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Trees of Irreducible Numerical Semigroups

Taryn M. Laird and José E. Martinez

Abstract.A 2011 paper by Blanco and Rosales describes an algorithm for constructing a directed tree graph of irreducible numerical semigroups of fixed Frobenius numbers. This paper will provide an overview of irreducible numerical semigroups and the directed tree graphs. We will also present new findings and conjectures concerning the structure of these trees.

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1 *Introduction*

There are many problems in which we unknowingly encounter numerical semigroups in mathematics. Two of the most well known examples of these problems are the *coin problem* and the *postage stamp problem*. Here is one of many such puzzles.

Example 1.1. Suppose we are given an infinite number of 4 cent, 6 cent, and 9 cent coins. We would like to determine the largest amount of exact change that we cannot make using combinations of these coins.

In order to find this maximum number, we must consider all possible combinations of the coins. That is,

$$\begin{aligned}\{\text{all possible amounts of change}\} &= \{4k_1 + 6k_2 + 9k_3 | k_1, k_2, k_3 \in \mathbb{N}\} \\ &= \{0, 4, 6, 8, 9, 10, 12, 13, 14, 15, 16, ...\}\end{aligned}$$

Since we have made 4 consecutive integer amounts (12 cents, 13 cents, 14 cents and 15 cents) we know that all integer amounts greater than 15 cents can be made by adding more 4 cent coins to these consecutive amounts. This means that the largest amount of exact change we cannot make is 11 cents.

The next problem is similar to Example 1.1, but it has an important difference.

Example 1.2. Suppose we are given an infinite number of 4 cent and 14 cent coins and we again wish to find the largest amount of change that we cannot make with these coins.

We again consider all possible combinations of the coins:

$$\begin{aligned}\{\text{all possible amounts of change}\} &= \{4k_1 + 14k_2 | k_1, k_2 \in \mathbb{N}\} \\ &= \{0, 4, 8, 12, 14, 16, 18, ...\}\end{aligned}$$

Note that we can never make an odd amount of change with our coins. This means that there cannot be a largest un-makeable amount.

The infinitely large set of change amounts in the first example is called a numerical semigroup, and the largest uncreatable number is called the Frobenius number of the numerical semigroup.

We now ask if there are other sets of coin denominations that have a largest un-makeable amount of 11 cents. It turns out that other such sets of denominations do share the same largest unmakeable amount and a special subset of the corresponding numerical semigroups can be recursively constructed in a directed tree graph.

In the following sections of the paper we will first define numerical semigroups and certain properties that are important to both the numerical semigroups themselves and the directed tree graphs. We will then briefly examine how numerical semigroups are connected to communative ring theory. In section 4 we will present an algorithm that creates the directed tree graphs and construct a few examples. To conclude the paper, we will present some new results that help describe the structure of these directed tree graphs.

2 Numerical Semigroups

We first need a solid grasp of what a numerical semigroup is, and more precisely, what it means to be an irreducible numerical semigroup.

Definition 2.1. A *numerical semigroup* is a subset S of the nonnegative integers $\mathbb{N}$ that satisfies the following:

1. S contains 0.
2. S is closed under addition.
3. $\mathbb{N}\backslash S$ is finite.

We say $\{a_1, a_2, ..., a_n\}$ is a *generating set* for S if $S = \{k_1a_1 + k_2a_2 + ... + k_na_n | k_1, k_2, ..., k_n \in \mathbb{N}\}$, and we call each a_i a *generator* of S. If no proper subset is a generating set for S, we say $\{a_1, a_2, ..., a_n\}$ is the *minimal generating set* and we write $S = \langle a_1, a_2, ..., a_n\rangle, 0 < a_1 < a_2 < ... < a_n$.

Example 2.1. Consider the numerical semigroup generated by 5, 6 and 13. Then

$$\begin{aligned} S = \langle 5, 6, 13\rangle &= \{5k_1 + 6k_2 + 13k_3 | k_1, k_2, k_3 \in \mathbb{N}\} \\ &= \{0, 5, 6, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, ...\} \\ &= \{0, 5, 6, 10, 11, 12, 13, 15, \rightarrow\} \end{aligned}$$

The arrow indicates that all integers greater than 15 are in S.

Associated with every numerical semigroup are two important integers which are important for our studies.

Definition 2.2. The *Frobenius number* of a numerical semigroup S, denoted $F(S)$, is the largest integer not in S. The *multiplicity* of S, denoted $m(S)$, is the smallest positive element of S.

Example 2.2. Consider once again the numerical semigroup generated by 5, 6, and 13. That is,

$$S = \langle 5, 6, 13\rangle = \{0, 5, 6, 10, 11, 12, 13, 15, \rightarrow\}$$

Here $F(S) = 14$ as it is the largest integer not in S, and $m(S) = 5$, as 5 is the smallest positive element in S.

Next we define the two special types of numerical semigroups that are the focus of our research.

Definition 2.3. A numerical semigroup S is said to be *symmetric* if and only if $F(S)$ is odd and if $x \in \mathbb{Z} \notin S$, then $F(S) - x \in S$.

Example 2.3. Consider, $S = \langle 6, 7, 10, 11 \rangle = \{0, 6, 7, 10, 11, 12, 13, 14, 16, \rightarrow\}$. Here $F(S) = 15$, which is odd, as required. In addition, for every integer $x \notin S$ we have $F(S) - x \in S$ as depicted in Figure 1 below. For example, if $x = 8$, we have $15 - 8 = 7 \in S$, and the same can be verified for all elements in $\mathbb{Z} \notin S$. Thus S is symmetric.

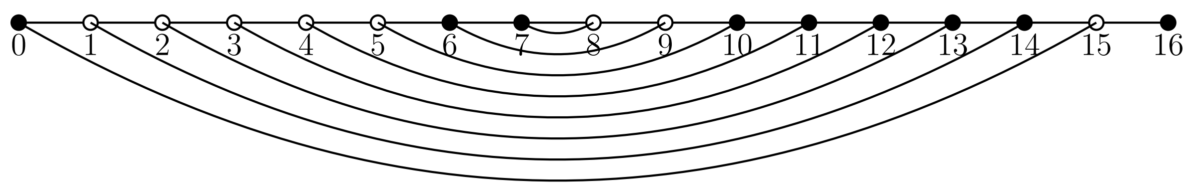

Figure 1

Definition 2.4. A numerical semigroup S is said to be *pseudosymmetric* if and only if $F(S)$ is even and if $x \in \mathbb{Z} \notin S$, then either $x = \frac{F(S)}{2}$ or $F(S) - x \in S$.

Example 2.4. Again consider $S = \langle 5, 6, 13 \rangle = \{0, 5, 6, 10, 11, 12, 13, 15, \rightarrow\}$. As previously discussed, $F(S) = 14$ is even as required. As indicated in Figure 2, for every integer $x \notin S$ we have either $x = \frac{F(S)}{2}$ or $F(S) - x \in S$, and so S is pseudosymmetric.

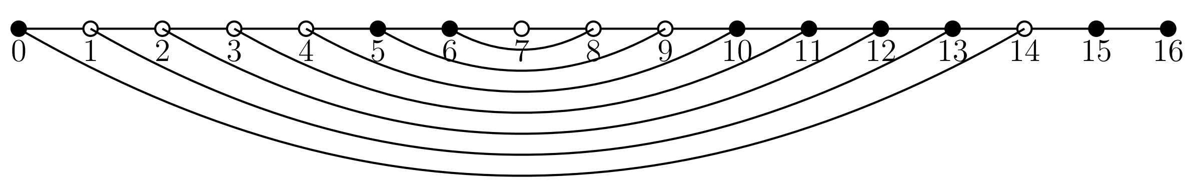

Figure 2

The reason for our emphasis on symmetric and pseudosymmetric numerical semigroups will be made clear in the remaining sections of the paper.

2.1 Irreducible Numerical Semigroups

Definition 2.5. A numerical semigroup S is *irreducible* if it cannot be expressed as an intersection of two numerical semigroups that properly contain S. For a given Frobenius number F, the set of all irreducible numerical semigroups with Frobenius number F is denoted $I(F)$.

In [2], Branco and Rosales showed that a numerical semigroup is irreducible if and only if it is symmetric or pseudosymmetric. The following theorem [1] defines a certain numerical semigroup unique to every Frobenius number.

Theorem 2.1. *For every positive integer F, there exists a unique irreducible numerical semigroup $C(F)$ whose generators are all larger than $\frac{F}{2}$. Moreover,*

$$
\begin{aligned}
C(F) &= \begin{cases} \{0, \frac{F+1}{2}, \rightarrow\}\backslash\{F\} & \textit{if } F \textit{ is odd,} \\ \{0, \frac{F}{2}+1, \rightarrow\}\backslash\{F\} & \textit{if } F \textit{ is even.} \end{cases} \\
&= \begin{cases} \langle \frac{F+1}{2}, \frac{F+3}{2}, ..., F-1\rangle & \textit{if } F \textit{ is odd,} \\ \langle \frac{F}{2}+1, \frac{F}{2}+2, ..., F-1, F+1\rangle & \textit{if } F \textit{ is even.} \end{cases}
\end{aligned}
$$

Example 2.5. Using Theorem 2.1 we will now construct $C(15)$.

$$
\begin{aligned}
C(15) &= \{0, \frac{15+1}{2}, \rightarrow\}\backslash\{15\} \\
&= \{0, 8, 9, 10, 11, 12, 13, 14, 15, 16, \rightarrow\}\backslash\{15\} \\
&= \{0, 8, 9, 10, 11, 12, 13, 14, 16, \rightarrow\} \\
&= \langle 8, 9, 10, 11, 12, 13, 14\rangle
\end{aligned}
$$

Note that when F is odd, $C(F)$ is generated by an interval of consecutive integers of $\mathbb{N}$. This type of numerical semigroup, first investigated by García-Sánchez and Rosales in [3], led to a new characterization of symmetry for interval-generated numerical semigroups. The interested reader can consult [3] for more information.

3 Graph Theory Necessities

The remainder of this paper will focus on graphs of families of specific numerical semigroups that share a common Frobenius number. Therefore, a brief review of basic graph theory terminology is appropriate.

Definition 3.1. A *graph* is a collection of points called *vertices* and lines called *edges* that connect pairs of vertices. Two vertices are *adjacent* if an edge connects them.

Definition 3.2. A *tree graph* is a graph in which there is a unique sequence of alternating edges and vertices "connecting" each pair of vertices. A *directed tree graph* is a tree graph in which there is a direction associated with each edge.

Example 3.1. In Figure 3 we see an example of a directed tree graph. We will reference this tree in order to clearly define the terms that we use to describe tree graphs.

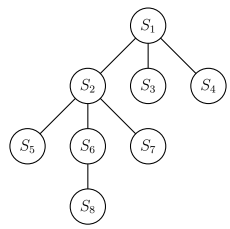

Figure 3

The designated vertex from which the tree appears to "hang" - i.e., the vertex (S_1) at "the top" of the tree - is called the *root* of the tree. The *level* of a vertex is the length of the path from the vertex to the root. For instance, in Figure 3 the level of S_2 is 1, and we say that S_2 is in the first level. The *height* of a tree is the maxiumum level in the tree. Note that the height of the tree in Figure 3 is 3. Given any pair of adjacent vertices, the vertex in the greater level is called the *child* of the vertex in the smaller level. For example, in Figure 3, S_3 is a child of S_1, and S_5, S_6, and S_7 are the children of S_2. A vertex with no children is called a *leaf.* A *branch* is the shortest "path" from the root to a leaf. As a result, the longest branch in a tree is equvalent to the height of the tree. Note that in Figure 3 the longest branch is the path from S_1 to S_8 passing through S_2 and S_6.

4 Directed Tree Graphs

In an attempt to catalogue all symmetric and pseudosymmetric numerical semigroups with a fixed Frobenius number, Blanco and Rosales [1] developed an algorithm for systematically finding all such irreducible numerical semigroups.

Theorem 4.1. *Let F be a positive integer. Then the elements of $I(F)$ comprise a directed tree graph, denoted $G(I(F))$, with root $C(F)$. If S is an element of $I(F)$, then the children of S are $(S\backslash\{x_1\}\cup\{F-x_1\}), (S\backslash\{x_2\}\cup\{F-x_2\}), ..., (S\backslash\{x_r\}\cup\{F-x_r\})$, where $\{x_1, ..., x_r\}$ is the set of minimal generators of S such that for each $x \in \{x_1, ..., x_r\}$ the following conditions are satisfied:*

1. $\frac{F}{2} < x < F$
2. $2x - F \notin S$

3. $3x \neq 2F$

4. $4x \neq 3F$

5. $F - x < m(s)$.

When a minimal generator, x, of $S \in (G(I(F))$ satisfies the five conditions, we say that x *spawns* a child of S, that child being $(S \backslash \{x\} \cup \{F - x\})$. A minimal generator of a given vertex that does not spawn a child is said to be *barren*.

Example 4.1. In order to build $G(I(11))$, the directed tree graph associated with Frobenius number 11, we begin by constructing the root, $C(11)$:

$$C(11) = \{0, \frac{F(S)+1}{2}, \rightarrow\} \backslash \{F\} = \{0, 6, 7, 8, 9, 10, 12, \rightarrow\}$$
$$= \langle 6, 7, 8, 9, 10 \rangle.$$

Using Theorem 4.1 and checking each minimal generator we find that the following generators spawn the given irreducible numerical semigroups in $G(I(11))$.

$x_1 = 8 : (\langle 6, 7, 8, 9, 10 \rangle \backslash \{8\}) \cup \{3\} = \{0, 3, 6, 7, 9, 10, 12, \rightarrow\} = \langle 3, 7 \rangle$
$x_2 = 7 : (\langle 6, 7, 8, 9, 10 \rangle \backslash \{7\}) \cup \{4\} = \{0, 4, 6, 8, 9, 10, 12, \rightarrow\} = \langle 4, 6, 9 \rangle$
$x_3 = 6 : (\langle 6, 7, 8, 9, 10 \rangle \backslash \{6\}) \cup \{5\} = \{0, 5, 7, 8, 9, 10, 12, \rightarrow\} = \langle 5, 7, 8, 9 \rangle$

These irreducible numerical semigroups are the children of the root $C(11)$ in $G(I(11))$. By applying the same algorithm we find that, in $G(I(11))$, $\langle 3, 7 \rangle$ is barren, whereas $\langle 4, 6, 9 \rangle$ has one child, $\langle 2, 13 \rangle$, and $\langle 5, 7, 8, 9 \rangle$ has one child, $\langle 4, 5 \rangle$ spawned by 9 and 7, respectively. The entire tree $G(I(11))$ is shown in Figure 4 below.

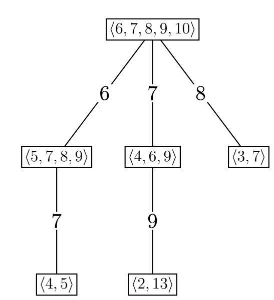

Figure 4

The following images of specific directed tree graphs were generated by a program written in Mathematica and compiled in QTikZ. For simplicity, the tree vertices are labels of the form S_1, S_2, S_3, etc. (corresponding to the order in which they were found by the program) and the minimal generator from which each child spawned is not shown.

Example 4.2. Figure 5 below is an image of $G(I(16))$. It should be noted that this tree is isomorphic to $G(I(18))$, and in all likelihood they are the only such pair of isomorphic trees in the entire family $G(I(F))$ for $F > 11$.

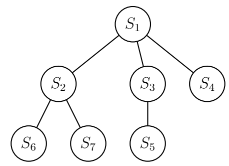

Figure 5

Example 4.3. Figure 6 is an image of $G(I(30))$.

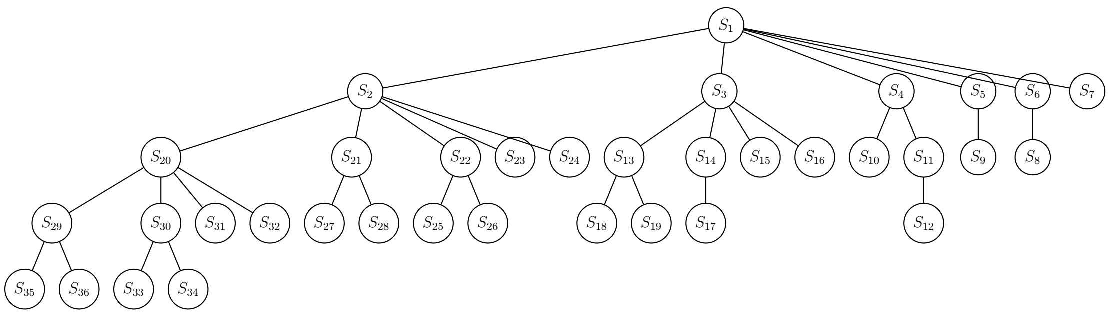

Figure 6

The tree in Figure 6 shows many of the obvious characteristics of the trees for $F > 12$, as the left side contains longer branches and more children. Moreover, the "width" and the number of vertices vary dramatically from tree to tree. Further exploration of the structure of the trees can be found in the next section.

Example 4.4. Figure 7 is an image of $G(I(27))$.

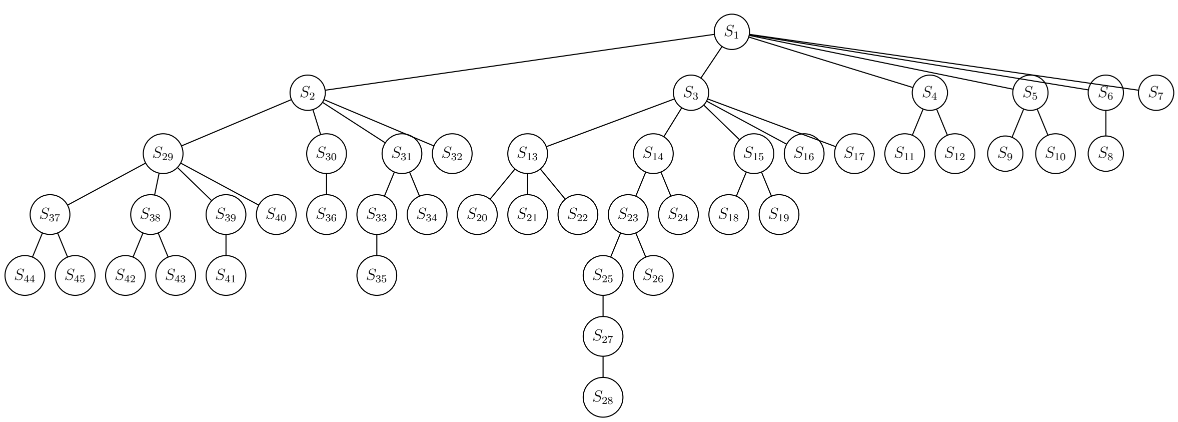

Figure 7

5 New Results

In our investigation we have been especially interested in finding explicit formulas for the number of vertices, the width and the height of any given tree. This has in general been difficult, so we have resorted to examining less complicated properties of the trees which may lead to finding of formulas. For instance, we have investigated which minimal generators will spawn children in $G(I(F))$ and how often, and we have examined the longest branches in the trees. To this end, the trees have been split into two categories, the odd Frobenius number trees and the even Frobenius number trees. The following properties of $G(I(F))$ have been found:

Proposition 5.1. *Let $F = 2k + 1$. Then:*

1. *If $3|F$, then the minimal generator $k + \frac{k+2}{3}$ does not spawn children in $G(I(F))$.*
2. *If k is even, then the minimal generators of $C(F)$ greater than or equal to $k + \frac{k}{2} + 1$ will not spawn children of $C(F)$.*
3. *If k is odd, then the minimal generators of $C(F)$ greater than or equal to $k + \frac{k+1}{2} + 1$ will not spawn children of $C(F)$.*

Proof. To prove part (1), note that $3(k + \frac{k+2}{3}) = 4k + 2 = 2F$. This violates condition 3 in Theorem 4.1, and thus the minimal generator $k + \frac{k+2}{3}$ is barren.

For (2), note that $2(k + \frac{k}{2} + 1) - (2k + 1) = k + 1$. Since $k + 1$ is a minimal generator of $C(F)$, $2(k + \frac{k}{2} + 1) - F \in C(F)$, which violates condition 2 in Theorem 4.1. Also, if $n > \frac{k}{2} + 1$, then $2(k + n) - F > k + 1$, which means that $2n - F \in C(F)$. Therefore, $k + n$ will not spawn children of $C(F)$.

Finally, for part (3) note that $2(k + \frac{k+1}{2} + 1) - (2k + 1) = k + 2$, and since $k + 2$ is a minimal generator of $C(F)$, $k + \frac{k+1}{2} + 1 - F \in C(F)$, which violates condition 2. Likewise for $n \geq \frac{k+1}{2} + 2$, note that $2(k + n) - F > k + 2$, so $2n - F \in C(F)$. Thus, $k + n$ will not spawn children of $C(F)$. $\square$

The following is another small, yet useful result we have found for studying the odd Frobenius number trees.

Proposition 5.2. *Let $F = 2k + 1, k > 1$. The smallest odd minimal generator of $C(F)$ will always spawn a child of $C(F)$.*

Proof. For this proof we must show that the smallest odd minimal generator of $C(F)$ satisfies all conditions of Theorem 4.1. For the first condition, suppose k is odd. The smallest odd minimal generator of $C(F)$ is then $k + 2$. Note that $\frac{F}{2} = k + \frac{1}{2}$, hence $\frac{F}{2} < k + 2 < F$. For the second condition, note that $2(k+2) - F = 3$, and since $F > 3, k > 2$. Thus $k+1 > 3$, so $2(k+1) - F \notin S$. For the third condition, suppose $3(k+2) = 2F$. Then $3k+6 = 4k+2$ and $k = 4$, which contradicts the assumption that k is odd. Therefore, $3(k + 2) \neq 2F$. To show that the fourth condition is satisfied, suppose that $4(k + 2) = 3F$. Then $4k + 8 = 6k + 3$, which implies that $2k = 5$, which is false. Therefore, $4(k+2) \neq 3F$. For the fifth condition, note that $F-(k+2) = k-1$ and $m(C(F)) = k+1$, thus $F-(k+2) < m(C(F))$. Therefore, $k+2$ will spawn a child of $C(F)$.

Next, suppose k is even. Then the smallest odd minimal generator of $C(F)$ is $k+1$. For the first condition, note that $\frac{F}{2} < k+1 < F$. For the second condition, note $2(k+1)-F = 1 < k+1$, hence $2(k+1)-F \notin C(F)$. For the third condition, note that $3(k+1) = 2F$ implies that $k = 1$, which is false because $k > 1$. Therefore, $3(k+1) \neq 2F$. For the fourth condition, $4(k+1) = 3F$ implies that $2k = 1$, which is false. Therefore, $4(k+1) = 3F$. Finally, for the fifth condition, note that $F-(k+1) = k$ and $m(C(F)) = k+1$, so $F-(k+1) < m(C(F))$. Therefore, $k+1$ will spawn a child of $C(F)$. $\square$

We have also derived analogous results for even Frobenius number trees:

Proposition 5.3. *Let $F = 2k$. Then:*

1. *If $4|F$, then the minimal generator $k + \frac{F}{4}$ does not spawn children in $G(I(F))$.*

2. *If $6|F$, then the minimal generator $k + \frac{F}{6}$ does not spawn children in $G(I(F))$.*

3. *If k is even, $4 \not| F$ and $6 \not| F$, then the minimal generators of $C(F))$, $k+n$, where $1 \leq n \leq \frac{k}{2}$ will each spawn a child of $C(F)$.*

Proof. For the proof of part (1), note that $4(k+\frac{F}{4}) = 4(k+\frac{2k}{4}) = 6k$, and $3F = 6k$. Thus, $4(k+\frac{F}{4}) = 3F$, which violates condition 4 in Theorem 4.1. Therefore, $k+\frac{F}{4}$ is barren in $G(I(F))$. For part (2), note that $3(k+\frac{F}{6}) = 3(k+\frac{2k}{6}) = 4k$, and $2F = 4k$. Thus, $3(k+\frac{F}{6}) = 2F$, which violates condition 3.

Finally, for part (3), note that to prove that a minimal generator spawns a child, it must be shown that the 5 conditions in Theorem 4.1 are satisfied. Let $1 \leq n \leq \frac{k}{2}$. For condition one, note that $\frac{F}{2} = k$, so $\frac{F}{2} < k+n < F$. For condition 2, note that $2(k+n)-F = 2n < m(C(F))$. For condition 3, note that if $3(k+n) = 2F$, then $3k+3n = 4k$, and $3n = k$, which contradicts the assumption $6 \not| F$. Also, for condition 4, if $4(k+n) = 3F$, then $4k+4n = 6k$ and $4n = F$, which contradicts the assumption $4 \not| F$. Finally, for condition 5, note that $F-k-n$, so $F-(k+n) \leq m(C(F))$ since n ≥ 1. Therefore $k+n, 1 \leq n \leq \frac{k}{2}$ will each spawn a child of $C(F)$. $\square$

We have devoted a great deal of effort toward determining the heights of these trees. We have been especially interested in the longest branches of trees with odd Frobenius number greater than 11. For this study, the following lemma has been found to be useful.

Lemma 5.1. *Let S be an irreducible numerical semigroup with Frobenius number $F = 2k+1$ such that S is spawned by $k+n$, where $k+n$ is an odd minimal generator of $C(F)$ for some positive integer n. Stated differently: $S = S^*\backslash\{k+n\} \cup \{F-(k+n)\}$, where S is a child of $S^* \in G(I(F))$. Then:*

1. *If k is even and w is the smallest odd such that $k+n < k+w < F$ and $k+w$ is the smallest odd minimal generator of S, then $k+w$ will spawn a child of S.*

2. *If k is odd and h is the smallest even such that $k+n < k+h < F$ and $k+h$ is the smallest odd minimal generator of S, then $k+h$ will spawn a child of S.*

Proof. First, suppose k is even. We must show that the minimal generators $k+n$ satisfiy conditions 1-5. For the first condition, note that $\frac{F}{2} = k + \frac{1}{2} < k + w$, so $\frac{F}{2} < k + w < F$. For the second condition, note that $2(k+w) - F = 2w - 1$ and since $w < k+1$, then $2w < k+w+1$ and $2w - 1 < k + w$, and since $k+w$ is the smallest odd in S, then $2w - 1 \notin S$. For the third condition, note that if $3(k+w) = 2F$, then $k = 3w - 2$. But k is even and $3w-2$ is odd. Thus, $3(k+w) \neq 2F$. To show that the fourth condition is satisfied, note that $4(k+w) = 3F$ implies that $4w = 2k+3$. But $2k+3$ is odd and $4w$ is even. Hence, $4(k+w) \neq 3F$. Finally, for the fifth condition, note that $F - (k+w) = k+1-w$, and since $w > n$, then $k+1-w < k+1-n$. Thus, $F-(k+w) < m(S)$. Therefore, $k+w$ will spawn a child of S.

For the second part, suppose that k is odd. For the first condition, note that $\frac{F}{2} = k + \frac{1}{2}$, so $\frac{F}{2} < k + h < F$. For the second condition, note that $2(k+h) - F = 2h - 1$ and recall that $h < k+1$, so $2h - 1 < k + h$. Since $k+h$ is the smallest odd in S, then $2h - 1 \notin S$. For the third condition, note that $3(k+h) = 2F$ implies that $k = 3h - 2$. But k is odd and $3h - 2$ is even. Thus, $3(k+h) \neq 2F$. For the fourth condition, note that if $4(k+h) = 3F$, then $4h - 3 = 2k$, but $4h - 3$ is odd, a contradiction. Thus, $4(k+h) \neq 3F$. Finally, for the fifth condition, note that $F - (k+h) = k+1-h$. Since $m < h$, then $k+1-h < k+1-m$. Thus, $k+1-h < m(S)$. Therefore, $k+h$ will spawn a child of S. □

To illustrate what Lemma 5.1 says, we present the following example:

Example 5.1. Let $F = 2k+1 = 11$. Then $k = 5$. Consider the child of $C(11)$ S spawned by $k+2 = 7$:

$$\begin{aligned} S &= C(11) \setminus \{7\} \cup \{11 - 7\} \\ &= \langle 4, 6, 9 \rangle. \end{aligned}$$

Note that k is odd, and 4 is the smallest even integer such that $k+2 < k+4 < 11$ and $k+4 = 9$ is the smallest odd minimal generator of S. Thus, by Lemma 5.1, the minimal generator 9 will spawn a child of S.

The study of the odd Frobenius number trees led to the following:

Theorem 5.1. *Let $F = 2k+1, k > 5$. Note that $C(F) = \langle k+1, k+2, k+3, ..., 2k \rangle$. Then:*

1. *If k is even, $G(I(F))$ contains a branch whose vertices are spawned by $k+1, k+3, ..., k+k-1$ in this order.*

2. *If k is odd, $G(I(F))$ contains a branch whose vertices are spawned by $k+2, k+4, ..., k+k-1$ in this order.*

Proof. Let k be even. Note that $C(F) = \langle k+1, k+2, ..., k+k \rangle$. By Proposition 5.2, $k+1$ will spawn a child of $C(F)$. Suppose $k+n$ is the smallest odd minimal generator of a numerical semigroup $S \in G(I(F))$, where S is spawned by $k+n-2$ and $1 \le k+n-2 \le 2k-1$. By Lemma 5.1, $k+n$ will spawn a child, S', of S.

Assuming the above, we must show that $k+n+2$ will spawn a child of S'. Note that $F-(k+n) = 2k+1-(k+n)$ is even and all minimal generators of S' less than $k+n$ are even. Thus $k+n+2$ is the smallest odd minimal generator of S'. By Lemma 5.1, $k+n+2$ will spawn a child of S', as long as $n+2 < 2k+3$, as desired. A similar argument is used to prove part 2 and the result follows. $\square$

Example 5.2. Figure 8 is an image of $G(I(19))$. Note that the longest branch is the one described in Theorem 5.1 .

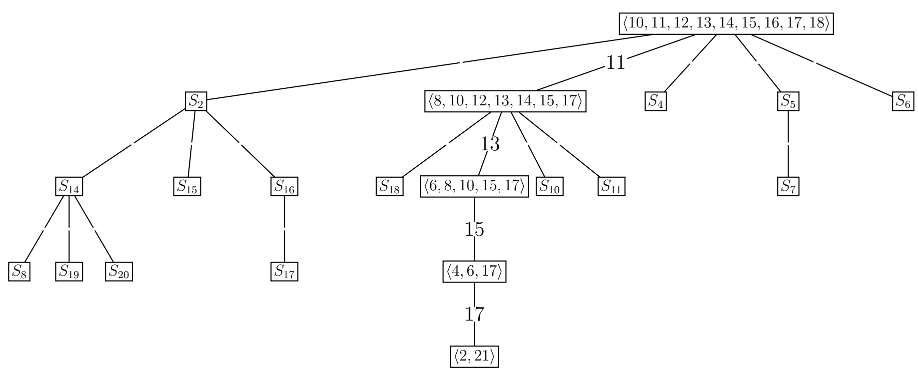

Figure 8

We have generated images for all odd Frobenius number trees between $F = 7$ and $F = 67$, and in all of these trees, where $F > 11$, the longest branch is the one described in Theorem 5.1 above. This leads us to the following conjecture.

Conjecture 5.1. *If $F = 2k+1$, then the branch described in Theorem 5.1 is the unique longest branch in $G(I(F))$.*

6 Open Questions

We have yet to find a formula for the number of vertices in a given tree. This is an especially mysterious and difficult problem to explore since the number of vertices in the trees is not strictly increasing as the Frobenius numbers increase, as can be seen in Figure 9 below which shows the number of levels and vertices in $G(I(F))$.

$F(S)$	Levels	Vertices
14	2	6
15	3	7
16	2	7
17	4	15
18	2	7
19	4	20
20	3	11
21	5	18
22	3	20
23	5	36
24	3	14
25	6	44
26	4	35

$F(S)$	Levels	Vertices
27	6	45
28	4	37
29	7	83
30	4	36
31	7	109
32	5	70
33	8	101
34	5	106
35	8	174
36	5	77
37	9	246
38	6	182
39	9	227

$F(S)$	Levels	Vertices
40	6	196
41	10	420
42	6	203
43	10	546
44	7	342
45	11	498
46	7	527
47	11	926
48	7	411
49	12	1182
50	8	844
51	12	1121
52	8	981

Figure 9

Note that the Frobenius numbers which appear to be responsible for the non-increasing behavior of the numbers of vertices appear to always be divisible by 4 or 6 in the even case, and divisible by 3 in the odd case, which correspond to Propositions 5.1 and 5.3, respectively. Interesting as they are, these observations have not led to a conjecture for the number of vertices in a tree. As shown in the table, the number of vertices grows quite rapidly, making it very difficult to generate images of the trees with larger Frobenius numbers. The largest tree for which we have an image of is $G(I(67))$, which has 11972 vertices.

Determining the height of the trees is slightly less mysterious, and has led to the following conjecture.

Conjecture 6.1. *The height of $G(I(F))$ is $\lfloor \frac{k}{2} \rfloor$ for $F = 2k+1$ (corresponding to a unique branch of this length) and $\lfloor \frac{k-1}{3} \rfloor$ for $F = 2k$ (corresponding to non-unique branches of this length) for $k > 6$.*

Note that the conjectured height of the odd trees corresponds to the length of the branch described in Theorem 5.1. In our future work we hope to prove this conjecture. We also hope to find a formula for, or relatively sharp bounds on, the number of vertices in a given tree.

References

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