{Cyc}(x)$ , is defined by

$$\text{Cyc}(x) = \{y \in G \mid \langle x, y \rangle \text{ is cyclic}\}.$$

In general,  $\text{Cyc}(x)$  is not a subgroup. For example, in the group  $\mathbf{Z}_2 \oplus \mathbf{Z}_4$ , we have  $\text{Cyc}((0, 2)) = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 3)\}$ , which is not a subgroup. Groups in which all the cyclicizers are subgroups are referred to as *tidy* groups; such groups are discussed in [1].

## 2 The Fundamental Sequence

It is clear that for all groups  $G$  and all  $x \in G$ ,

$$\langle x \rangle \subseteq \text{Cyc}(x) \subseteq C(x) \subseteq N(x).$$

We now consider necessary and sufficient conditions for equality of terms in the above sequence.

### 2.1 Groups in which $\langle x \rangle = \text{Cyc}(x)$

**Theorem 1** *A group  $G$  satisfies  $\langle x \rangle = \text{Cyc}(x)$  for all  $x \in G^\times$  if, and only if, every element of  $G^\times$  is of prime order.*

*Proof:* Suppose there exists  $x \in G^\times$  such that  $x$  is not of prime order.

Choose  $k$  such that  $k$  properly divides the order of  $x$ . Then  $x^k \neq e$  and  $x \in \text{Cyc}(x^k)$ , since  $\langle x, x^k \rangle = \langle x \rangle$ , but  $x \notin \langle x^k \rangle$ . Hence  $\langle x^k \rangle \neq \text{Cyc}(x^k)$ .

Conversely, suppose every non-identity element of  $G$  is of prime order. Then every non-identity element  $x$  of  $G$  is contained in exactly one cyclic subgroup of  $G$ , namely  $\langle x \rangle$ . Therefore,  $\text{Cyc}(x) = \langle x \rangle$ .  $\square$

Finite groups in which every non-identity element has prime order are classified in [2]; the only such groups are  $p$ -groups of exponent  $p$ , certain non-nilpotent groups of order  $p^a q$  (where  $p$  and  $q$  are distinct primes), and  $A_5$ .

## 2.2 Groups in which $\langle x \rangle = \text{Cyc}(x) = C(x)$

We will refer to groups satisfying this condition as *anti-abelian*.

**Theorem 2** *A group  $G$  satisfies  $\langle x \rangle = \text{Cyc}(x) = C(x)$  for all  $x \in G^\times$  if, and only if,  $G$  is either cyclic of prime order or nonabelian of order  $pq$ , where  $p$  and  $q$  are distinct primes.*

*Proof:* If  $|G| = p$ , then  $\langle x \rangle = G = C(x)$  for all  $x \in G^\times$ . If  $|G| = pq$ , and  $G$  is not abelian, then  $|Z(G)| = 1$ , since  $G/Z$  is not cyclic. Therefore, for any  $x \in G^\times$ , we have  $\langle x \rangle \leq C(x) < G$ . But  $[G : \langle x \rangle]$  is prime, so we must have  $\langle x \rangle = C(x)$ .

Conversely, suppose  $G$  is anti-abelian, but not cyclic of prime order. First, we note that for any prime  $p$ , we have  $p^2 \nmid |G|$ . Otherwise,  $G$  contains a subgroup  $H$  of order  $p^2$ , which must be abelian, and must contain an element  $h$  of order  $p$ . But then  $C(h)$  includes  $H$ , so  $|C(h)| \geq p^2$ , a contradiction since  $|h| = p$ . Therefore,  $G$  is of square-free order, and hence [3] must be metacyclic, with generators  $a$  and  $b$  such that  $|a||b| = |G|$ . It follows by Theorem 1 that  $|a| = p$  and  $|b| = q$ , where  $p$  and  $q$  are distinct primes. Finally, if  $G$  is abelian, then  $\langle a \rangle = C(a) = G$ , a contradiction.  $\square$

### 2.3 Groups in which $\text{Cyc}(x) = C(x)$

We will arrive at the classification of such groups via a series of lemmas. But first we define a property of finite groups.

**Definition:** For all  $n \geq 2$ , a group has the *cyclic if commutative* property, denoted  $CC_n$ , if for each  $n$ -tuple  $(x_1, \dots, x_n)$  of group elements,  $\langle x_1, \dots, x_n \rangle$  abelian implies  $\langle x_1, \dots, x_n \rangle$  cyclic.

Note that the condition that  $\text{Cyc}(x) = C(x)$  for all  $x \in G^\times$  is equivalent to  $CC_2$ .

We now show that the  $CC_n$  condition is independent of the choice of  $n$ .

**Lemma 3** *For all groups  $G$  and all  $n \geq 2$ ,  $G$  has property  $CC_n$  if, and only if,  $G$  has property  $CC_2$ .*

*Proof:* Assume  $G$  has property  $CC_n$ , and let  $x, y \in G$  be given such that  $xy = yx$ . Then the  $n$ -tuple  $(x, y, e, \dots, e)$  generates an abelian subgroup, and hence, by hypothesis, a cyclic one. But  $\langle x, y, e, \dots, e \rangle = \langle x, y \rangle$ , so  $\langle x, y \rangle$  is cyclic. Hence  $G$  has property  $CC_2$ .

We prove  $CC_2 \Rightarrow CC_n$  by induction on  $n$ . Let  $n > 2$  be given, and assume that  $CC_2$  implies  $CC_{n-1}$  for  $G$ . Consider  $G$  with property  $CC_2$ , and let  $x_1, \dots, x_n$  be given such that  $\langle x_1, \dots, x_n \rangle$  is abelian. Then  $\langle x_1, \dots, x_{n-1} \rangle \leq C(x_n) = \text{Cyc}(x_n)$ . But  $\langle x_1, \dots, x_{n-1} \rangle$  is cyclic by the inductive hypothesis; let  $g$  be a generator. Then  $g \in \text{Cyc}(x_n)$ , so  $\langle x_1, \dots, x_n \rangle = \langle g, x_n \rangle$  is cyclic. Hence  $G$  has property  $CC_n$ .  $\square$

Call a group  $G$  a *CC-group* if  $G$  satisfies the  $CC_n$  condition for all  $n \geq 2$ . Next we give an alternative characterization of the  $CC$  property, one that will be more useful when we attempt to classify such groups.

**Lemma 4** *For all groups  $G$ ,  $G$  is a CC-group if, and only if, each abelian subgroup of  $G$  is cyclic.*

*Proof:* Suppose each abelian subgroup of  $G$  is cyclic. Let  $x, y \in G$  be given such that  $xy = yx$ . Consider  $H = \langle x, y \rangle$ .  $H$  is abelian, and therefore

by hypothesis is cyclic. So  $x$  and  $y$  generate a cyclic subgroup, and thus  $\text{Cyc}(x) = C(x)$  for all  $x \in G^\times$ . So  $G$  is a  $CC$ -group.

Conversely, suppose  $G$  is a  $CC$ -group. Then any abelian subgroup of  $G$  is finitely generated, and therefore cyclic.  $\square$

We also need the following result:

**Proposition 5** *A  $p$ -group has all its abelian subgroups cyclic if, and only if, it is cyclic or a generalized quaternion group. [4]*

Now we may give a more precise characterization of  $CC$ -groups.

**Theorem 6**  *$G$  is a  $CC$ -group if, and only if, every Sylow subgroup of  $G$  is either cyclic or a generalized quaternion group.*

*Proof:* If  $G$  is a  $CC$ -group, then every subgroup of  $G$  must also have this property; in particular so must the Sylow subgroups of  $G$ . Thus, by Lemma 4, all the abelian subgroups of the Sylow subgroups must be cyclic, and by Proposition 5, all of the Sylow subgroups of  $G$  must be cyclic or generalized quaternion groups.

To prove the converse, suppose that all the Sylow subgroups of  $G$  are either cyclic or generalized quaternion groups. Let an abelian subgroup  $A$  of  $G$  be given. Then  $A$  is a direct product of its Sylow subgroups, as follows:

$$A = A_{p_1} \oplus \cdots \oplus A_{p_k}.$$

But then for each  $i$  we have  $A_{p_i} \leq P_i$ , where  $P_i$  is a Sylow subgroup of  $G$ , so by Proposition 5,  $A_{p_i}$  is cyclic. Hence,  $A$  is cyclic. Thus, since every abelian subgroup of  $G$  is cyclic,  $G$  is a  $CC$ -group by Lemma 4.  $\square$

## 2.4 Groups in which $\text{Cyc}(x) = C(x) = N(x)$

Such groups have a simple classification:

**Theorem 7** *A group  $G$  satisfies  $\text{Cyc}(x) = C(x) = N(x)$  for all  $x \in G^\times$  if, and only if,  $G$  is cyclic.*

*Proof:* Cyclic groups clearly satisfy the condition. For the converse, suppose that  $G$  is a non-cyclic group in which  $\text{Cyc}(x) = C(x)$  for all  $x \in G^\times$ . Then by Theorem 6 there are two cases.

CASE 1: All of the Sylow subgroups of  $G$  are cyclic. Then  $G$  must be metacyclic, and is given by the presentation

$$\langle a, b \mid a^m = b^n = e, b^{-1}ab = a^r \rangle$$

for certain values of  $m, n$  and  $r$ . Then  $b \in N(a)$  but  $b \notin C(a)$ .

CASE 2:  $G$  has a generalized quaternion 2-Sylow subgroup. This subgroup is given by the presentation

$$\langle x, y \mid x^{2^n} = e, y^2 = x^{2^{n-1}}, y^{-1}xy = x^{-1} \rangle$$

for some value of  $n > 1$ . Then  $y \in N(x)$  but  $y \notin C(x)$ .  $\square$

## 2.5 Groups in which $\langle x \rangle = \text{Cyc}(x) = C(x) = N(x)$

These groups also belong to a very restrictive class.

**Theorem 8** *A group  $G$  satisfies  $\langle x \rangle = \text{Cyc}(x) = C(x) = N(x)$  for all  $x \in G^\times$  if, and only if,  $G$  is cyclic of prime order.*

*Proof:* If  $G$  is cyclic of prime order, then  $\langle x \rangle = \text{Cyc}(x) = C(x) = N(x) = G$  for all  $x \in G^\times$ . Conversely, if  $\langle x \rangle = \text{Cyc}(x) = C(x) = N(x)$  for all  $x \in G^\times$ , then by Theorem 2,  $G$  is either cyclic of prime order or nonabelian of order  $pq$ . Furthermore, Theorem 7 implies  $G$  is cyclic, so  $G$  is cyclic of prime order.  $\square$



## 2.6 Groups in which $C(x) = N(x)$

Unfortunately, we cannot classify groups which satisfy  $C(x) = N(x)$  for all  $x \in G^\times$ . However, we do know some facts about this property.

- If  $G$  is abelian, then  $C(x) = N(x) = G$  for all  $x \in G^\times$ .
- If  $G$  is a group with this property, then any normal cyclic subgroup of  $G$  must be contained in the center of  $G$  (since if  $h$  is a generator of  $H \triangleleft G$ , we must have  $C(h) = N(h) = G$ ).
- If  $G$  is a nonabelian  $p$ -group with this property, then there is no element  $x \in G$  such that  $[G : \langle x \rangle] = p$  (since if such an  $x$  exists, we must have  $N(x) = G$ , so  $C(x) = G$ , which implies  $\langle x \rangle \leq Z(G)$ ; but then  $G/Z(G)$  is cyclic, a contradiction).
- The smallest nonabelian group with this property is  $A_4$ .

## 3 Some special classes of centralizers

We will now give some results on groups in which we restrict the centralizers of the group.

### 3.1 Groups in which the centralizers are cyclic

We first prove the following lemma about groups with cyclic centralizers:

**Lemma 9** *A group  $G$  is such that  $Cyc(x)$  is a cyclic subgroup for all  $x \in G^\times$  if, and only if, each element  $x \in G^\times$  is contained in exactly one maximal cyclic subgroup.*

*Proof:* Suppose every element of  $G$  is contained in exactly one maximal cyclic subgroup. Let  $x \in G^\times$  be given, and choose  $y$  such that  $x \in \langle y \rangle$ ,

where  $\langle y \rangle$  is a maximal cyclic subgroup. Then if  $g \in \text{Cyc}(x)$ , we must have  $\langle x, g \rangle \leq \langle y \rangle$ , so  $g \in \langle y \rangle$ . Also, if  $h \in \langle y \rangle$ , then  $\langle x, h \rangle \leq \langle y \rangle$ , so  $\langle x, h \rangle$  is cyclic, giving  $h \in \text{Cyc}(x)$ . Therefore  $\text{Cyc}(x) = \langle y \rangle$ , and hence  $\text{Cyc}(x)$  is cyclic.

Conversely, suppose there exists  $x \in G^\times$  such that  $x \in \langle y \rangle$  and  $x \in \langle z \rangle$ , where  $\langle y \rangle$  and  $\langle z \rangle$  are distinct maximal cyclic subgroups. Suppose  $\text{Cyc}(x)$  is cyclic. Then since  $\langle y \rangle \leq \text{Cyc}(x)$ , we have  $\langle y \rangle = \text{Cyc}(x)$ . Similarly  $\langle z \rangle = \text{Cyc}(x)$ , and hence  $\langle y \rangle = \langle z \rangle$ , a contradiction.  $\square$

This allows us to give a characterization of all groups in which  $C(x)$  is cyclic for all  $x$ . Clearly, if  $G$  is abelian, then  $C(x)$  is cyclic for all  $x \in G^\times$  if, and only if,  $G$  is cyclic. The following theorem deals with the case in which  $G$  is nonabelian. ( $(m, n)$  denotes the greatest common divisor of  $m$  and  $n$ .)

**Theorem 10**  *$G$  is a nonabelian group such that  $C(x)$  is cyclic for all  $x \in G^\times$  if, and only if,  $G$  is given by the presentation*

$$\langle a, b \mid a^m = b^n = e, b^{-1}ab = a^r \rangle,$$

where  $(m, n) = 1, (r^j - 1, m) = 1$  for  $1 \leq j < n$ , and  $r^n \equiv 1 \pmod{m}$ .

*Proof:* Suppose  $G$  is nonabelian, and that  $C(x)$  is cyclic for all  $x \in G^\times$ . Then  $\text{Cyc}(x) = C(x)$  for all  $x \in G^\times$ , since if  $xy = yx$ , then  $\langle x, y \rangle$  is a subgroup of  $C(x)$ , and is thus cyclic. Therefore, by Theorem 6, every  $p$ -Sylow subgroup of  $G$  must be either cyclic or a generalized quaternion group. But  $G$  cannot have a generalized quaternion subgroup, since it would then contain a centralizer which is not cyclic (specifically, the centralizer of any element in the center of a quaternion group is the entire quaternion group, which is not cyclic). Therefore, every  $p$ -Sylow subgroup of  $G$  is cyclic, and hence  $G$  is metacyclic, with presentation

$$G = \langle a, b \mid a^m = b^n = e, b^{-1}ab = a^r \rangle,$$

where  $(m, n) = 1, (r - 1, m) = 1$  and  $r^n \equiv 1 \pmod{m}$ . [3]

Now suppose there exists  $1 < j < n$  such that  $(r^j - 1, m) = k \neq 1$ . Let  $l = m/k$ . Then the cyclic subgroup generated by  $b$  is  $\{e, b, b^2, \dots, b^{n-1}\}$ , and the cyclic subgroup generated by  $a^l b a^{-l}$  is  $\{e, b a^{l(r-1)}, b^2 a^{l(r^2-1)}, \dots, b^n a^{l(r^{n-1}-1)}\}$ . In particular,  $b^j = b^j a^{l(r^j-1)}$ , so these two maximal cyclic subgroups have a

nontrivial intersection, which contradicts Lemma 9. Therefore,  $(r^j - 1, m) = 1$  for all  $1 \leq j < n$ .

For the converse, suppose that  $G$  is given by the presentation above. Then every element of  $G$  is expressible in the form  $b^i a^j$  for suitable  $i, j$ , and furthermore  $b^i a^j b^k a^l = b^{i+k} a^{r^k j + l}$  for all  $0 \leq i, k < n$  and  $0 \leq j, l < m$ . Thus,  $b^i a^j$  and  $b^k a^l$  commute if and only if  $r^k j + l \equiv r^i l + j \pmod{m}$ , which is equivalent to  $(r^k - 1)j \equiv (r^i - 1)l \pmod{m}$ .

So let  $g = b^i a^j \in G^\times$  be given, and consider an element  $b^k a^l$  in  $C(g)$ . We assume that  $0 \leq i, k < n$  and  $0 \leq j, l < m$ . There are two cases:

CASE 1:  $i = 0$ . Then we must have  $(r^k - 1)j \equiv 0 \pmod{m}$ , so since  $j \neq 0$  (we are only concerned with non-identity elements of  $G$ ), we must have  $(r^k - 1, m) \neq 1$ . But then  $k = 0$ , by hypothesis, so  $b^k a^l = a^l$ , and hence  $C(g) \leq \langle a \rangle$ .

CASE 2:  $i \neq 0$ . Then for each value of  $k$  in  $0, \dots, n - 1$ , we must have  $l \equiv \frac{r^k - 1}{r^i - 1} j \pmod{m}$ , since  $(r^i - 1, m) = 1$ . Therefore there is exactly one element in  $C(g)$  for each value of  $k$  in  $0, \dots, n - 1$ , and hence  $|C(g)| = n$ . But we know that  $\langle g \rangle \leq C(g)$ , and  $|\langle g \rangle| = n$ , so  $C(g) = \langle g \rangle$ .

In either case,  $C(g)$  is cyclic.  $\square$

### 3.2 Groups in which the centralizers are normal

We cannot fully classify these groups. We do, however, have the following:

**Theorem 11** *If  $G$  is a group such that  $G' \leq Z(G)$ , then  $C(x)$  is normal for every  $x \in G$ .*

*Proof:* Suppose  $G' \leq Z(G)$ , and let  $x \in G$  be given. Then  $G' \leq Z(G) \leq C(x)$ , so  $C(x) \triangleleft G$ .  $\square$

It is conjectured that the converse is also true; i.e.  $C(x)$  is normal for all  $x \in G$  if, and only if,  $G' \leq Z(G)$ . This conjecture has been verified, via CAYLEY, for all groups of order less than or equal to 100.

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