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Christopher Ang   Alex Schulte

Abstract. The directed graph of a ring is a graphical representation of its additive and multiplicative structure. Using the directed edge relationship $(a, b) \to (a + b, a \cdot b)$, one can create a directed graph for every ring. This paper focuses on the structure of the sources in directed graphs of commutative rings with identity, with special concentration in the finite and reduced cases.

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1 Introduction

The use of graph theory to obtain ring theoretic information has been extensive; some examples are zero-divisor graphs [9], total graphs [7], and commuting graphs [8]. Digraphs of rings were considered by Lipkovski in [6], and by Hauskin and Skinner in [5]. Hauskin and Skinner were able to show, among other results concerning the general structure of digraphs of commutative rings, that from the digraph of a commutative ring we may determine whether the ring is an integral domain, and also whether a given ideal is prime. They also gave two examples of rings which were non-isomorphic, but which have directly isomorphic digraphs.

This paper is a continuation of the work done by Hauskin and Skinner, and we will be using the same conventions and notations used in their paper. Where necessary, results from that paper will be reproduced here. In this paper we place special emphasis on sources and looped vertices, since in a digraph these are the most readily identifiable vertices, and in order for two digraphs to be directly isomorphic the number of sources and looped vertices in both digraphs must be the same.

In Section 2, we will give the necessary background to understand the paper. We will explore sources in finite fields in Section 3, with a couple of results given for integral domains. In Section 4, we will show that there is a fair amount of structure retained between the digraph of a ring and the digraphs of its factor rings. In Section 5, we will show that given a digraph of a ring, it is possible to tell whether the underlying ring is reduced or not, and how many fields the ring is "made up of." This leads to the main conjecture of the paper, that unique finite reduced rings, up to isomorphism, produce unique digraphs. Finally, in the last Section we will give some ideas for further inquiry.

2 Background

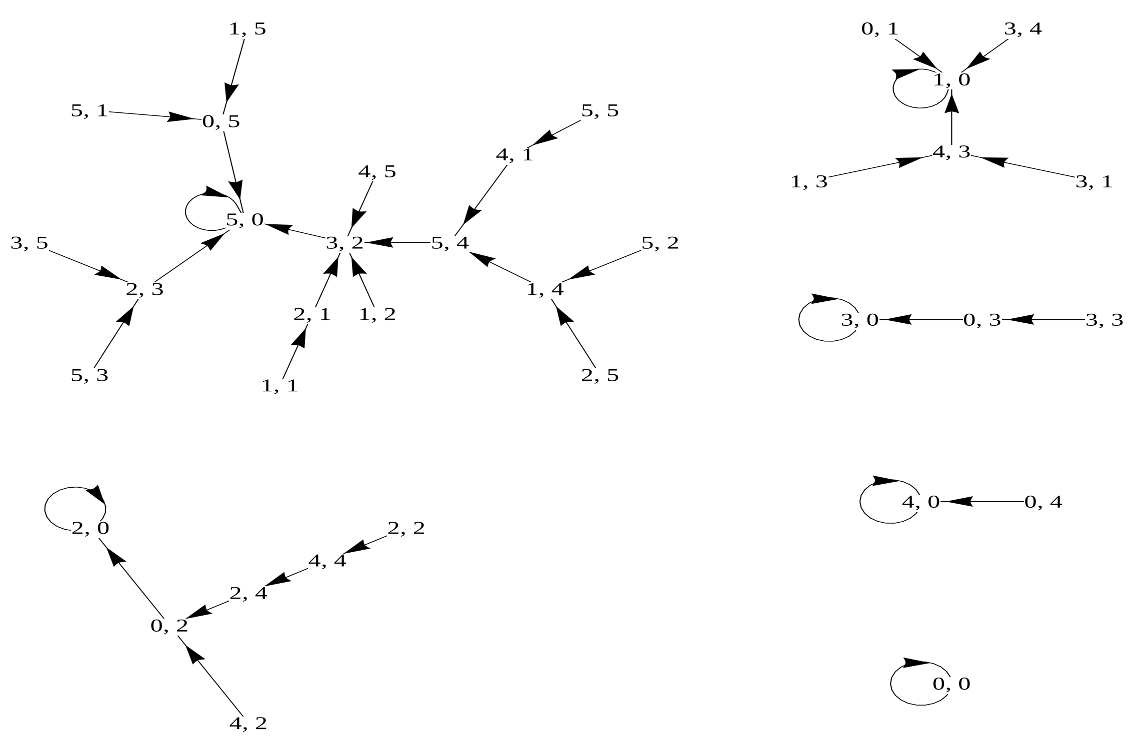
Throughout, R will denote a commutative ring with identity. While it is only feasible to graph finite rings, results are given for all rings, unless otherwise specified. By an ideal, it is meant a proper ideal of a ring. We will denote the finite field of order p^n, p prime, by $GF(p^n)$. An element $a \in R$ is a *zero-divisor* if and only if there exists a non-zero $b \in R$ such that $ab = 0$.

For a graph G, the set of vertices is denoted $V(G)$ and the set of edges is denoted $E(G)$, where an element in $E(G)$ is a pair u, v with $u, v \in V(G)$. Edges are often denoted $u - v$. It is said that G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') = \{v - w : v, w \in V(G')$ and $v - w \in E(G)\}$. Note that what we have defined is typically referred to as an *induced subgraph.* If $a_1, a_2, \ldots, a_n$ are vertices in G, then $a_1 - a_2 - \cdots - a_n$ denotes a *walk* in G. A *path* is a walk consisting of distinct vertices. A *connected* graph is one in which there is a path between any two distinct vertices. A *connected component* of a graph G, denoted $\mathcal{C}$, is a maximal connected subgraph of G (For a graph theory reference, see [3]). A *directed edge* is an ordered pair of vertices, which we will denote as $u \to v$, for $u, v \in V(G)$. We will also say that u *points* at v to mean that there exists a directed edge $u \to v$. A *directed graph*, or *digraph* for brevity, is a graph where all edges are directed edges.

Let D be a digraph. A vertex in D has *incoming degree* n if there are n distinct vertices pointing to it. A *source* is a vertex with incoming degree zero. If $a_1, a_2, \ldots, a_n$ are vertices in D, then $a_1 \to a_2 \to \cdots \to a_n$ denotes a *directed walk* in D from vertex a_1, to vertex a_n, where a_i is *directionally adjacent* to a_{i+1} for $1 \leq i \leq n-1$. If $a_1, a_2, \ldots, a_n$ are distinct, then the directed walk is a *directed path.* A *cycle of length* n is a directed walk of the form $a_1 \to a_2 \to \cdots \to a_n \to a_1$, where $a_1 \to a_2 \to \cdots \to a_n$ is a directed path. A *looped vertex* is a vertex a such that (a, a) is a directed edge. Any digraph has a canonical graph obtained by replacing all directed edges $u \to v$ with a corresponding undirected edge $u - v$. A digraph is *connected* if its canonical graph is connected, and a component of a digraph is a *connected component* if in the canonical graph the corresponding component is a connected component.

For a ring R, the digraph of R, denoted $\Psi(R)$, is the graph with $V(\Psi(R)) = R \times R$, and for $(a, b), (c, d) \in V(\Psi(R))$, $\Psi(R)$ contains the directed edge $(a, b) \to (c, d)$, if and only if $a + b = c$ and $a \cdot b = d$. The set of all sources in the digraph of a ring is denoted $\mathcal{S}(\Psi(R))$. As an example, see Figure 1 for the digraph of $\mathbb{Z}_6$.

Figure 1: $\Psi(\mathbb{Z}_6)$



3 Finite Fields and Integral Domains

In this Section we consider digraphs of finite fields as well as integral domains, beginning by classifying certain types of sources. As a byproduct of this investigation, we also compute the incoming degree of certain types of sources in digraphs of integral domains. Finally, we show that one can count the number of sources in the digraph of a finite field if one knows the order of the field. Before moving on to results, it is first necessary to restate a couple of propositions from Number Theory as statements in Algebra. The reader should be able to deduce these results after referring to [4, Theorem 22.2].

Lemma 3.1. *Let $F = GF(p^n)$, where p is an odd prime, and $n \in \mathbb{N}$.*

1. *If $p^n = 4k + 1$ where $k \in \mathbb{N}$, then there exists $c \in F$ such that $c^2 = -1$.*
2. *If $p^n = 4k + 3$ where $k \in \mathbb{N}$, then there does not exist a $c \in F$ such that $c^2 = -1$. Furthermore, for $a \in F$, there exists a $b \in F$ such that $a = b^2$ if and only if $-a \neq c^2$ for any $c \in F$*

The following two results give necessary and sufficient conditions for when vertices of the form $(0, a)$ are sources in the digraph $\Psi(F)$ of the finite field F. These conditions depend on the congruence class of $|F|$ modulo 4. Note that since $(0, 0) \to (0, 0)$ is a directed edge in $\Psi(R)$ for any ring R, $(0, 0)$ will always have an incoming degree of at least one, and thus can never be a source in the digraph of any ring.

Theorem 3.2. *Let p be a prime and $n \in \mathbb{N}$ such that $p^n = 4k + 1$ for some $k \in \mathbb{N}$. Then $(0, a) \neq (0, 0)$ is a source in $\Psi(GF(p^n))$ if and only if $a \neq b^2$ for any $b \in GF(p^n)$.*

Proof. ($\Rightarrow$) Let $a = b^2$ for some $b \in GF(p^n)$. By Lemma 3.1, since p^n is a prime power of the form $4k+1$, there exists a $c \in GF(p^n)$ such that $c^2 = -1$. Thus the vertex $(bc, -bc)$ points to the vertex $(0, -(bc)^2)$. Since $GF(p^n)$ is commutative and $c^2 = -1$, $-(bc)^2 = -b^2c^2 = b^2 = a$, and so $(0, -(bc)^2) = (0, a)$. Thus $(0, a)$ is not a source.
($\Leftarrow$) Suppose now that $(0, a)$ is not a source. Then there exists a vertex $(b, -b)$ which points to $(0, a)$. That is, $-b^2 = a$. By Lemma 3.1 there exists $c \in GF(p^n)$ such that $c^2 = -1$. Then $-b^2 = c^2b^2 = (bc)^2 = a$.

□

Theorem 3.3. *Let p be a prime and $n \in \mathbb{N}$ such that $p^n = 4k + 3$ for some $k \in \mathbb{N}$. Then $(0, a)$ is a source in $\Psi(GF(p^n))$ if and only if there exists a non-zero $b \in GF(p^n)$ such that $a = b^2$.*

Proof. ($\Rightarrow$) Assume that $a \neq b^2$ for any $b \in GF(p^n)$. Then, by Lemma 3.1, there exists $c \in GF(p^n)$ such that $c^2 = -a$. Then $-c^2 = a$ and the vertex $(c, -c)$ points to $(0, -c^2) = (0, a)$. Thus $(0, a)$ is not a source in $\Psi(GF(p^n))$.
($\Leftarrow$) Suppose now that $(0, a)$ is not a source in $\Psi(GF(p^n))$. Then there exists a vertex $(c, -c) \in \Psi(GF(p^n))$ such that $(c, -c) \to (0, a)$ is a directed edge. Thus $-c^2 = a$, and by Lemma 3.1 there does not exist a $b \in GF(p^n)$ such that $a = b^2$.

□

Corollary 3.4. *In a finite field F of odd order, there are $\frac{|F|-1}{2}$ sources of the form $(0, a)$ in $\Psi(F)$.*

Proof. Let $|F|= q$. Since F is a finite field, there exists an $r \in A = (F\backslash\{0\}, \cdot)$ such that $|r| = q-1$. Thus, any element of A may be represented by a distinct power of r. If $r^m = b^2$, then $m = 2 + k(q-1)$, where $k \in \mathbb{Z}$. Since q is odd, $2 \mid (q-1)$, and thus $2 \mid m$. The set of numbers which form the distinct powers of r is $\{1, 2, 3, ..., q-2, q-1\}$. Of these, exactly half are divisible by two. Thus there are exactly $\frac{q-1}{2}$ powers of r which are equal to b^2 for some $b \in F$, and $\frac{q-1}{2}$ powers of r which are not equal to b^2 for any $b \in F$. By Theorems 3.2 and 3.3, there are $\frac{|F|-1}{2}$ sources of the form $(0, a)$. □

The following two theorems examine what possible incoming degrees a vertex may have in a digraph of a domain. They allow the counting of the total number of sources in a digraph of a finite field, and will be referred to in Section three when examining digraphs of factor rings.

Theorem 3.5. *Let D be an integral domain with $a \in D$. Then the vertex $(2a, a^2)$ in $\Psi(D)$ has incoming degree 1.*

Proof. It is always the case that $(a, a) \to (2a, a^2)$, and thus $(2a, a^2)$ has incoming degree of at least 1.

Let char $D \neq 2$, and suppose $(b, b) \to (2a, a^2)$ for some $b \in D$. Then $2b = 2a$, and by the cancellation property $a = b$.

Now suppose that char $D = 2$, and again suppose that $(b, b) \to (2a, a^2)$. If $a = 0$, then $a^2 = 0 = b^2$, and thus $b = 0$; if $b = 0$, then $a = 0$. If $a, b \neq 0$, then $a^2 = b^2$, so $a^2 - b^2 = (a - b)^2 = 0$, hence $a - b = 0$, and $a = b$.

Finally, suppose $(c, d) \to (2a, a^2)$, where c, d are distinct elements in D. Then $c + d = 2a$ and $cd = a^2$. Then $c^2 + 2cd + d^2 = 4a^2$, and so $c^2 - 2cd + d^2 = (c - d)^2 = 0$, hence $c = d$, contradicting our assumption. Thus $(2a, a^2)$ has incoming degree of one.

□

Theorem 3.6. *Let D be an integral domain. Then all non-source vertices in $\Psi(D)$ which are not of the form $(2a, a^2)$ have incoming degree 2.*

Proof. Suppose (c, d) is a vertex which is not of the form $(2a, a^2)$ and is not a source. Then $(c, d) = (e + f, ef)$, for some $e, f \in D$ with $e \neq f$. Then the vertices (e, f) and (f, e) both point at (c, d), thus the incoming degree of (c, d) is at least 2.

Suppose now that there exists a vertex $(g, h) \in \Psi(D)$ such that $(g, h) \notin \{(a, b), (b, a)\}$ and $(g, h) \to (c, d)$. If $g = b$, then $(h, g) = (a, b)$ and $(g, h) = (b, a)$, and thus the vertices $(g, h), (h, g)$ are not distinct from $(a, b), (b, a)$. Similarly when $h = a$, $g = a$, or $h = b$.

Thus, a, b, g, h are all distinct in D. Then

$$a + b = c = g + h \tag{3.1}$$

and

$$ab = d = gh. \tag{3.2}$$

Then $a^2 + 2ab + b^2 = g^2 + 2gh + h^2$. Using (3.2) this equation becomes $a^2 + 2ab + b^2 = g^2 + 2ab + h^2$, so that $a^2 - g^2 = h^2 - b^2$, hence $(a + g)(a - g) = (h + g)(h - g)$. Then since $a-g = h-b$ by equation (3.1), $(a+g)(h-b) = (h-b)(h+b)$, and since $h \neq b$ by assumption, $a + g = h + b$. Adding (3.1) to this last equation yields $2a = 2h$. If char $D \neq 2$, then $a = h$, a contradiction.

Suppose then that char $D = 2$. Then from equations (3.1), (3.2), $ab = gh$ and $a = b+g+h$. Substituting the second into the first yields $b^2 + bg + bh = gh$, so $b^2 + bg = hb + hg$. Since char $D = 2$ and $b \neq g$ by assumption, $b(b + g) = h(b + g)$, thus $b = h$, a contradiction. Thus no such vertex (g, h) exists. □

The previous two theorems give insight into the structure of both finite and infinite domains, and, as previously stated, they permit the counting of the total number of sources in a finite field, as the next result shows.

Theorem 3.7. *Let F be a finite field. Let $q = |F|$. Then $|\mathcal{S}(\Psi(F))| = \frac{q^2-q}{2}$.*

Proof. All vertices in $\Psi(F)$ which are not sources will be of the form $(2a, a^2)$ or $(a+b, ab)$ for distinct $a, b \in F$. Counting the number of vertices of these two forms will yield the desired result.

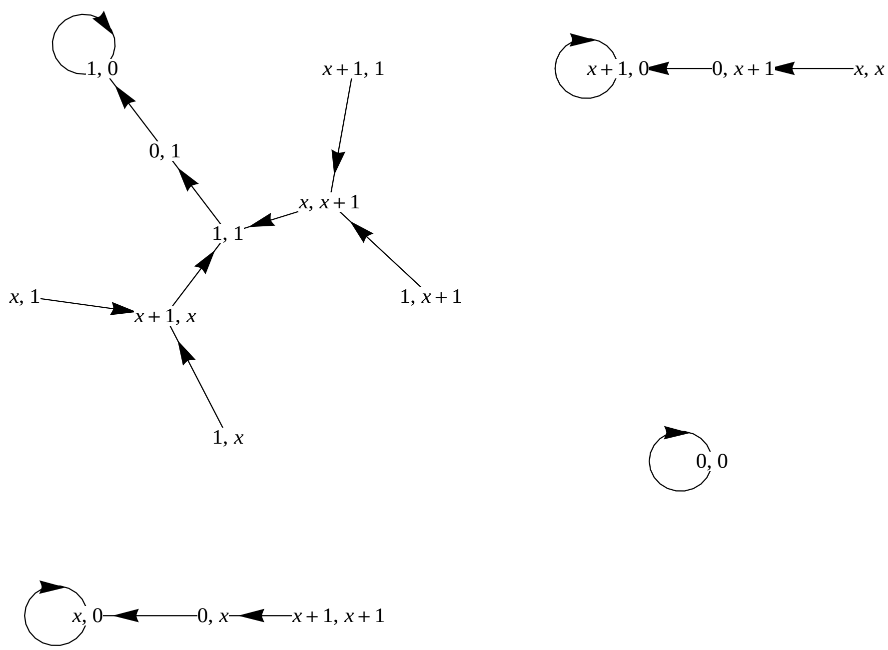
By Theorem 3.5 vertices in $\Psi(F)$ of the form $(2a, a^2)$ have incoming degree of one, and are pointed at only by the vertex (a, a). There are q distinct vertices of the form (a, a), thus there are q vertices of the form $(2a, a^2)$.

Let $a, b \in F$ be distinct. By Theorem 3.6 vertices in $\Psi(F)$ of the form $(a + b, ab)$ have incoming degree of 2, and are pointed to by the vertices (a, b) and (b, a). Since there are q^2 vertices in $\Psi(F)$ and of these there are q vertices of the form (a, a), there are $q^2 - q$ vertices of the form (a, b). Now since each vertex of the form $(a + b, ab)$ is pointed at only by two vertices, and there $q^2 - q$ vertices which point to vertices of the form $(a + b, ab)$, there are $\frac{q^2-q}{2}$ vertices of the form $(a + b, ab)$.

Thus there are $\frac{q^2-q}{2} + q = \frac{q^2+q}{2}$ vertices in $\Psi(F)$ which are not sources. Therefore there are $q^2 - \frac{q^2+q}{2} = \frac{q^2-q}{2}$ vertices which are sources. □

Example 3.8. Consider the digraph of $GF(4)$ (see Figure 2). Note that the vertices of the form $(2a, a^2)$ have incoming degree one, and all other non-source vertices have incoming degree 2. Also since $|GF(4)| = 4$, there are 6 sources in $\Psi(GF(4))$.

Figure 2: $\Psi(GF(4))$



4 Digraphs of Factor Rings

Just as information about a ring may be obtained by examining its factor rings, so too may information about the digraph of a ring be obtained from examining the digraphs of its factor rings. The following results are powerful tools in this regard.

Lemma 4.1. *Let R be a ring, and I be an ideal of R. Then $(a+I, b+I) \to (c+I, d+I)$ is a directed edge in $\Psi(R/I)$ if and only if for each $i_1, i_2 \in I$ there exists some $i_3, i_4 \in I$ such that $(a+i_1, b+i_2)$ points at $(c+i_3, d+i_4)$.*

Proof. ($\Rightarrow$) Let $(a+I, b+I) \to (c+I, d+I)$ be a directed edge in $\Psi(R/I)$. Then $a+b+I = c+I$ and $ab + I = d + I$. Now consider $(a + i_1, b + i_2)$ in $\Psi(R)$ for some $i_1, i_2 \in I$. Then $(a + i_1, b + i_2)$points at $(a + b + i_1 + i_2, ab + ai_2 + bi_1 + i_1i_2)$.

Now $a+b+i_1+i_2 \in a+b+I = c+I$ and $ab+ai_2+bi_1+i_1i_2 \in ab+I = d+I$. Thus there exists $i_3, i_4 \in I$ such that $(a+i_1, b+i_2)$ points at $(c+i_3, d+i_4)$.
($\Leftarrow$) Suppose that for each $i_1, i_2 \in I$ there exists some $i_3, i_4 \in I$ such that $(a+i_1, b+i_2)$ points at $(c+i_3, d+i_4)$. Then $a+b+i_1+i_2 = c+i_3$ and $ab+ai_2+bi_1+i_1i_2 = d+i_4$.

Then $(a+b+I) \cap (c+I) \neq \emptyset$ so $a+b+I = c+I$. Similarly, $ab+I = d+I$. Thus $(a+I, b+I) \to (c+I, d+I)$ is a directed edge in $\Psi(R/I)$. $\square$

Part of the structure which is preserved from the digraph of the factor ring are the sources, as the next result demonstrates.

Theorem 4.2. *Let R be a ring and I be an ideal of R. Then $(a+I, b+I)$ is a source in $\Psi(R/I)$ if and only if $(a+i_1, b+i_2)$ is a source in $\Psi(R)$ for all $i_1, i_2 \in I$.*

Proof. ($\Rightarrow$) Suppose that $(a+I, b+I)$ is a source in $\Psi(R/I)$ and suppose that for some $i_1, i_2 \in I$ there exists a vertex (c, d) that points to $(a+i_1, b+i_2)$ in $\Psi(R)$. Then $c+d = a+i_1$ and $cd = b+i_2$. Hence $c+d+I = a+I$ and $cd+I = b+I$. Then $(c+I, d+I)$ points to $(a+I, b+I)$ in $\Psi(R/I)$, a contradiction. Thus, $(a+i_1, b+i_2)$ is a source in $\Psi(R)$ for all $i_1, i_2 \in I$.
($\Leftarrow$) Suppose now that $(c+I, d+I)$ points to $(a+I, b+I)$ in $\Psi(R/I)$. Then by Lemma 4.1, there exist $i_1, i_2, i_3, i_4 \in I$ such that $(c+i_1, d+i_2) \to (a+i_3, b+i_4)$ is a directed edge in $\Psi(R)$. $\square$

The previous result shows that sources are preserved when moving from the digraph of the factor ring to the ring itself. However, it is not always the case that a non-source vertex in the digraph of the factor ring will correspond to non-source vertices in the digraph of the ring, as the following example will show.

Example 4.3. Consider $\mathbb{Z}_6$ and its maximal ideal (2). In $\Psi(\mathbb{Z}_6/(2))$ the vertex $(1+(2), 0+(2))$ has incoming degree two. However, in $\Psi(\mathbb{Z}_6)$ the vertex $(5, 2)$ is a source. Thus, local is a necessary condition, as illustrated in Figures 3 and 4.

Figure 3: $\Psi(\mathbb{Z}_6/(2))$

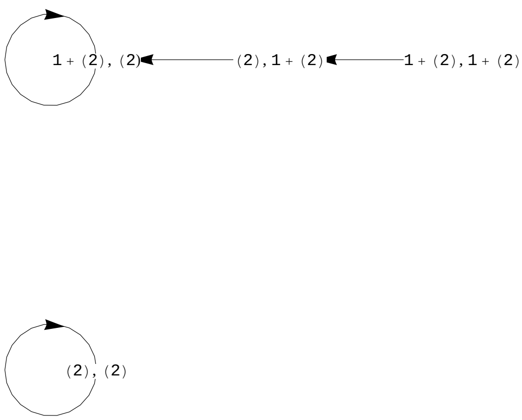
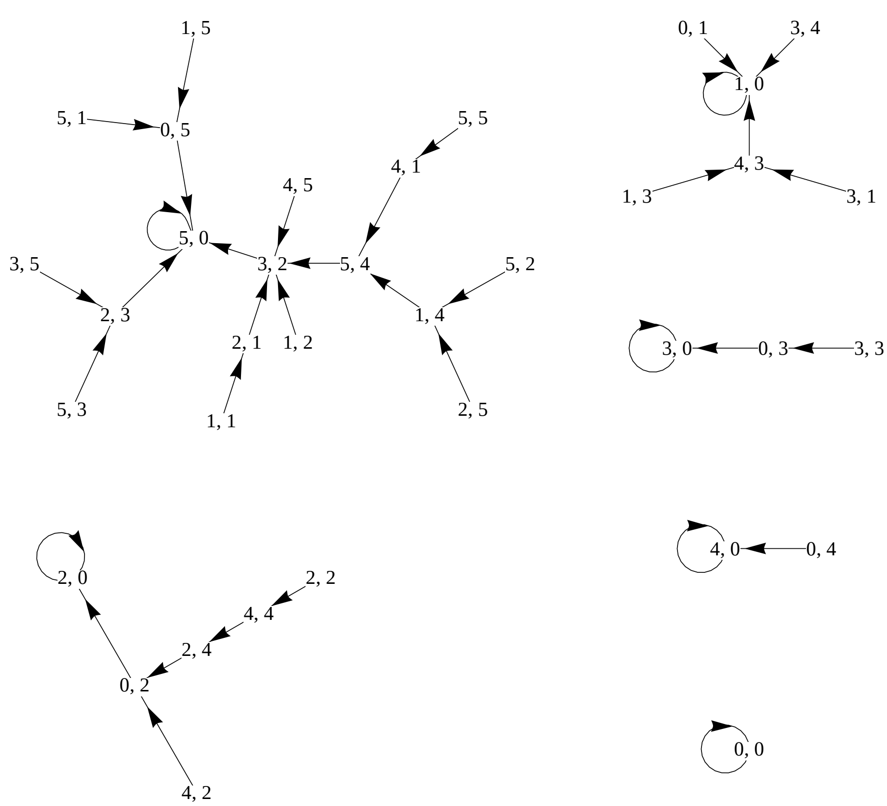


Figure 4: $\Psi(\mathbb{Z}_6)$



The following Theorem will show what conditions are necessary for a non-source vertex in the digraph of a factor ring to correspond to non-source vertices in the digraph of the ring.

Theorem 4.4. *Let R be a finite local ring with maximal ideal M. If $(a+M, b+M)$ has incoming degree 2 in $\Psi(R/M)$, then $(a+m_1, b+m_2)$ is not a source in $\Psi(R)$ for any $m_1, m_2 \in M$.*

Proof. Suppose that $(a+M, b+M)$ has incoming degree 2 in $\Psi(R/M)$. Since R/M is a finite field and $(a+M, b+M)$ is not of the form $(2e+M, e^2+M)$ for some $e \in R$, by Theorems 3.6 and 3.5, there exist vertices $(c+M, d+M)$ and $(d+M, c+M)$ with $c+M \neq d+M$ that point at $(a+M, b+M)$.

Now suppose for the sake of contradiction that there exists a source of the form $(a+m_1, b+m_2)$ in $\Psi(R)$. Then, by the pigeon hole principle there exists $\bar{a} \in a+M, \bar{b} \in b+M$ such that $(\bar{c}, \bar{d}) \to (\bar{a}, \bar{b})$ and $(\bar{c}+m_3, \bar{d}+m_4) \to (\bar{a}, \bar{b})$, where $\bar{c}, \bar{c}+m_3 \in c+M; \bar{d}, \bar{d}+m_4 \in d+M$ and either $m_3 \neq 0$ or $m_4 \neq 0$.

Suppose that $m_3 = 0$. Then $a = \bar{c}+\bar{d} = \bar{c}+\bar{d}+m_4$, and thus $m_4 = 0$, a contradiction. A similar contradiction occurs if $m_4 = 0$. Thus, it may be assumed that $m_3, m_4 \neq 0$. Then $\bar{c}+\bar{d} = \bar{c}+\bar{d}+m_3+m_4$, hence $m_3 = -m_4$. Also, $\bar{c}\bar{d} = \bar{c}\bar{d}+\bar{c}m_4+\bar{d}m_3+m_3m_4$, which implies $\bar{c}m_4+\bar{d}m_3+m_3m_4 = 0$. Replacing m_3 with $-m_4$, and reorganizing will yield $(\bar{c}-\bar{d})m_4 = m_4^2$.

Now since R is local, m_4 is nilpotent, and $Z(R) = M$ [2, Theorem 2.3]. Let n be the least positive integer such that $m_4^n \neq 0$, but $m_4^{n+1} = 0$. If $n = 1$, then $(\bar{c}-\bar{d})m_4 = m_4^2 = 0$, which implies that $(\bar{c}-\bar{d}) \in Z(R)$. Thus $\bar{c}+M = \bar{d}+M$, a contradiction. If $n > 1$, then

$$(\bar{c}-\bar{d})^n m_4^n = (m_4^2)^n = 0,$$

in which case $(\bar{c}-\bar{d})^n \in Z(R)$, and thus $(\bar{c}-\bar{d}) \in Z(R)$. Again, $\bar{c}+M = \bar{d}+M$, a contradiction.

Thus there cannot be a vertex $(a+m_1, b+m_2)$ which is a source in $\Psi(R)$ for any $m_1, m_2 \in M$. $\square$

Example 4.5. Consider $\mathbb{Z}_8$ and the ideal (4). In $\Psi(\mathbb{Z}_8/(4))$ the vertex $(0+(4), 3+(4))$ has incoming degree two. However, in $\Psi(\mathbb{Z}_8)$ the vertex $(4, 7)$ is a source. Thus, the ideal must be the maximal ideal. Figures 5 and 6 will demonstrate this.

Figure 5: $\Psi(\mathbb{Z}_8/(4))$

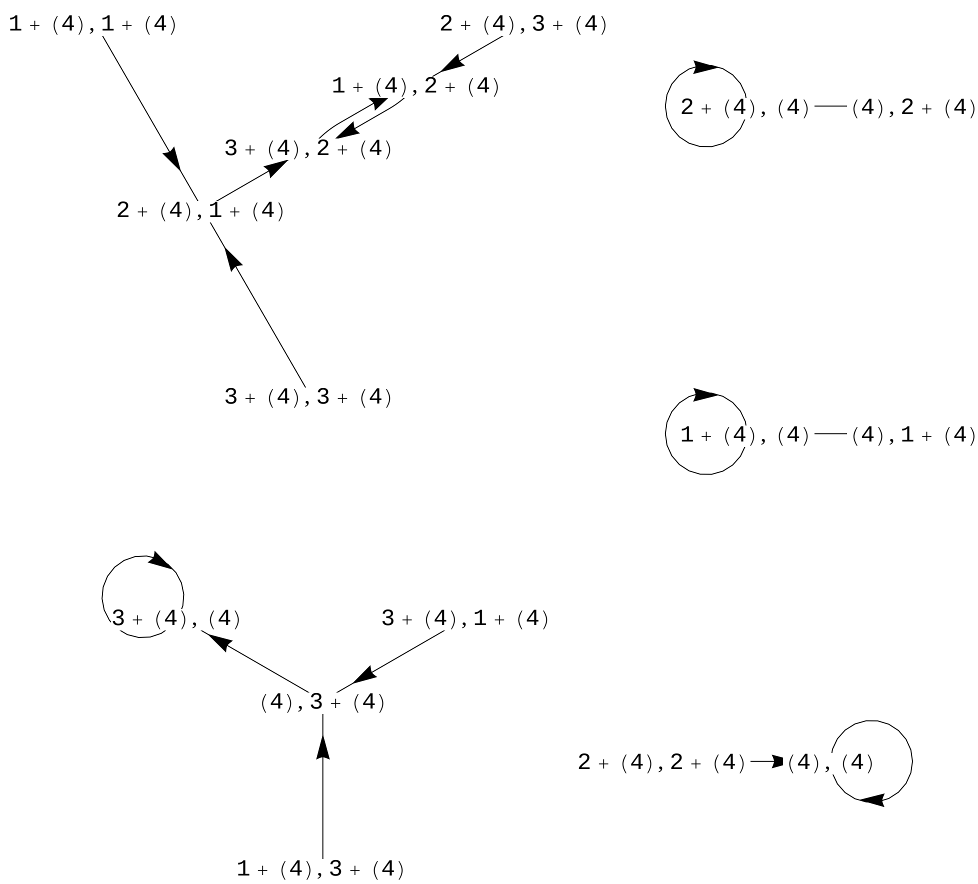
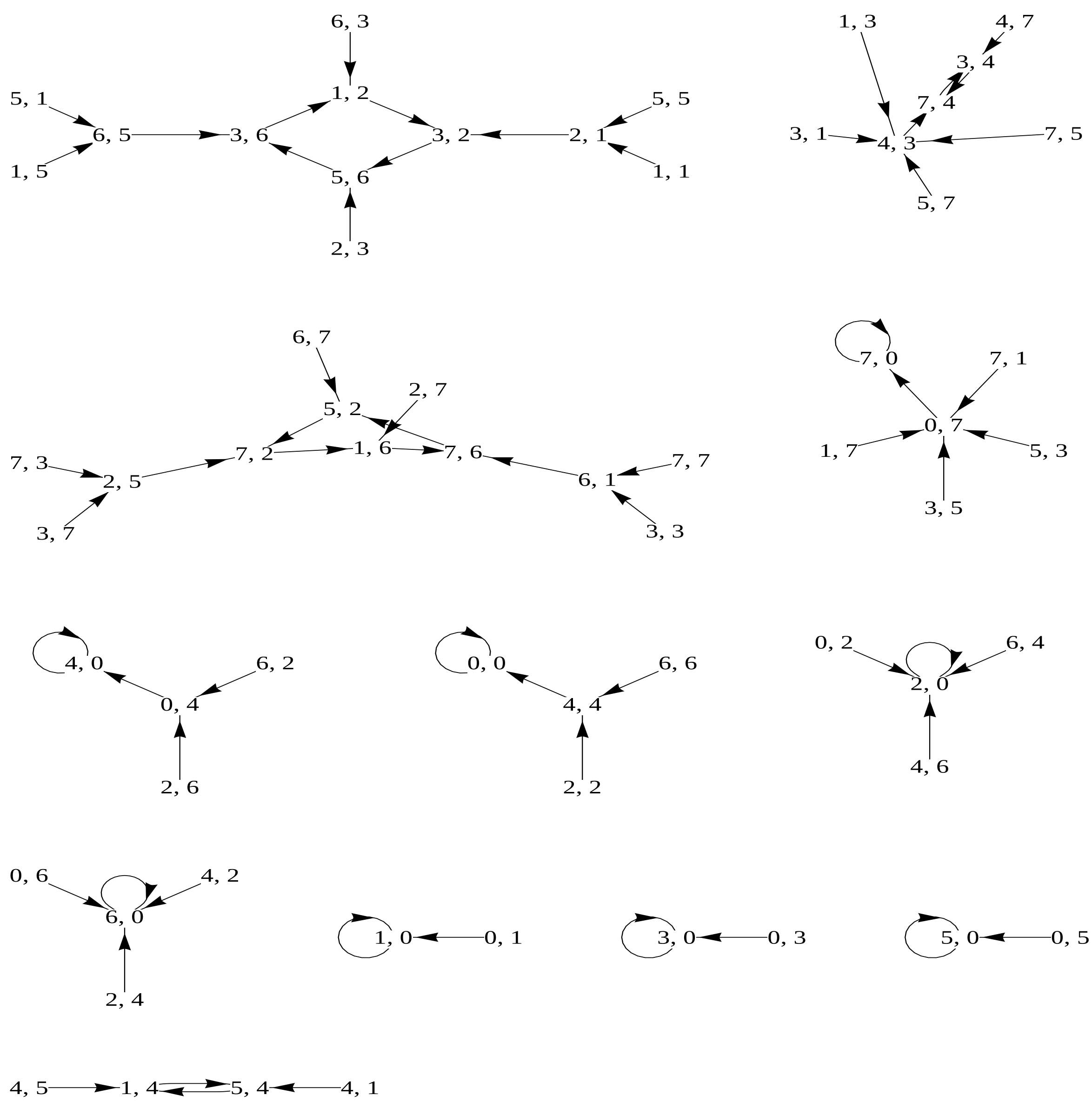


Figure 6: $\Psi(\mathbb{Z}_8)$



The following Theorem shows that in order for a vertex in the digraph of the factor to correspond only to non-source vertices in the digraph of the ring, it must have an incoming degree of two.

Theorem 4.6. *Let R be a finite local ring with maximal ideal M. Let n denote the least positive integer such that $M^n = \{0\}$. Of the vertices of the form $(2a+m_1, a^2+m_2)$ in $\Psi(R)$, where $m_1, m_2 \in M$, at least $|M|^2 - \frac{|M|^2}{|M^{n-1}|}$ are sources.*

Proof. By Lemma 4.1, since $(a+M, a+M) \to (2a+M, a^2+M)$ in $\Psi(R/M)$, there exist $m_1, m_2, m_3, m_4 \in M$ such that $(a+m_1, a+m_2) \to (2a+m_3, a^2+m_4)$ in $\Psi(R)$. Now let $b \in M^{n-1} \subseteq \{c \mid cM = \{0\}\}$. Then $(a+m_1+b, a+m_2-b) \to (2a+m_3, a^2+m_4)$, where $a+m_1 \neq a+m_1+b$, and $a+m_2 \neq a+m_2-b$ for any non-zero b.

Now, for any distinct $b, c \in M^{n-1}$, $a+m_1+b \neq a+m_1+c$, and thus each distinct $b \in M^{n-1}$ will yield a distinct vertex in $\Psi(R)$ of the form $(a+m_1+b, a+m_2-b)$. Thus there are $|M^{n-1}|$ distinct vertices of the form $(a+m_1+b, a+m_2-b)$ which point to $(2a+m_3, a^2+m_4)$.

Now, there are $|M|^2$ vertices of the form $(a+m_1, a+m_2)$ in $\Psi(R)$ and likewise there are $|M|^2$ vertices of the form $(2a+m_3, a^2+m_4)$. If $(2a+m_3, a^2+m_4)$ is not a source it is pointed to by at least $|M^{n-1}|$ vertices of the form $(a+m_1, a+m_2)$, hence at least $|M|^2 - \frac{|M|^2}{|M^{n-1}|}$ vertices of the form $(2a+m_3, a^2+m_4)$ are sources in $\Psi(R)$. □

5 Reduced Rings

This final Section will examine digraphs of reduced rings. A reduced ring is one in which there are no nonzero nilpotent elements–elements $x \in R$ such that $x^n = 0$, for some $n \in \mathbb{N}$. We see that by [1, Theorem 8.7] it is possible to decompose any reduced ring into a direct product of fields.

It is necessary at this point to generalize a couple of definitions and results by Hauskin and Skinner [5, Definition 6.1]. Let $\Psi(R_1 \times R_2 \times \cdots \times R_n)$ be the digraph of the direct product of rings. The subgraph $\Psi(\{0\} \times \{0\} \times \cdots \times \{0\} \times R_i \times \{0\} \cdots \times \{0\})$ is called the *canonical subgraph of R_i in $\Psi(R_1 \times R_2 \times \cdots \times R_n)$* and is denoted by $\Psi'(R_i)$. The digraphs of two rings, $\Psi(R)$ and $\Psi(S)$, are *directly isomorphic*, denoted $\Psi(R) \gtrsim \Psi(S)$, if and only if there exists a bijection $f : V(\Psi(R)) \to V(\Psi(S))$ such that for any two vertices $a, b \in \Psi(R)$ $a \to b$ if and only if $f(a) \to f(b)$ with $f(a), f(b) \in \Psi(S)$. The following theorem shows some of the structure of a canonical digraph.

Theorem 5.1. *[5, Theorem 6.2]*
Let $\Psi(R_1 \times R_2 \times \cdots \times R_n)$ be the digraph of the ring $R = R_1 \times R_2 \times \cdots \times R_n$, let $\mathcal{C}$ be the union of all connected components whose intersection with $\Psi'(R_i)$ is non-empty, and let $v \in V(\Psi(R))$ with $v \neq (0,0,...,0)$. Then, $\mathcal{C} \gtrsim \Psi(R_i)$ if and only if $v \to ((0,0,...,0),(0,0,...,0))$ is not a directed edge in $\Psi(R)$.

Proof. ($\Rightarrow$) Suppose $\mathcal{C} \gtrsim \Psi(R_i)$. This implies $|\mathcal{C}| = |\Psi(R_i)|$, and since $|\Psi'(R_i)| = |\Psi(R_i)|$, $|\Psi'(R_i)| = |\mathcal{C}|$. Thus $v \in V(\Psi'(R_i))$ if and only if $v \in V(\mathcal{C})$.

Every element of $\Psi'(R_i)$ has the form $((0, 0, ..., 0, x_i, 0, ..., 0), (0, 0, ..., 0, y_i, 0, ..., 0))$ where $x_i, y_i \in R_i$. A vertex not an element of $V(\Psi'(R_i))$, say $((0, 0, ..., 0, x_i, 0, ..., 0, x_j, 0, ..., 0), (0, 0, ..., 0, y_i, 0, ..., 0, y_j, 0, ..., 0))$, where $i \neq j$ can only point at a vertex of $\Psi'(R_i)$ if $x_j + y_j = 0$ and $x_j \cdot y_j = 0$. This means that $((0, 0, ..., 0, x_j, 0, ..., 0), (0, 0, ..., 0, y_j, 0, ..., 0))$ points at $((0, 0, ..., 0), (0, 0, ..., 0))$.

($\Leftarrow$) Now suppose that for all $j \neq i$ there does not exist $(0, 0) \neq (x_j, y_j) \in V(\Psi(R_j))$ such that (x_j, y_j) points at $(0, 0)$. Thus $x_j + y_j \neq 0$ or $x_j \cdot y_j \neq 0$ for all $(x_j, y_j) \neq (0, 0)$. Since every element of $\Psi'(R_i)$ has the form $((0, 0, ..., 0, x_i, 0, ..., 0), (0, 0, ..., 0, y_i, 0, ..., 0))$, this implies that there cannot be a $((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)) \in \Psi(R_1 \times R_2 \times ... \times R_n)$ which is not an element of $V(\Psi'(R_i))$ such that $((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n))$ points at $((0, 0, ..., 0), (0, 0, ..., 0))$. □

The next Theorem shows that one can determine whether a ring is reduced or not merely by observing the structure of its digraph. Note that an *unlabeled* digraph is one whose vertex coordinates are not given.

Theorem 5.2. *Let $\Psi(R)$ be an unlabeled digraph of a ring R. Then R is reduced if and only if $\Psi(R)$ has a connected component consisting of a looped vertex.*

Proof. ($\Leftarrow$) Let R be reduced. All looped vertices have the form $(a, 0)$ in $\Psi(R)$. If $a \neq 0$, then $(0, a)$ is a vertex distinct from $(a, 0)$ which points to $(a, 0)$. In the case where $a = 0$, $(a, 0) = (0, a)$; thus $(0, 0)$ is the only looped vertex which could be a connected component. Hence if $\Psi(R)$ has a connected component consisting of a looped vertex, it must be the vertex $(0, 0)$. Then there does not exist any vertex $(a, b) \in V(\Psi(R))$, with $a \neq 0, b \neq 0$, which points at $(0, 0)$. Suppose R is not reduced. Then there exists a non-zero $x \in R$ such that for some minimal $n \in \mathbb{N}$, $x^n = 0$. Then $x^{n-1} \neq 0$ and $(x^{n-1}, -x^{n-1}) \to (0, 0)$, a contradiction.
($\Rightarrow$) If $(a, b) \to (0, 0)$, this means that $a = -b$, and thus $-a^2 = 0$. Thus $a^2 = 0$. Since R is reduced, $a = 0$ and thus $(a, b) = (0, 0)$.

□

The previous two theorems allows one to conclude that in a reduced ring R, the digraph of a finite field which is part of the direct product of R appears in $\Psi(R)$.

Corollary 5.3. *Let $R = F_1 \times F_2 \times \cdots \times F_n$ be a reduced ring expressed as a direct product of n fields. Let $\mathcal{C}_i$ be the set of connected components in $\Psi(R)$ whose union with $\Psi'(F_i)$ is non-empty. Then $\mathcal{C}_i \gtrsim \Psi(F_i)$.*

The next few theorems explore the structure of looped vertices in digraphs of reduced rings. Recall that the only vertices which are looped in $\Psi(R)$ are of the form $(a, 0)$.

Theorem 5.4. *Let $R = F_1 \times F_2 \times \cdots \times F_n$ be a reduced ring, where F_i is a field. Then there exists a looped vertex of incoming degree 2^a in $\Psi(R)$ for each $0 \leq a \leq n$.*

Proof. *Case 1*: Let $a = 0$. Since R is reduced, by Theorem 5.2, $(0, 0)$ has incoming degree 1.
Case 2: Let $1 \leq a \leq n$. Consider the vertex $Q = ((x_1, x_2, ..., x_a, 0, 0, ...0), (0, 0, ..., 0))$ in $\Psi(R)$, where $x_i \neq 0$ for each $i \in \{1, 2, ..., a\}$. Note that Q is a looped vertex. Now any vertex of the form $((y_1, y_2, ..., y_a, 0, 0, ..., 0), (w_1, w_2, ..., w_a, 0, 0, ..., 0))$, where $\{y_i, w_i\} = \{0, x_i\}$, will point to Q, and there are no other vertices which point to Q. Furthermore there are 2^a such vertices. □

The previous theorem allows one to identify the number of finite fields in the composition of any reduced ring, provided that the ring is a finite direct sum of finite fields. The following proposition gives a counting argument for the looped vertices of incoming degree 2. Similar arguments can be made for the number of looped vertices of higher incoming degrees.

Proposition 5.5. *Let R be a reduced ring such that $R = F_1 \times F_2 \times \cdots \times F_n$, where F_i is a field. Then the number of looped vertices with incoming degree 2 in $\Psi(R)$ is equal to $p_1^{a_1} - 1 + p_2^{a_2} - 1 + \cdots + p_n^{a_n} - 1$ where $|F_i| = p_i^{a_i}$.*

Proof. There are $p_1^{a_1} - 1 + p_2^{a_2} - 1 + \ + p_n^{a_n} - 1$ looped vertices of the form $((0, 0, ..., 0, x_i, 0, ..., 0), (0, 0, ..., 0))$, where $x_i \neq 0$. Any looped vertex of this form will be pointed to by itself and the vertex $((0, 0, ..., 0), (0, 0, ..., 0, x_i, 0, ..., 0))$. It remains to be shown that looped vertices of this form are the only vertices with incoming degree 2. Since R is reduced, by 5.2 $((0, 0, ..., 0), (0, 0, ..., 0))$ has incoming degree 1.

Now any other looped vertex will have the form $((x_1, x_2, ..., x_n), (0, 0, ..., 0))$, where at least two of the x_i's are non-zero. The set of vertices which point to $((x_1, x_2, ..., x_n), (0, 0, ..., 0))$ are of the form $((y_1, y_2, ..., y_n), (w_1, w_2, ..., w_n))$ where $\{y_i, w_i\} = \{0, x_i\}$. Thus the incoming degree will be 2^m, where m is the number of non-zero x_i's. Since $m \geq 2$, there are no other looped vertices of incoming degree 2 other than the ones of the form $((0, 0, ..., 0, x_i, 0, ..., 0), (0, 0, ..., 0))$.

□

6 Future Research

Because the idea of directed graphs is relatively new, future research could be taken in a number of different directions. We offer the following conjecture as one avenue:

Conjecture 1. *Let R, S be finite reduced rings. Then $\Psi(R) \gtrsim \Psi(S)$ if and only if $R \cong S$.*

This conjecture seems to the authors to be true from a number of different accounts. In addition to determining the number of 1-cycles of incoming degree two, one could repeat such a counting argument for any 1-cycle of incoming degree 2^a, for $n \geq a \geq 0$, where n is the number of finite fields in the reduced ring. The two reduced rings, in order to be directly isomorphic, need to have the same number of 1-cycles of each incoming degree in both digraphs. Furthermore, the number of sources in both digraphs must be the same. Theorem 3.7 gives the number of sources in a finite field, and [5, Theorem 6.3] permits one to count the number of sources in a direct product when one knows the number of sources in each constituent ring. Thus it is possible to count the number of sources in a reduced ring. So in order for the digraphs of two reduced rings to be directly isomorphic, they must have the same number of sources, and this number is obtainable. It seems highly implausible for two finite reduced rings to satisfy these conditions, yet not be isomorphic.

Some other questions that could be explored are as follows:

Question 1. *Can an infinite path or cycle be defined or described?*

Question 2. *How many sources does a connected component have and are there bounds on this number?*

Question 3. *What sort of ring structure do the sources in a digraph retain?*

Question 4. *Is there a maximal length of a cycle in a finite commutative ring with identity?*

Question 5. *What is the structure of a subring that is not an ideal in $\Psi(R)$?*

References

[1] Atiyah, M., *Introduction to Commutative Algebra*, Westview Press, (1994).

[2] Axtell, M., Stickles, J., Trampbachls, Zero-Divisor Ideals and Realizable Zero-Divisor Graphs, (2008).

[3] Bollobás, B., *Modern Graph Theory*, Springer, New York, (1998).

[4] Gallian, J., *Contemporary Abstract Algebra*, Houghton Mifflin Company, Boston, MA, (2006).

[5] Hauskin, S., Skinner, J., Directed Graphs of Commutative Rings, *Pre-print*, (2011).

[6] Lipkovski, A.T., Digraphs associated with finite rings, *Pre-print*, (2010).

[7] Akbari, S. Kiani, D. Mohammadi, F. Moradi, S.: The total graph and regular graph of a commutative ring. J. Pure Appl. Algebra 213(12), 2224-2228 (2009)

[8] Akbari, S. Mohammadian, A. Radjavi, H. Raja, P.: On the diameters of commuting graphs. Linear Algebra Appl. 418(1), 161-176 (2006)

[9] Anderson D.F., Axtell M., Stickles J.: Zero-divisor Graphs in Commutative Rings: A Survey. Edited by Fontana et al, Com