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WITH RATIONAL MEDIANS

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Abstract: A characterization of all integer-sided triangles with a rational median is given, similar to the characterization of Pythagorean triangles. An infinite family of integer-sided triangles with two rational medians is given along with several examples with three rational medians. All examples come from solutions to systems of quadratic Diophantine equations.

§1

Let A, B, and C be the lengths of the sides of a triangle, as in Figure 1. If x is twice the length of the median to C, then A, B, C, and x satisfy the well-known equation

$$(1) \qquad x^2 = 2A^2 + 2B^2 - C^2$$

which follows easily from the law of cosines. If A, B, and C are integers, then x is the square root of an integer, so if x is rational, it must be an integer, and hence, the length of the median is an integer or half an integer. A complete solution to the Diophantine equation (1) is given in Theorem 1 below.

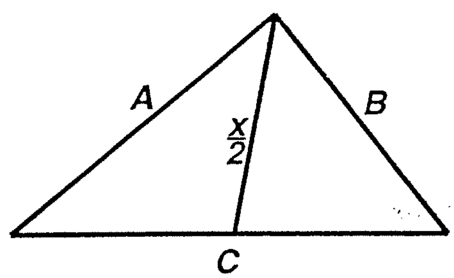

Fig. 1

As an illustration, consider the triple $(A,B,C) = (4,7,9)$. Equation (1) gives $x = 7$. Similarly, (5,10,13), (6,13,17), and (7,16,21) yield 9,11, and 13, respectively, for x. Notice that these four triples form arithmetic progressions in each of the components, as do the double medians. If we assume that solutions to the Diophantine equation (1) can be grouped into families in arithmetic progression, then the necessary (and sufficient!) condition in (2) becomes evident.

<u>Theorem 1:</u> Let $b > d > 0$ be integers. Define integers a and c by

$$(2) \qquad \frac{a}{c} = \frac{b-d}{b+d}$$

in lowest terms. Then for $n = 1,2,3,\ldots$ the triple

$$(A,B,C) = \big(a\,n + b,\ c\,n + d,\ (a + c)\,n + (b - d)\big)$$

yields an integer solution for x in equation (1). Further, every solution to (1) is a member of such a one-parameter family of triples.

The proof is straight-forward: First notice that with a, b, c, d, A, B and C so defined, we have $A+B-C = 2d>0$, $A+B-C = 2an+2(b-d)>0$, and $B+C-A = 2cn>0$, so (A, B, C) is indeed a triangle.

Now we need to show that $2A^2 + 2B^2 - C^2$, as given, can be written as a perfect square. Note that (2) gives

$$a\,b + a\,d = b\,c - c\,d$$
$$4a\,b + 4a\,d = 4b\,c - 4c\,d$$
$$2a\,b + 2a\,d - 2b\,c + 6c\,d = -2a\,b - 2a\,d + 2b\,c + 2c\,d \qquad (3)$$

Now,

$$\begin{aligned} x^2 &= 2(a\,n + b)^2 + 2(c\,n + d)^2 - ((a + c)n + (b-d))^2 \\ &= (a^2 - 2a\,c + c^2)n^2 + (2a\,b + 2a\,d - 2b\,c + 6c\,d)n + (b^2 + 2b\,d + d^2) \\ &= (a - c)^2 n^2 + (-2a\,b - 2a\,d + 2b\,c + 2c\,d)n + (b + d)^2 \\ &= [\,(a - c)n - (b + d)\,]^2, \end{aligned}$$

applying equation (3) at the second step.

Conversely, if A,B,C, and x satisfy (1), let

$$d = \frac{A + B - C}{2} \quad \text{and} \quad b = \frac{A - B + x}{2}$$

If $A \equiv B \pmod 2$, then from (1) we have $x^2 + C^2 \equiv 2(A^2 + B^2) \equiv 0 \pmod 4$ so x and C must both be even, and if $A \not\equiv B \pmod 2$, then $x^2 + C^2 \equiv 2 \pmod 4$, so x and C must both be odd. In either case, d and b turn out to be integers. It is easy to check that these are the right values of b and d in the construction above.

Note that A and B could be switched, giving a different value for b. Thus, each of these triangles belong to two of the one-parameter families.

As an example, take $b = 4$ and $d = 1$. Then, $\frac{a}{c} = \frac{4-1}{4+1} = \frac{3}{5}$ and the triples

$(3n+4,\ \ 5n+1,\ \ 8n+3)$, $n = 1,2,3,\ldots$ represent an infinite family of integer-sided triangles with a rational median.

§2

We can use the solution above to search for integer-sided triangles with two rational medians. Given a family $(an + b,\ cn + d,\ (a + c)n + (b - d)) = (A,B,C)$, we know that x, the median drawn to C, is and integer. If y is twice the median drawn to B, could y also be an integer? We must have

$$y^2 = 2A^2 + 2C^2 - B^2$$

$$= (2a + c)^2n^2 + (8ab - 4ad + 4cb - 6cd)n + (2b - d)^2$$

$$= \left((2a + c)n + \frac{4ab - 2ad + 2cb - 3cd}{2a + c}\right)^2 + (2b - d)^2 - \left(\frac{4ab - 2ad + 2cb - 3cd}{2a + c}\right)^2$$

by completing the square in n. Clearing fractions and simplifying yields

$$((2a + c)y)^2 = ((2a + c)^2n + 4ab - 2ad + 2cb - 3cd)^2 + 24abcd.$$

Since $24abcd$ is positive, in order for the right side to be a perfect square, like the left side, we must have

$$((2a + c)^2n + 4ab - 2ad + 2cb - 3cd)^2 + 24abcd$$
$$\geq ((2a + c)^2n + 4ab - 2ad + 2cb - 3cd + 1)^2,$$

since there can't be a perfect square between m^2 and $(m+1)^2$. This inequality reduces to

$$n \le \frac{24abcd + 6cd + 4ad - 4bc - 8ab - 1}{2(2a + c)^2} = B_1$$

Similarly, if $n > B_2$, $z^2 = 2B^2 + 2C^2 - A^2$ cannot be a square for an appropriate bound B_2.

Now, if we choose $b > d$ in Theorem 1, we need only check $n \le max\{B_1, B_2\}$ in the corresponding infinite family of triangles for triangles with two rational medians. We have computed a large list of examples using this technique, and there are about 5,000 triangles with integer sides, two rational medians, and perimeter less than 2,000. The next theorem says there are infinitely many integer triangles with 2 rational medians.

Theorem 2: For $n = 1,2,3, \ldots$ the triple

$$\left(2n + 5, \frac{n^2 + 5n + 2}{2}, \frac{n^2 + 5n + 8}{2}\right) = (A, B, C)$$

represents a solution to the Diophantine system:

$$x^2 = 2A^2 + 2B^2 - C^2$$
$$z^2 = 2B^2 + 2C^2 - A^2$$

The proof is straight-forward algebra and is omitted. The first few members of this family are $(A, B, C, x, z) = (7,4,7,9,9)$, $(9,8,11,17,13)$, $(11,13,16,27,18)$, ...

§3

Among the 5,000 triangles with two rational medians and perimeter less than 2,000, there are 24 which actually have three rational medians. Each of these represents a solution to the Diophantine system:

$$x^2 = 2A^2 + 2B^2 - C^2$$
$$y^2 = 2A^2 + 2C^2 - B^2$$
$$z^2 = 2B^2 + 2C^2 - A^2$$

Listed below are the triples (A,B,C), with $A+B+C < 2000$, which are primitive, that is, $gcd\,(A,B,C) = 1$, along with the double medians x, y, *and* z.

Table 1

	A	B	C	x	y	z
1.	68	85	87	158	131	127
2.	113	243	290	523	367	244
3.	127	131	158	261	255	204
4.	142	463	529	984	621	435
5.	145	207	328	529	463	142
6.	159	314	325	619	404	377
7.	208	659	683	1326	765	699
8.	233	255	442	683	659	208
9.	244	367	523	870	729	339
10.	277	446	477	881	640	569
11.	327	386	409	725	632	587
12.	377	404	619	975	942	477
13.	381	393	474	783	765	612
14.	466	491	807	1252	1223	515

Notice that (x, y, z) of line 5 above is the same as (A, B, C) of line 4. This is typical and is a consequence of the following construction, where a new triangle is formed from the double medians of a triangle with sides α, β, γ:

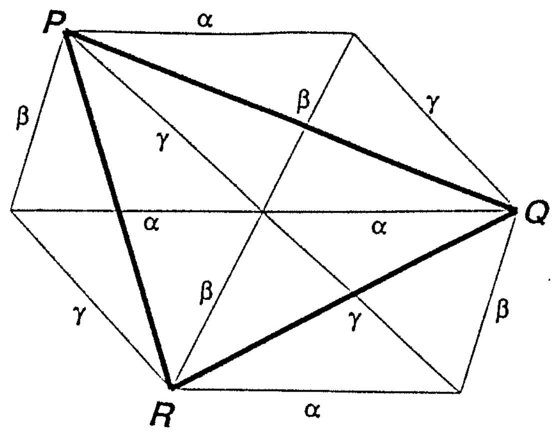

Fig. 2

PQ is a diagonal of a parallelogram, and therefore bisects the other diagonal, so half of *PQ* is a median of the triangle. It is evident that the new triangle *PQR* has sides of length equal to twice the medians of the smaller triangle, and that the new medians have length $^3/_2\alpha$, $^3/_2\beta$, $^3/_2\gamma$. So the triangles with 3 rational medians occur in pairs, related by this construction.

§4

Using Heron's formula for the area of a triangle, we see that existence of a triangle with three rational medians and integer (or rational) area implies a solution to the Diophantine system:

$$x^2 = 2A^2 + 2B^2 - C^2$$
$$y^2 = 2A^2 + 2C^2 - B^2$$
$$z^2 = 2B^2 + 2C^2 - A^2$$
$$(AREA)^2 = s\,(s - A)(s - B)(s - C)$$

where $s = {}^1/_2\,(A + B + C)$ is the semiperimeter.

There is no solution known to this system at this time. The problem is similar to the more famous unsolved problem of finding a box with edges, face diagonals and body diagonal all rational. Such a box also involves seven quantities which must satisfy a system of four Diophantine equations:

$$d^2 = a^2 + b^2$$
$$e^2 = a^2 + c^2$$
$$f^2 = b^2 + c^2$$
$$g^2 = a^2 + b^2 + c^2$$

where a, b, and c are the lengths of the edges. See Guy [1].

REFERENCE

1. Richard K. Guy, A Dozen Difficult Diophantine Dilemmas, *American Mathematical Monthly* 95(1988) 31-36