3-1993

Hypergraph Representations and Orders of Cwatsets

Julie Kerr
Rose-Hulman Institute of Technology

Advisors:
Gary Sherman

MSTR 93-02

HYPERGRAPH REPRESENTATIONS AND ORDERS OF CWATSETS

Julie Kerr

MS TR 93-02

March 1993

Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute, IN 47803

FAX(812) 877-3198 Phone: (812) 877-8391

Hypergraph Representations and Orders of Cwatsets

J.B. Kerr

March 18, 1993

1 Introduction

Definition. *A subset, C, of $\mathbf{Z}_2^d$ (binary d-space) with order n is a cwatset of degree (or dimension) d and order n if, for each element* b *of C, there exists a permutation σ of S_d such that $(C + \mathrm{b})^\sigma = C$.*

The cwatsets in $\mathbf{Z}_2^d$ have been characterized as the projections in $\mathbf{Z}_2^d$ of the subgroups of the wreath product of $\mathbf{Z}_2$ by S_d [2]. This characterization and Cayley were used to construct all cwatsets of degrees 3, 4, and 5 that are not subgroups of $\mathbf{Z}_2^d$. These cwatsets are listed in the appendix.

The list provides answers to two of the questions posed in [2]. The 4-dimensional cwatset C,

0000
1111
0011
1100
0110
1001,

provides a counterexample to the conjecture that every cwatset can be written as a direct sum of cyclic cwatsets. (A cyclic cwatset is generated by a single element of $\mathbf{Z}_2^d$ and a single element of S_d.) It is easy to check that C is not cyclic. Thus (since there are no 1-dimensional or 2-dimensional cwatsets which are not groups) the only other possibility is that it is the direct sum of a 1-dimensional cwatset and a 3-dimensional cyclic cwatset. The only possibility is $\{0, 1\} \bigoplus \{000, 101, 011\}$, which does not equal C. However, the idea of somehow 'reducing' cwatsets does appear promising. In this paper, we will introduce *perfect cwatsets*, another kind of 'irreducible' cwatset.

Sherman and Wattenberg proved in [2] that if there exists a cwatset of degree d and order k, then k divides $2^d d!$. They also asked if the converse is true; if $k \leq 2^d$ and if k divides $2^d d!$, must there exist a cwatset of degree d and order k? The appendix shows that the answer is no, for there is no cwatset of degree 5 and order 15. Although we originally found this counterexample by making an exhaustive list of cwatsets of degree 5, the result also comes as a special case of a general theorem. Before stating this theorem, we note that it is useful to represent a cwatset by lising one element on each line, as in the 4-dimensional example above. This way, we may speak of the m^{th} column of the cwatset. The *weight* of a column is the number of 1's in that column, and a *k-column* is a column of weight k.

Theorem 1 *A cwatset of odd order has order at most $\binom{d}{(d-1)/2}$ if d is odd or $\binom{d}{(d-2)/2}$ if d is even.*

Proof. Every column of a cwatset of odd order n is either a *large column* (a column of weight greater than $n/2$) or a *small column* (a column of weight less than $n/2$). Since adding 1 to a large column changes it to a small column and adding 1 to a small column changes it to a large column, then each element of the cwatset must have the same number of 1's in large columns as in small columns. Suppose that there are m large columns. Then, each element can be uniquely represented by the set of large columns with 0's in that element and small columns with 1's in that element. Each such set has order m, and therefore, there are at most $\binom{d}{m}$ possible elements in the cwatset. Clearly, m is less than $d/2$, and the result follows. □

2 Perfect Cwatsets and Hypergraph Representations

Definition. *A perfect cwatset is a cwatset in which every column is either a k-column or an $(n-k)$-column, where n is the order of the cwatset.*

The 4-dimensional cwatset in the introduction is perfect because every column has weight 3. Another perfect cwatset is

$$\begin{matrix} 000 \\ 101 \\ 011 \end{matrix}$$

in which each column has either weight 1 or weight 2. The cwatset

$$\begin{matrix} 00000 \\ 11000 \\ 00101 \\ 11101 \\ 00011 \\ 11011 \end{matrix}$$

is an example of a cwatset which is not perfect, because the columns have weights 2, 3, and 4.

Any perfect cwatset with $k \neq n/2$ can be represented as a hypergraph in the following manner. Each column is represented by one vertex. Each element, $\mathbf{b}$, is represented by an edge whose vertices correspond to k-columns with a 0 in $\mathbf{b}$ and $(n-k)$-columns with a 1 in $\mathbf{b}$. For example, the cwatset

$$\begin{array}{c} 0000 \\ 1100 \\ 1010 \\ 1001 \end{array}$$

is represented by the hypergraph on four vertices with edges $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$, and the cwatset

$$\begin{array}{c} 0000000 \\ 1001000 \\ 1101100 \\ 1111110 \\ 1110111 \\ 0110011 \\ 0010001 \end{array}$$

is represented by the hypergraph on seven vertices with edges $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,5\}$, $\{4,5,6\}$, $\{5,6,7\}$, $\{1,6,7\}$, and $\{1,2,7\}$.

Lemma 1 *Each element of a perfect cwatset has the same number of 1's in k-columns as in $(n-k)$-columns.*

Proof. If $k = n/2$ the result is trivial. Thus, we need only consider the case when $k \neq n/2$. When an element of a cwatset is added to the cwatset, the number of columns of a given weight must remain the same. Since adding 1 to a k-column changes it to an $(n-k)$-column, adding 1 to an $(n-k)$-column changes it to a k-column, and adding 0 to any column leaves it the same, the result follows. $\square$

Theorem 2 *The hypergraph, H, of a perfect cwatset, C, has the following properties:*

(i) H is regular.

(ii) H is uniform.

(iii) For any two edges $\mathbf{e}$ *and* $\mathbf{f}$ *in H, there is a permutation of the vertices of H which sends edges onto edges and, in particular, sends* $\mathbf{e}$ *onto* $\mathbf{f}$.

Proof. (*i*) Every k-column of C contains $(n-k)$ 0's, and every $(n-k)$-column contains $(n-k)$ 1's. Therefore, each vertex of H has degree $(n-k)$, and hence H is regular. (*ii*) Suppose that C has m k-columns. By Lemma 1, each element, $\mathbf{b}$, of C has s 1's in k-columns and s 1's in $(n-k)$-columns for some s. Thus, the edge representing $\mathbf{b}$ is incident to $(m-s)+s=m$ vertices. Therefore, H is uniform. (*iii*) Let $\mathbf{f}$ be the set of vertices corresponding to the k-columns of C. Since C contains the $\mathbf{0}$ element (see [2]), $\mathbf{f}$ is in fact an edge in H. Let $\mathbf{e}$ be any edge in H, and suppose that $\mathbf{e}$ corresponds to the element $\mathbf{b}$, with $(C+\mathbf{b})^\sigma = C$. In order to prove (*iii*), it suffices to show that the permutation σ sends edges onto edges and sends $\mathbf{e}$ onto $\mathbf{f}$. Let H' represent the hypergraph of $C+\mathbf{b}$ and let H'' represent the hypergraph of $(C+\mathbf{b})^\sigma$. Since $(C+\mathbf{b})^\sigma = C$, then $H = H''$. The following chart shows that for any element $\mathbf{b}'$, the representation of $\mathbf{b}'$ in H is the same as the representation of $\mathbf{b}+\mathbf{b}'$ in H'. The eight columns of the chart represent the eight possible combinations of 0's and 1's in $\mathbf{b}$ and $\mathbf{b}'$ and k-columns and $(n-k)$-columns of C. The k-columns are shown by dots, and the $(n-k)$-columns are unmarked. Similarly, the vertices lying in an edge are shown by dots, and the vertices not lying in the edge are unmarked.

k-columns of C	•	•	•	•				
b	0	0	1	1	0	0	1	1
b'	0	1	0	1	0	1	0	1
$b+b'$	0	1	1	0	0	1	1	0
k-columns of $C+b$	•	•					•	•
b' in H	•		•			•		•
$b+b'$ in H'	•		•			•		•

Therefore, $H = H'$. It follows that $H' = H''$. Thus, σ sends edges onto edges. It also sends the k-columns of $C+\mathbf{b}$ onto the k-columns of C. But the k-columns of $C+\mathbf{b}$ are the vertices of $\mathbf{e}$. Therefore, σ sends $\mathbf{e}$ onto $\mathbf{f}$, which completes the proof. □

Theorem 3 *Any hypergraph satisfying conditions (i) - (iii) of Theorem 2 is the representative of some cwatset.*

Proof. Let H be a hypergraph. For edges $\mathbf{g}$ and $\mathbf{h}$ in H, define $\mathbf{g_h}$ as the element of $(\mathbf{Z}_2)^d$ containing a 1 in any column whose corresponding vertex is in exactly one of $\mathbf{g}$ and $\mathbf{h}$, and a 0 in all other columns. Define $H_{\mathbf{h}}$ as $\{\mathbf{g_h} : \mathbf{g} \in H\}$. Now let $\mathbf{f}$ be an edge in H, and let $S = H_{\mathbf{f}}$. For any $\mathbf{b}$ in S, let $\mathbf{e}$ be the edge of H for which $\mathbf{b} = \mathbf{e_f}$. Finally, for any $\mathbf{b}'$ in S, let $\mathbf{e}'$ be the edge of H for which $\mathbf{b}' = \mathbf{e}'_{\mathbf{f}}$. The following chart shows that $\mathbf{b}' + \mathbf{b} = \mathbf{e}'_{\mathbf{e}}$.

f	•	•	•	•				
b	0	0	1	1	0	0	1	1
b'	0	1	0	1	0	1	0	1
e	•	•					•	•
e'	•		•			•		•
$b'+b$	0	1	1	0	0	1	1	0
e'_e	0	1	1	0	0	1	1	0

It follows that $S + \mathbf{b} = H_{\mathbf{e}}$. Let σ be the permutation of the vertices of H which sends edges onto edges and sends $\mathbf{e}$ onto $\mathbf{f}$. We have

$$(S+\mathbf{b})^\sigma = (H_{\mathbf{e}})^\sigma = (H^\sigma)_{\mathbf{e}^\sigma} = H_{\mathbf{f}} = S.$$

Therefore, S is a cwatset, and it is easy to check that H is the hypergraph of S. □

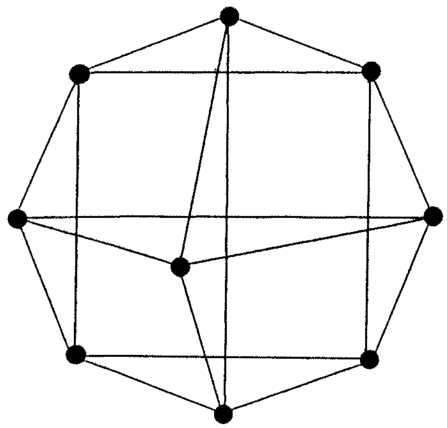
3 Finding $(d, 2)$-cwatsets

Definition. *A (d,m)-cwatset is a perfect cwatset ($k \neq n/2$) of degree d having exactly m columns which are k-columns.*

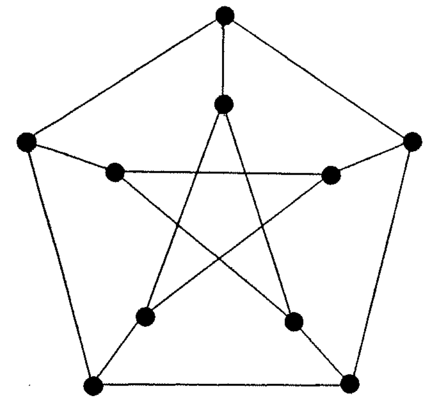
The $(d, 2)$-cwatsets are especially easy to study because their hypergraphs are just graphs. It is clear that all cyclic graphs and all complete graphs meet conditions (i) - (iii) of Theorem 2. Therefore, there exist $(d, 2)$-cwatsets of orders d, for $d \geq 3$, and $\binom{d}{2}$, for $d \geq 2$. Now let $d = s \cdot t$, and suppose that there exists a graph G with t vertices and n edges which meets conditions (i) - (iii). Then we can create a graph with d vertices and $n \cdot s$ edges which satisfies conditions (i) - (iii) by simply combining s disjoint copies of G. Also, we can create a graph H with d vertices and $n \cdot s^2$ edges which meets the conditions by representing each vertex, u, in G as a set, U, of s vertices in H, and by representing each edge, uv, in G as the complete bipartite graph on U and V in H. Therefore, if there exists a $(t, 2)$-cwatset of order n, then there also exist $(d, 2)$-cwatsets of orders $n \cdot s$ and $n \cdot s^2$. This information is enough to give us a sizeable list of orders of $(d, 2)$-cwatsets:

degree	**orders**
2	1
3	3
4	2, 4, 6
5	5, 10
6	3, 6, 9, 12, 15
7	7, 21
8	4, 8, 12, 16, 24, 28
9	9, 27, 36
10	5, 10, 20, 25, 40, 45
11	11, 55
12	6, 12, 18, 24, 30, 36, 48, 54, 60, 66

In addition, there are several graphs which meet conditions (i) - (iii) but don't fit into any nice category. The following two graphs show that there exist a (9,2)-cwatset of order 18 and a (10,2)-cwatset of order 15:

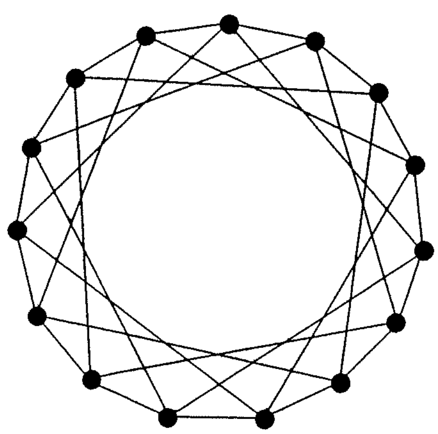


Degree 9, Order 18

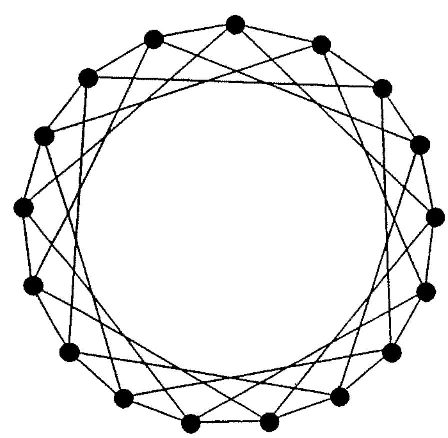


Degree 10, Order 15

There also exist $(d^2-1,2)$-cwatsets of order $2(d^2-1)$ and $(d^2+1,2)$-cwatsets of order $2(d^2+1)$, as shown in this example with $d=4$:



Degree 15, Order 30



Degree 17, Order 34

4 Limitations on the Orders of (d, m)-cwatsets and $(d, 2)$-cwatsets

Theorem 4 *Any (d, m)-cwatset has order which is at most $\binom{d}{m}$ and is divisible by $d/\gcd(d, m)$.*

Proof. Let n be the order of the cwatset C. Suppose that the vertices of the hypergraph representation of C have degree a. Since each edge is incident to m vertices, we have $d \cdot a = n \cdot m$. Thus,

$$\frac{d}{\gcd(d,m)} a = n \frac{m}{\gcd(d,m)},$$

and so $d/\gcd(d, m)$ divides n. Furthermore, there are only $\binom{d}{m}$ possible edges for the hypergraph, and so n is at most $\binom{d}{m}$. $\square$

By Theorem 4, the order of a $(d, 2)$-cwatset must be divisible by $d/2$ if d is even or d if d is odd, and the order must be at most $\binom{d}{2}$. We have already seen that a large portion of these possibilities are indeed the orders of $(d, 2)$-cwatsets. However, there also exist numbers which meet the conditions of Theorem 4 yet are not the orders of any $(d, 2)$-cwatset. The proofs that there are no $(d, 2)$-cwatsets of a given degree and order vary from case to case and are derived mostly by trial and error. There are, however, a few facts which are essential in all of these proofs.

- Since any edge of a cwatset's graph can be mapped onto any other edge without changing the structure of the graph, each edge must be contained in the same number of 3-cycles (i.e. triangles).

• The number of edges times the number of triangles per edge is equal to three times the number of triangles in the graph.

• It follows that if the number of edges in the graph is not divisible by three, then the number of triangles per edge must be divisible by three.

Theorem 5 *Let m be an integer, $0 < m < d$, such that $m \cdot d$ is even. If $m > (d-m-1)(d-m-2)$ and if $d-m$ does not divide d then there is no $(d,2)$-cwatset of order $m \cdot d/2$.*

Proof. Suppose that there exists such a cwatset, and let H be the hypergraph of the cwatset. Since H is regular, each vertex of H must have degree m. Let v be a vertex of H, let S be the set of m vertices adjacent to v, and let T be the set of $d-m-1$ vertices non-adjacent to v. Suppose that each edge is contained in x 3-cycles of H. This means that each vertex in S is adjacent to exactly x other vertices in S and hence is adjacent to exactly $m-x-1$ vertices in T. Since the sum of the degrees of the vertices in T is equal to the number of edges between S and T plus twice the number of edges between vertices of T, then

$$m(d-m-1) = m(m-x-1) + r,$$

where $0 \le r \le (d-m-1)(d-m-2)$. Since $m > (d-m-1)(d-m-2)$, then $r = 0$ and $d-m-1 = m-x-1$. It follows that every vertex of S is adjacent to every vertex of T and that no two vertices of T are adjacent. In other words, in the complement of H, there exist $d-m$ vertices disjoint from the rest of the vertices, which form a complete subgraph. Notice that v is one of the vertices in this complete subgraph. But v was chosen arbitrarily. Therefore, every vertex must be part of a complete $(d-m)$-subgraph of the complement of H which is disjoint from all the other vertices. However, this can only happen if $d-m$ divides d. Thus, since $d-m$ does not divide d, then H cannot exist. □

Theorem 5 tells us that there is no (8,2)-cwatset of order 20 and no (10,2)-cwatset of order 35. Proofs for other degrees and orders follow in a similar manner. Here is a more complete list of which numbers satisfying Theorem 4 are the orders of $(d,2)$-cwatsets and which are not.

d	$\binom{d}{2}$	orders	not orders	unknown
2	1	1		
3	3	3		
4	6	$2,4,6$		
5	10	$5,10$		
6	15	$3,6,9,12,15$		
7	21	$7,21$	14	
8	28	$4,8,12,16,24,28$	20	
9	36	$9,18,27,36$		
10	45	$5,10,15,20,25,40,45$	35	30
11	55	$11,55$	$33,44$	22
12	66	$6,12,18,24,30,36,48,54,60,66$	42	

5 Further Questions

Definition. *A perfect cwatmultiset is a multiset in which every element of a perfect cwatset is duplicated the same number of times.*

Let k be any integer, $0 \le k < n$. When an element $\mathbf{b}$ is added to a cwatset C of order n, each k-column becomes either a k-column or an $(n-k)$-column, depending on whether b contains a 0 or a 1 in that column. Thus, σ may only send a k-column to a k-column or to an $(n-k)$-column. Therefore, if we cross out all but the k-columns and $(n-k)$-columns of C, the remaining set will form a perfect cwatset or perfect cwatmultiset. This indicates that perfect cwatsets may be the key to understanding the properties of cwatsets in general. It is still not entirely clear, though, how, once all of the perfect cwatsets have been categorized, we can use that information to find all of the cwatsets.

There are two types of perfect cwatsets: those in which every column has the same weight ($k = n/2$), and those in which the columns are of two distinct weights ($k \neq n/2$). It is clear from the list of 5-dimensional cwatsets that there are far more perfect cwatsets of the first type than of the second type. In this paper we have focused on the second type ($k \neq n/2$) of perfect cwatsets, which have the useful property that they can be represented as hypergraphs with a high degree of symmetry. Does a similar representation exist for the perfect cwatsets in which each column has weight $n/2$? Is there a more general method for finding the orders of $(d, 2)$-cwatsets or (d, m)-cwatsets?

6 References

1. J.E. Atkins and G.J. Sherman, Sets of typical subsamples, *Statistics and Probability Letters* **14** (1992) 115-117.

2. G.J. Sherman and M. Wattenberg, Introducing ... Cwatsets!, *Mathematics Magazine* (to appear).

7 Appendix

7.1 The Cayley code.

The wreath product of $\mathbf{Z}_2$ by S_d is the set $S_d \times \mathbf{Z}_2^d$ together with the binary operation $(\sigma_1, \mathbf{b}_1) * (\sigma_2, \mathbf{b}_2) = (\sigma_1\sigma_2, \mathbf{b}_1^{\sigma_2} + \mathbf{b}_2)$.

In other words, when $(\sigma_2, \mathbf{b}_2)$ is added to $(\sigma_1, \mathbf{b}_1)$, $\mathbf{b}_1$ is permuted by σ_2 before $\mathbf{b}_2$ is added. Recall that in a cwatset C, for every element $\mathbf{b}$ there exists a σ such that $(C + \mathbf{b})^\sigma = C$, or, in other words, $C = C^{\sigma^{-1}} - \mathbf{b} = C^{\sigma^{-1}} + \mathbf{b}$. Thus for any subgroup of the wreath product, we can view a single element as an ordered pair of one of the elements of a cwatset and the inverse of the corresponding permutation.

Here is the Cayley code used for generating cwatsets:

```
library cwatsets;


D = 5;

B = cyclic(2);

S = symmetric(D);

W = wreath(B,S);

C = subgroups(W);
```

Puts all of the subgroups of the wreath product into array C.

```
All = null;
```

This set will contain all column permutations of the cwatsets encountered.

```
For Sbs = 1 to length(C) do

 G = C[Sbs];
 E = elements(G);
 K = null;
 for each x in E do
  Bin = conseq(0,D);
  for A = 1 to D do
   Bin[A] = ((2*A)^x) mod 2;
   end;
  K = K join [Bin];
  end;
```

This section converts subgroup G to a cwatset K by extracting just the binary part from the subgroup elements. Cayley stores each element of the wreath product as a permutation x of the symmetric(2D) group, and the Ath digit of the binary part can be recovered by checking whether x sends 2A to an odd or even number.

```
 Od = order(All);
 All = All join [K];
```

```
if (Od ne order(All)) then
 for each w in S do
  Equiv = null;
  for each y in K do
   Bin = conseq(0,D);
   for A = 1 to D do
    Bin[A] = y[A^w];
    end;
   Equiv = Equiv join [Bin];
   end;
  All = All join [Equiv];
  end;
```

Adds all equivalent cwatsets of K to All by finding all column permutations of K.

```
K1 = setseq(K);
Grp = K;
for H = 1 to length(K1) do
 x = K1[H];
 for J = H to length(K1) do
  y = K1[J];
  z = conseq(0,D);
```

```
for A = 1 to D do
 z[A] = (x[A] + y[A]) mod 2;
 end;
Grp = Grp join [z];
end;
end;
```

Computes the group generated by the elements of cwatset K. If the group has the same number of elements as the original cwatset, then that cwatset was itself a group. These sets will not be listed as cwatsets.

```
if (order(K) ne order(Grp)) then
 print(Sbs);
 print(K);
 end;
```

Prints the index and elements of the cwatsets which are not themselves groups and which are not the equivalent of any cwatset previously encountered.

```
end;
end;

finish;
```

7.2 Cwatsets of degree 3

Order 3	Order 6
000	000
101	100
011	110
	111
	011
	001

7.3 Cwatsets of degree 4

Order 3	Order 4	Order 6			Order 8	Order 12
0000	0000	0000	0000	0000	0000	0000
0011	0011	1111	0011	1000	1111	1011
1001	0101	0011	1001	0011	0110	0011
	1001	1100	0111	1011	1001	1000
		0110	0100	0010	0111	0010
		1001	1101	1001	1000	1001
					0010	0100
					1101	1111
						0111
						1100
						0110
						1101

7.4 Cwatsets of degree 5

Order 3	Order 4		Order 5		Order 6				
00000	00000	00000	00000	00000	00000	00000	00000	00000	00000
00101	00111	00011	01111	01100	11000	11111	00101	00101	00101
00011	10010	01001	10111	10100	00101	00101	01111	00011	00111
	01110	10001	01001	00101	11101	11010	01010	01101	00010
			10100	00110	00011	00011	01100	01000	00100
					11011	11100	00011	01011	00011

Order 8				
00000	00000	00000	00000	00000
11111	10111	10111	11011	00110
00111	01011	00101	01001	01011
11000	11100	10010	10010	10011
01110	01001	01001	01000	00111
10001	11110	11110	10011	00001
01101	00100	01010	00010	01010
10010	10011	11101	11001	10010

Order 10			
00000	00000	00000	00000
11111	11111	01110	00110
01111	01100	10111	10111
10000	10011	11001	10001
01000	01011	10011	01111
10111	10100	01001	01100
01001	00101	01111	01001
10110	11010	10100	10100
01011	00110	00010	00011
10100	11001	11100	11101

Order 12				
00000	00000	00000	00000	00000
01010	00101	00101	00101	00101
00110	00011	00011	00011	00011
11000	00111	01100	11011	01100
11110	00010	01111	11101	01111
10100	00100	01010	11000	01010
00001	11000	11011	01011	01011
01011	11100	11101	01101	01101
00111	11011	11000	01000	01000
11001	11111	10111	10000	00111
11111	11010	10100	10101	00100
10101	11101	10010	10011	00010

Order 16		Order 20	Order 24
00000	00000	00000	00000
10001	00001	11011	00001
11000	11000	10101	01110
01000	10110	00001	11101
11001	00110	11110	01111
10111	10111	11111	11010
00110	01110	01010	00101
00111	11010	00100	10000
11111	00011	10111	00010
01110	11011	01011	10001
01111	01011	00011	11110
00010	10101	11000	11111
01010	11101	11001	10101
10101	01100	10001	00110
11101	10000	01110	10010
10000	01101	00110	00111
		00111	01000
		11100	01001
		01000	10110
		10100	10111
			01101
			11000
			01010
			11001