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Equilateral Dimension of Riemannian Manifolds with Bounded Curvature

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Abstract. The equilateral Dimension of a riemannian manifold is the maximum number of distinct equidistant points. In the first half of this paper we will give upper bounds for the equilateral dimension of certain Riemannian Manifolds. In the second half of the paper we will introduce a new metric invariant, called the *equilateral length*, which measures size of the equilateral dimension. This will then be used in the recognition program in Riemannian geometry, which seeks to identify certain Riemannian manifolds by way of metric invariants such as diameter, extent, or packing radius.

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An equilateral set in a given metric space, X, is a set E such that the distance between any pair of distinct points in E is r, where r is some positive constant. The equilateral dimension, denoted $e(X)$ can thus be defined to be the maximum cardinality of all equilateral sets. It is important to note that for general metric spaces the equilateral dimension need not be finite. A simple example of this is $\mathbb{Z}$, with the discrete metric:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y; \\ 0 & \text{if x=y} \end{cases}$$

The equilateral dimension of $\mathbb{R}^n$ is simply $n+1$. This can be seen by noting that every point of an equilateral set with m members must lie on the intersection of the boundary of $m-1$ metric balls. Similiarly, the equilateral dimension of S^n, with metric induced from its canonical Riemannian metric, is $n+2$. Here we see that while S^n locally "looks" like R^n, its equilateral dimension is nonetheless greater. This simple example illlustrates how geometric invariants of Riemannian manifolds, such as sectional or Ricci curvature, can manifest in the very coarse notion of the equilateral dimension, even if the two manifolds are diffeomorphic.

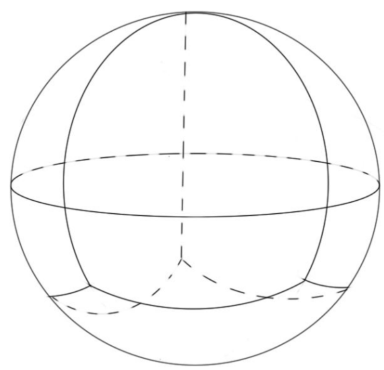

Figure 1: Four equidistant points in S^2

These considerations naturally lead to the question: Can the equilateral dimension of an arbitrary Riemannian manifold be bounded by some constant? The case of $\mathbb{R}^n$, where $e(\mathbb{R}^n) = n+1$, demonstrates that this bound will have to depend on the topological dimension of the manifold. The difficulty of answering this question arises because the equilateral dimension is an inherently global notion, and while the distance may be smooth inside the injectivity radius, we cannot always form uniform lower bounds on the injectivity radius. Hence a proof similiar to the case of $\mathbb{R}^n$ will fail for general manifolds, in which the boundary of open balls need not be, for example, smooth submanifolds.

The problem of the equilateral dimension of a Riemannian manifold was initally posed by Richard Kusner in [Gu]. Kusner conjectured that the equilateral dimension of a Riemannian manifold of dimension n will always be $\leq n+2$. This paper will give sufficient conditions for this upper bound to hold, as well as other upper bounds which depend on the manifold's dimension. Our main tool will be various comparison theorems, and a key ingredient is the assumption that the manifold is complete.

This paper is divided into three parts. First, we will examine the case of nonnegative Ricci curvature using the Bishop-Gromov Volume Comparison Theorem to derive an upper bound of $e(M)$. The fact that the Ricci curvature is nonnegative is necessary for the upper bound to depend solely on the dimension. A weaker result holds for Ricci curvature bounded below, but in this case the upper bound will depend on the distance between the points in the equilateral sets.

The second part will be devoted to the case of non-negative sectional curvature, and in this case, Topogonov's Theorem will be used for the same purpose. These upper bounds will depend solely on the dimension of the manifold, and will depend on the metric structure of S^n.

Last, we will obtain various results which translate Grove's rigidity results concerning the packing radius of Alexandrov Spaces to equilateral sets, and introduce a metric invariant which measures the "length" of equilateral sets of a particular order. In other words, we will discuss the repercussions of the existence of various types of equilateral sets on the geometric and topological structure of the manifold. Results concerning the packing radius will be used extensively throughout the last section.

This paper will assume a knowledge of elementary Riemannian Geometry that may be found in [Pe],[dC], [Le], or [CE].

1 The Case of Non-negative Ricci Curvature

In this section, we derive an upper bound for the equilateral dimension of complete manifolds whose Ricci curvature is non-negative. Our main tool will be Gromov's generalization of Bishop's Volume Comparison Theorem. For a more general statement of the theorem the reader should consult [Z]. In general, volume of metric balls in Riemannian manifolds can be extremely difficult to compute, but the Bishop-Gromov Volume Comparison restricts the growth of these balls, which will be the pith of the proof of our first main theorem.

The following theorem gives an upper bound for the growth of balls centered at a given point using spaces whose volumes can be more easily computed: the simply connected space forms of constant curvature, such as $\mathbb{H}^n, \mathbb{R}^n$, and S^n. In this paper, any expression involving a superscript H denotes the analogous quantity in the simply connected space forms of constant curvature H.

Gromov's Relative Volume Comparison Theorem 1. *Let (M^n, g) be a complete Riemannian manifold of dimension n, and $\mathrm{Ric}(M) \geq (n-1)H$, then for any $p \in M^n$ and any $0 < r \leq R$*

$$\frac{\mathrm{vol}(B_p(R))}{\mathrm{vol}(B_p(r))} \leq \frac{\mathrm{vol}^H(B(R))}{\mathrm{vol}^H(B(r))}$$

This theorem comes from a comparison theorem concerning the distance function from a fixed point. The comparison theorem gives an upper bound for the rate of change of the area form of geodesic spheres along radial geodesics in terms of the corresponding quantity in the model space. The assumption that the manifold is complete is necessary in order to obtain the previous global result. The classical Hopf-Rinow Theorem states that completeness, in the metric sense, is equivalent to any two points being connected by a length-minimizing geodesic. This allows one to express the volume of a ball by integrating the area form along radial geodesics, while avoiding the points in which the distance function may not be smooth, thereby yielding a global result.

Using this result, we may obtain the following upper bound for the equilateral dimension. We cannot say if this upper bound is sharp. The proof should be compared with the proof of Gromov's Packing Lemma in [Z],[Gr], or [Pe].

Main Lemma 1. *Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 2$ and* $\mathrm{Ric}(M) \geq 0$. *Then given any equilateral set $E = \{p_1, p_2, \ldots, p_m\}$, then $m \leq 3^n$.*

Proof. Let E be a equilateral set of distinct points $p_1, p_2, \ldots, p_m$ such that $d(p_i, p_j) = r$ for all distinct $1 \leq i, j \leq m$. Then by the triangle inequality we have that for all $1 \leq i, j \leq m$ $B_{p_i}(\frac{r}{2}) \subseteq B_{p_j}(\frac{3r}{2})$, and when i, j are distinct $B_{p_i}(\frac{r}{2}) \cap B_{p_j}(\frac{r}{2}) = \emptyset$. Without loss of generality we may assume that $\mathrm{vol}(B_{p_1}(r)) = \min_{1 \leq i \leq m} \mathrm{vol}(B_{p_i}(r))$.

$$\mathrm{vol}(B_{p_1}\left(\tfrac{3r}{2}\right)) \geq \sum_{i=1}^{m} \mathrm{vol}(B_{p_i}(\tfrac{r}{2})) \geq m \cdot \mathrm{vol}(B_{p_1}(\tfrac{r}{2}))$$

As $\mathrm{Ric}(M^n) \geq 0$, by the previous theorem we can bound m using the volume of balls in $\mathbb{R}^n$. Therefore,

$$m \leq \frac{\mathrm{vol}(B_{p_1}(\frac{3r}{2}))}{\mathrm{vol}(B_{p_1}(\frac{r}{2}))} \leq \frac{\omega_n(\frac{3r}{2})^n}{\omega_n(\frac{r}{2})^n} = 3^n$$

where ω_n denotes the volume of a ball of radius one in $\mathbb{R}^n$. □

Here we should note that the proof of this theorem relies solely on the packing type of complete Riemannian manifolds of dimension n and nonnegative Ricci curvature. For more concerning the packing type of a metric space the reader may consult [Gr].

Our main theorem is a simple corollary of this lemma:

Main Theorem 1. *Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 2$ and* $\mathrm{Ric}(M^n) \geq 0$, *then $e(M^n) \leq 3^n = C(n)$.*

2 The Case of Positive Sectional Curvature

We next shift our attention to the investigation of the equilateral dimension of manifolds with nonnegative and positive sectional curvature. Throughout this section our main tool will be the classical Topogonov's Theorem, which is a global generalization of the Rauch Comparison Theorems. For a detailed proof the reader should consult [CE].

In the proceeding definition all indices are to be taken modulo 3. Let M be a Riemannian manifold, then a *geodesic triangle* in M is a triplet of geodesic segments, parameterized by arc length $(\gamma_1, \gamma_2, \gamma_3)$ of lengths (l_1, l_2, l_3) such that $\gamma_i(l_i) = \gamma_{i+1}(0)$ and $l_{i+1} \geq l_{i+2}$, for all $i = 1, 2, 3$. We will let

$$\alpha_i = \measuredangle(-\gamma_i'(l_i), \gamma_{i+1}'(0))$$

We now state part Topogonov's Theorem:

Topogonov's Theorem 1. *Let M^n be a complete manifold, with $\sec(M^n) \geq H$.*

A. Let $(\gamma_1, \gamma_2, \gamma_3)$ be a geodesic triangle in M^n such that all geodesics are minimizing and if $H > 0$ then $l_i \leq \frac{\pi}{\sqrt{(H)}}$. Let M^H denote the 2-dimensional space form of constant curvature H. Then there exists a geodesic triangle $(\bar{\gamma_1}, \bar{\gamma_2}, \bar{\gamma_3})$, such that for all $i = 1, 2, 3$ $l_i = \bar{l_i}$ and $\alpha_i \leq \bar{\alpha}_i$.

B. Let γ_1, γ_2 be minimizing normal geodesics in M^n such that $\gamma_1(l_1) = \gamma_2(0)$, of lengths l_1, l_2 and $\alpha = \measuredangle(-\gamma_1'(l_i), \gamma_2'(0))$. We call such a configuration a hinge. Additionally, if $H > 0$ assume that

$$l_2 \leq \frac{\pi}{\sqrt{H}}$$

Then there exists a hinge $(\bar{\gamma_1}, \bar{\gamma_2}, \alpha)$ of length l_1, l_2 in M^H, such that

$$d(\gamma_1(0), \gamma_2(l_2)) \leq d(\bar{\gamma_1}(0), \bar{\gamma_2}(l_2))$$

C If equality holds in the above equation, then γ_1, γ_2 spans a surface with a totally geodesic interior which is isometric to the triangular surface spanned by $\bar{\gamma_1}, \bar{\gamma_2}$

Roughly speaking, the first part states that the angles of any geodesic triangle in a complete Riemannian manifold with sectional curvature $\geq H$ will always be greater than the corresponding angles formed by a triangle of the same lengths in the corresponding model space. The second part states that the distance between the endpoints of two geodesics (emanating from a common point with a given angle) will grow no faster than two geodesics with the same lengths and angle in the corresponding model space.

Throughout the rest of the paper we will mainly use result (A) of Topogonov's theorem. As we are to be dealing with equilateral sets, the angle of a spherical equilateral triangle will be of particular interest. Here the angles are easily computed using the spherical law of cosines.

Theorem 2.1. *Let $(\gamma_1, \gamma_2, \gamma_3)$ be an equilateral geodesic triangle of length l in a complete Riemannian manifolds M^n with $\sec(M^n) \geq 1$. Then for all $i = 1, 2, 3$*

$$\cos \alpha_i \leq \frac{\cos l}{1 + \cos l}$$

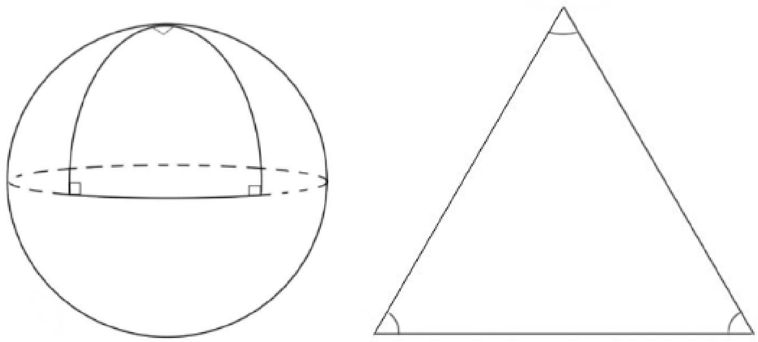

Figure 2: Equilateral triangles of length $\frac{\pi}{2}$ in S^2 and $\mathbb{R}^2$.

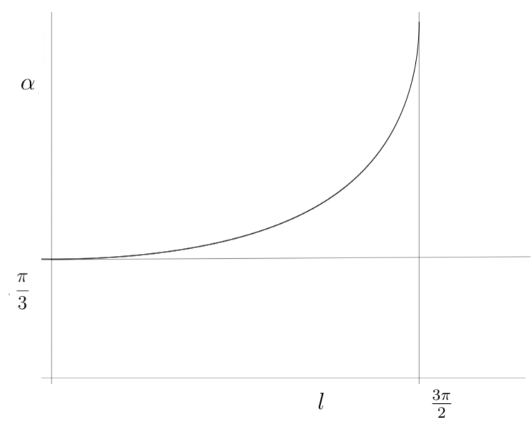

Figure 3: $\cos\alpha = \frac{\cos l}{1+\cos l}$

Proof. By Topogonov's theorem, there exists $\{a, b, c\} \subset S^2$ such that $|ab| = |bc| = |ac| = l$, of angle $\bar{\alpha}$, satisfying $\bar{\alpha} \leq \alpha_i$ for $i = 1, 2, 3$. By the spherical law of cosines we have that

$$\cos l = \cos^2 l + \sin^2 l \cos\bar{\alpha}$$

Therefore by simple algebra, this implies that for $i = 1, 2, 3$,

$$\cos\alpha_i \leq \cos\bar{\alpha} = \frac{\cos l}{1 + \cos l}$$

□

From this formula, it immediately follows that any equilateral geodesic triangle in S^2 must have length less than $\frac{3\pi}{2}$.

Given any compact metric space X, we define the *capacity* of X, denoted $\mathrm{Cap}_X(r)$, to be the maximum number of disjoint balls of radius $\frac{r}{2}$ contained in X. For more about the capacity and its relation to the Gromov-Hausdorff topology, the reader should consult Chapter 10 of [Pe].

Main Lemma 2. *Let M^n be a complete Riemannian manifold of nonnegative sectional curvature. Given any equilateral set $E = \{p_1, p_2, \dots, p_m\}$ of length r, Then*

$$m \leq \mathrm{Cap}_{S^{n-1}}\left(\frac{\pi}{6}\right) = C(n)$$

If in addition $\sec(M^n) \geq 1$, *then*

$$m \leq \mathrm{Cap}_{S^{n-1}}\left(\arccos\left(\frac{\cos r}{1+\cos r}\right)\right)$$

Proof. Suppose $E = \{p_1, p_2, \dots, p_m\}$ is an equilateral set of length r in a Riemannian Manifold of non-negative sectional curvature. As M^n is complete, it follows from the classical Hopf-Rinow theorem that each pair of points $\{p_i\}_{i=1}^m$ can be joined by a length-minimizing geodesic, and for all $2 \leq i < j \leq m$, the points (p_1, p_i, p_j) form an equilateral geodesic triangle of length r. For each $i \geq 2$, let γ_i be a normal minimizing geodesic from p_1 to p_i. It then follows that $\|\gamma_i'(0)\| = 1$ for each i.

Let $I : T_{p_1}M^n \to \mathbb{R}^n$ be a linear isometry, then under this isometry we can identify $\{\gamma_i'(0)\}_{i=2}^n$ as a subset of $S^{n-1} \subset \mathbb{R}^n$. We also let, for $2 \leq i < j \leq m$, $\theta_{i,j} = \measuredangle(\gamma_i'(0), \gamma_j'(0)) = d^{S^{n-1}}(\gamma_i'(0)), \gamma_j'(0))$, where $d^{S^{n-1}}$ is the canonical metric on S^{n-1}. Then by Topogonov's theorem, we have that for distinct i, j and for all $2 \leq i, j \leq m$, $\theta_{i,j} \geq \frac{\pi}{3}$, as the angle of an equilateral triangle in $\mathbb{R}^2$ is equal to $\frac{\pi}{3}$. Hence each

$$B^{S^{n-1}}_{\gamma_i'(0)}\left(\frac{\pi}{6}\right) \subseteq B^{S^{n-1}}_{\gamma_i'(0)}(\theta_{i,j})$$

and for all distinct i, j and $2 \leq i, j \leq m$, $B^{S^{n-1}}_{\frac{\pi}{6}}(\gamma_i'(0)) \cap B^{S^{n-1}}_{\frac{\pi}{6}}(\gamma_j'(0)) = \emptyset$. It then trivially follows from the definition of Cap_{S^n} that $m \leq \mathrm{Cap}_{S^n}(\frac{\pi}{6})$.

The next statement follows from an identical argument as before, we just use Topogonov's theorem with $H = 1$, in which case,

$$\theta_{i,j} \geq \arccos\left(\frac{\cos r}{\cos r + 1}\right)$$

And hence,

$$m \leq \mathrm{Cap}_{S^{n-1}}\left(\arccos\left(\frac{\cos r}{\cos r + 1}\right)\right)$$

□

An immediate consequence of the previous lemma gives us the main theorem of this section:

Main Theorem 2. *Let M^n be a complete Riemannian manifold of dimension n, with $sec(M^n) \geq 0$, then there exists a constant $C(n) = Cap_{S^{n-1}}(\frac{\pi}{6})$, depending only on n such that $e(M^n) \leq C(n)$.*

3 Packing Radius and Equilateral Length

Using theorems established in [GM1,2], throughout this section we will explore the relationships between the geometry of complete manifolds with sectional curvature ≥ 1 and the existence of certain equilateral sets. There have been many deep results concerning complete manifolds with sectional curvature ≥ 1, including Bonnet's Theorem and Topogonov's Maximal Diameter Theorem, which states that for such manifolds, the diameter must be $\leq \pi$, with equality holding only if the manifold is isometric to S^n.

We shall use various applications of the *q-th packing radius*, denoted $pack_q X$, of a compact metric space (X, d). This is defined to be the largest $r > 0$ such that X contains q disjoint balls of radius r, or

$$\operatorname{pack}_q X = \frac{1}{2} \max_{x_1,\ldots,x_q} \min_{1 \leq i < j \leq q} d(x_i, x_j)$$

An important aspect of the packing radius is that it is non-increasing in q, that is,

$$\frac{1}{2} \operatorname{diam} X = \operatorname{pack}_2 X \leq \operatorname{pack}_3 X \leq \ldots \operatorname{pack}_q X \leq \ldots$$

By induction, it is not difficult to prove that $\operatorname{pack}_{n+2} S^n = \frac{1}{2}\arccos(\frac{-1}{n+1})$, and $\operatorname{pack}_{n+3} S^n = \frac{\pi}{4}$. A proof of this result is provided in [GW]. For convenience, throughout this section we shall let $l_n = \arccos(\frac{-1}{n+1})$. We will state without proof a few theorems concerning the packing radius of Riemannian manifolds with sectional curvature ≥ 1 which will be used throughout the rest of this paper. For a proof of these theorems the reader should consult [GM1,2], or [GW]. These theorems will not be stated in their fullest generality, as they hold for any Alexandrov space with curvature ≥ 1.

Lemma 3.1. *If M^n is a complete Riemannian manifold with $sec(M^n) \geq 1$, then for all $q \geq 2$,*

$$\operatorname{pack}_q(M^n) \leq \operatorname{pack}_q(S^n)$$

The proof of this lemma follows from part (B) of Topogonov's Theorem.

We shall now define another metric invariant which will be useful in our next applications of the packing radius. Let X be a compact metric space of equilateral dimension n. Then, for $q \leq e(X)$, we define the q-th equilateral length, denoted eq_q to the maximum length of an equilateral set with q elements, or

$$\operatorname{eq}_q(X) = \max\{\operatorname{diam}(E) | E \text{ is an equilateral set of order } q\}$$

Similiar to the packing radius, it easily follows from the definition of eq_q that

$$\operatorname{eq}(X) = \operatorname{eq}_{e(X)} \leq \operatorname{eq}_{e(X)-1} \leq \ldots \leq \operatorname{eq}_2(X) = \operatorname{diam}(X)$$

where $\operatorname{eq}(X) = \operatorname{eq}_{e(X)}$ is the *equilateral length* of X. In our applications, it will be important to note that $\operatorname{eq}_{e(S^n)}(S^n) = 2\operatorname{pack}_{n+2} S^n$.

The equilateral length differs from the packing radius, as well as other metric invariants such as the extent as defined in [GM1,2] or [GW]. For example, $\mathrm{pack}_q(X)$ is defined for any $q \geq 2$ and any compact metric space X. while, $\mathrm{eq}_q(X)$ is only defined for $2 \leq q \leq e(X)$. Additionally, by Main Theorem 2 we know that when $X = M^n$ is a complete Riemannian manifold of dimension n and sectional curvature ≥ 1, we have that for this class of functions,

$$q \leq \mathrm{Cap}_{S^n}\left(\arccos \frac{\cos(\mathrm{eq}(M^n))}{1+\cos(\mathrm{eq}(M^n))}\right) \leq \mathrm{Cap}_{S^n}\left(\frac{\pi}{6}\right)$$

The following theorems apply the packing radius to the $(n-1)$-sphere imbedded in the n-dimensional tangent space at a point in the equilateral set, which gives us a more reasonable upper bound of the equilateral dimension.

Main Lemma 3. *Let M^n be a complete manifold with $\sec(M^n) \geq 1$. If $\mathrm{eq}_q(M^n) > \frac{\pi}{2}$ for some $q \geq 2$, then $q \leq n+2$.*

Proof. For the sake of contradiction, we assume that $q > n+2$, and without loss of generality we may assume that $q = n+3$. Using the same notation and identification of $T_{p_1}M^n$ with $\mathbb{R}^n$ as in Main Lemma 2, we have that, as $\mathrm{eq}_q(M^n) = r > \frac{\pi}{2}$

$$\cos(\theta_{i,j}) \leq \cos l_n < 0$$

Which implies that

$$\begin{aligned} \frac{\pi}{2} &=< \min_{2\leq i<j\leq n+2} \theta_{i,j} \\ &= \min_{2\leq i<j\leq n+2} d^{S^{n-1}}(\gamma_i'(0)), \gamma_j'(0)) \\ &\leq 2\,\mathrm{pack}_{n+2}(S^n) = \frac{\pi}{2} \end{aligned}$$

This is a contradiction. Hence we have that $q \leq n+2$. □

The following corollary is a simple consequence of the previous lemma of the definition of the equilateral length. This gives a particular instance in which Kusner's conjecture holds ([Gu]).

Main Theorem 3. *Let M^n be a complete manifold with $\sec(M^n) \geq 1$. If $\mathrm{eq}(M^n) > \frac{\pi}{2}$, then $e(M^n) \leq n+2$.*

One should note that the assumptions in the previous theorem imply that $\mathrm{diam}\, M^n > \frac{\pi}{2}$, and hence M^n is homeomorphic to S^n, by Grove and Shiohama's Generalized Sphere Theorem in [GS].

The final main theorem will be a rigidity result, which is a straight-forward consequence of the following result that is proven in [GW].

Lemma 3.2. *Let M^n be a Riemannian manifold with* $\sec(M^n) \geq 1$ *then if*

$$\text{pack}_{n+2}(M^n) = \text{pack}_{e(S^n)}(S^n) = \frac{1}{2} l_n$$

then M^n is isometric to S^n.

With this lemma we can state and prove the following rigidity result, the proof of which rests on the fact that $\text{pack}_{n+2}(S^n) = \frac{1}{2}\,\text{eq}(S^n)$.

Main Theorem 4. *Let M^n be a complete manifold with* $\sec(M^n) \geq 1$, $e(M^n) \geq n+2$. *If* $\text{eq}(M^n) = l_n$, *then M^n is isometric to S^n, and $e(M^n) = n+2$.*

Proof. As $e(M^n) \geq n+2$, and $\text{eq}(M^n) = l_n$, there exists an equilateral set $E = \{p_1, p_2, \ldots, p_{n+2}\} \subset M^n$ of order $n+2$ with $d(p_i, p_j) = l_n$ for all $1 \leq i < j \leq n+2$, this implies that

$$\begin{aligned} l_n = \min_{1 \leq i < j \leq n+2} d(p_i, p_j) &\leq \max_{(x_1, \ldots, x_q)} \min_{1 \leq i < j \leq n+2} d(x_i, x_j) \\ &= 2\,\text{pack}_{n+2}(M^n) \\ &\leq 2\,\text{pack}_{n+2} S^n = l_n \end{aligned}$$

Where the last inequality follows by Lemma 4.1. Hence all inequalities are equalities, and we have that $\text{pack}_{n+2}(M^n) = \text{pack}_{n+2}(S^n)$. Therefore, by the previous theorem, M^n is isometric to S^n. $\square$

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