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Abstract

Let T be a property which holds for a group independent of whether or not this group is imbedded in a group G or in a p-Sylow of G. Using a generalization of Sylow's second theorem, we prove that if for any p group P the number of subgroups of P satisfying T is congruent to 1 mod p, then for any group G, the number of p subgroups satisfying T is also congruent to 1 mod p. As an application, we give simple proofs of several theorems, including the well-known Frobenius theorem.

Theorem 1, the main theorem of this paper, allows us to transfer certain theorems about p groups to all finite groups. In the statement of Theorem 1, T is a group property, which is true for a p group P whether or not P is viewed as a subgroup of a larger group G or as a subgroup of a p-Sylow of G. For instance all isomorphism-invariant group properties T (such as a group being a p group, or a group containing a subgroup of order 8) satisfy this condition.

Theorem 1 *Let T be a group property as above. Suppose that for any p group P, the number of subgroups of P satisfying T is congruent to* 1 *mod p. Then for any finite G, the number of p subgroups satisfying T is congruent to* 1 *mod p.*

In order to prove Theorem 1, we require Theorem 2, a known [1] generalization of Sylow's second theorem, namely that the number of p-Sylows containing a fixed p group is congruent to 1 mod p. Theorem 2 is normally proven by double coset arguments. We give a simple action based proof which makes use of the following Lemma. $N(G)$ denotes the normalizer of a group G.

Lemma 1 *A p subgroup H of G is contained in $N(P)$, where P is a p-Sylow of G, if and only if $H \subset P$.*

PROOF: If $H \subset P$, then clearly $H \subset N(P)$.

If $H \subset N(P)$, then since H is a p group it is contained in some p-Sylow of $N(P)$. However P is the unique p-Sylow of $N(P)$, since P is normal in $N(P)$ and all p-Sylows in any group, in particular $N(P)$, are conjugate. Thus $H \subset P$. □

Theorem 2 *Let H be a p-subgroup of a group G. Then the number of p-Sylows containing H is congruent to 1 mod p.*

PROOF: Let H act by conjugation on the set of all p-Sylows $P_1, \ldots, P_n$ of G. Now H fixes a p-Sylow P_i if and only if $H \subset N(P_i)$. Lemma 1 thus implies H fixes a p-Sylow P_i if and only if $H \subset P_i$. Since H is a p group, all orbits of this action have size a power of p. This implies that the number of fixed points of this action is congruent to n mod p. The result follows since $n \equiv 1$ mod p, by Sylow's second theorem. □

Now we can prove Theorem 1.

PROOF: Let G be an arbitrary group. The idea of the proof is to count (in two different ways) the number, n, of pairs (H, S) where H is a p group satisfying property T and S is a p-Sylow of G containing H (recall that each p group is contained in some p-Sylow). Intuitively, one gets two different counts by projecting onto the H and S coordinates. Note that by hypothesis we need not be concerned with whether or not H is viewed as a subgroup of S or G.

For the first count, we fix a p-Sylow S_i of G. Since S_i is a p group the number of H contained in S_i satisfying property T is congruent to 1 mod p. By Sylow's second theorem, the number of p-Sylows of G is congruent to 1 mod p. It follows that $n \equiv 1(p)$.

For the second count, we fix a p subgroup H of G with property T. By Theorem 1, the number of p-Sylows of G containing H is congruent to 1 mod p. This implies that n is equal to the number of p subgroups H of G with property T, modulo p.

The result follows by equating the two counts. □

To see the power of Theorem 1, we give a new proof of a fairly difficult result, Frobenius' theorem. A short Lemma is required (the Lemma actually holds if G is any group, but the proof [2] is harder).

Lemma 2 *Let G be an abelian p group. Then the number of subgroups of order p is congruent to 1 mod p.*

PROOF: Since G is abelian, it is a direct product of say k cyclic groups. Note that the elements of order p in G are products of elements of the form $a_1, \ldots, a_k$ where each a_i is contained in the factor i and has order 1 or p, but not all a_i have order 1. Since cyclic groups of order a power of p have precisely p elements of order 1 or p (these elements are the p elements in the unique subgroup of order p) we see that the number of elements of order p in G is $p^k - 1$. Since groups of order p are either equal or disjoint, the number of subgroups of order p of G is $(p^k - 1)/(p - 1)$ which is

$$p^{k-1} + p^{k-2} + \cdots + 1,$$

which is congruent to 1 mod p. □

Theorem 3 *(Frobenius) If p^m divides the order of G, then the number of subgroups of G of order p^m is congruent to 1 mod p.*

PROOF: The property of having order p^m is isomorphism invariant, so by Theorem 1, we can assume G is a p group. Let G act on the set of all subgroups of order p^m by conjugation. Since G is a p-group, the orbits of the action have size a power of p. The fixed points of this action are normal groups of G, so it suffices to prove that the number of normal subgroups of G of order p^m is congruent to 1 mod p. We proceed by induction on the order of G. The base case, where G is cyclic of order p is trivial.

Let n be the size of the set of pairs of the form (A, N) where A is a group of order p contained in the center Z of G, (Z is non-trivial since G is a p group), and N is a normal subgroup of order p^m of G containing A. We then count n is two different ways.

For the first count, fix N. Then the number of pairs of the form (A, N) is simply the number of groups of order p in $Z \cap N$. (Since G is a p-group, any normal subgroup H of G has non-trivial intersection with Z. This can be seen by having G act on the non-identity elements of H by conjugation, and noting that this action must have fixed points). Now since $Z \cap N$ is abelian, by Lemma 2 the number of subgroups of order p of $Z \cap N$ is congruent to 1 mod p. Thus, n is equal to the number of normal subgroups N of order p^m, modulo p.

For the second count, fix A of order p in Z. We want to count the number of normal subgroups N of order p^m containing A. Note that the homomorphism $G \to G/A$ preserves normality, so that in G/A we want the number of normal subgroups of order p^{m-1}. By the induction hypothesis, this is congruent to 1 mod p. Also, since Z is abelian, by Lemma 2 the number of A of order p in Z is congruent to 1 mod p. Thus $n \equiv 1(p)$.

The theorem now follows by equating the two counts. □

The following theorem is in fact true for all P, not just for normal P [2]. The proof for normal P, however, is in the spirit of this paper.

Theorem 4 *If p^m divides the order of G, and P is a normal subgroup of order p^k where $k \leq m$, then the number of subgroups of order p^m containing P is congruent to* 1 *mod p.*

PROOF: If P is normal in G, then it is certainly normal in the p-Sylows of G. The property of containing a fixed subgroup P satisfies the conditions of Theorem 1, so it suffices to prove this theorem for p groups G.

We prove this result by induction on k. The base case $k = 0$ is Frobenius' theorem. Assume the result for $k-1$. Now a normal subgroup of order p^k has non-trivial intersection with the center $Z(G)$. (See parenthetical comment in paragraph 3 of the previous theorem for a proof). Let A be a subgroup of order p contained in $P \cap Z$. Then the number of subgroups of G of order p^m containing P is equal to the number of subgroups of G/A of order p^{m-1} containing P/A. Note that P/A is normal and has order p^{k-1}. The result follows by the induction hypothesis. □

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References

[1] Suzuki, Group Theory 1, p. 96

[2] Searcoid, A Reordering of the Sylow Theorems, American Math Monthly, Vol. 94, Number 2, Feb. 1987