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Abstract

For all $g \geq 2$ there is a Riemann surface of genus g whose automorphism group has order $8g + 8$, establishing a lower bound for the possible orders of automorphism groups of Riemann surfaces. Accola and Maclachlan established the existence of such surfaces; we shall call them Accola-Maclachlan surfaces. Later Kulkarni proved that for sufficiently large g the Accola-Maclachlan surface was unique for $g = 0, 1, 2 \bmod 4$ and produced exactly one additional surface (the Kulkarni surface) for $g = 3 \bmod 4$. In this paper we determine the symmetries of these special surfaces, computing the number of ovals and the separability of the symmetries. The results are then applied to classify the real forms of these complex algebraic curves. Explicit equations of the real forms of the curves are given in all except one case.

1. Introduction.

In the 1960's Accola [1] and Maclachlan [12] proved independently that for every $g \geq 2$ there is a Riemann surface X_g (*Accola-Maclachlan surface*) of genus g whose automorphism group has order $8g+8$, by explicitly giving the equation of the surface, $w^2 = z^{2g+2} - 1$, and calculating its automorphism group. This result is interesting in that it is the largest order of an automorphism group that can be uniformly constructed for every g. This extended a result of Wiman in the last century in which he explicitly constructed a surface (*Wiman surface*) of genus g, $w^2 = z^{2g+1} - 1$ with an

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automorphism of maximal order $4g+2$. Much later, Kulkarni [8] considered the question of uniqueness of the surfaces, i.e., whether they were the only surfaces of genus g with an automorphism group of order $8g+8$. Kulkarni demonstrated uniqueness for $g = 0, 1, 2 \bmod 4$ and g sufficiently large. For $g = 3 \bmod 4$ and sufficiently large g he also proved, that in addition to Accola-Maclachlan surface, there is exactly one other surface of genus g whose automorphism group has order $8g+8$, though he did not give explicit equations for the surfaces. We shall call this surface the *Kulkarni surface* denoted by Y_g. He also showed that the conformal automorphism group of X_g has the following presentation $\langle a, b \mid a^{2(g+1)}, b^4, (ab)^2, ab^2a^{-1}b^2\rangle$ and the one of Y_g has the following presentation $\langle a, b \mid a^{2(g+1)}, b^4, (ab)^2, b^2ab^2a^g\rangle$.

In this paper we prove that both the Accola-Maclachlan and the Kulkarni surfaces are symmetric and we determine their symmetries. The results are given in the Table in the Theorem 3.2. Our purpose is not only examine the symmetries themselves, but also to determine the real forms of these two special families of Riemann surfaces. Before proceeding, let us recall a few facts about symmetries and real forms. Let X be a surface of genus g. The group of conformal automorphisms, of X is denoted by $\mathrm{Aut}^+(X)$, and the group of all automorphisms of X, both conformal and anti-conformal, is denoted by $\mathrm{Aut}^\pm(X)$. A *symmetry* σ of X is an anti-conformal, involutary automorphism, i.e., an element of $\mathrm{Aut}^\pm(X)\backslash\mathrm{Aut}^+(X)$ of order 2. The fixed point set $X_\sigma \subset X$ of a symmetry σ is a finite number of disjoint curves diffeomorphic to circles, which are called *ovals* of the symmetry. We denote by $k(X,\sigma)$ the number of ovals of σ. Define the symbol $\varepsilon(X,\sigma)$ to be 0 if $X \setminus X_\sigma$ is not connected - we say that σ is *separating* - and 1 otherwise - we say that σ is *non-separating.* To abbreviate we define the *species of* (X,σ) as

$$\mathrm{sp}(X,\sigma) = \begin{cases} +k(X,\sigma) & \text{if} \quad \varepsilon(X,\sigma) = 0, \\ -k(X,\sigma) & \text{if} \quad \varepsilon(X,\sigma) = 1. \end{cases} \tag{1}$$

Two pairs (X,σ) and (Y,τ) are said to be topologically equivalent if there exists a homeomorphism $h : X \to Y$ such that $\tau h = h\sigma$. Weichold [16] proved that (X,σ) and (Y,τ) are topologically equivalent if and only if X and Y have the same genus and $\mathrm{sp}(X,\sigma) = \mathrm{sp}(Y,\tau)$. Later, Harnack [5] proved that given a triple of integers (g,k,ε) there exists a pair (X,σ) such that g is the genus of X, $k = k(X,\sigma)$ and $\varepsilon = \varepsilon(X,\sigma)$ if and only if

$$1 \leq k+\varepsilon \leq g+1 \; ; \; k \equiv g+1 \bmod (2-\varepsilon) \tag{2}$$

Clearly, two conjugate symmetries σ and τ on X with respect to $\mathrm{Aut}^{\pm}(X)$ have the same species since the pairs (X,σ) and (X,τ) are topologically equivalent. The *symmetry type* of a surface is the list of species of all conjugacy classes of symmetries occurring in $\mathrm{Aut}^{\pm}(X)$.

There is a 1-1 correspondence between symmetries and real forms of algebraic curves. In particular, if an algebraic curve can be defined by polynomial equations with real coefficients then complex conjugation defines a symmetry of the corresponding Riemann surface. The ovals of a symmetry are the connected components of the corresponding real form. A real form is separating if it disconnects its complexification, and non-separating otherwise.

We shall determine the symmetry types of the Accola-Maclachlan surfaces and the Kulkarni surfaces, computing the number of ovals and separability of the symmetries. The results are then applied to determine the real forms of these complex algebraic curves. Explicit equations of the real forms of the curves are given in all cases but one.

2. Number of ovals.

From Riemann's uniformization theorem, every compact Riemann surface X of genus $g \geq 2$ can be represented as the quotient H/Γ of the upper half complex plane H under the action of a fuchsian surface group Γ. Thus there exists a Fuchsian group Δ containing Γ as a normal subgroup such that $G = \mathrm{Aut}^{+}(X) = \Delta/\Gamma$. Although the method we are going to outline works in a more general setting, as appears in [4], for $X = X_g$ or Y_g it is sufficient to analyze what happens when Δ is generated by three elliptic elements of even distinct orders k, l and m (actually in our case $k = 2(g+1), l = 4, m = 2$).

The unique NEC (non-euclidean crystallographic) group Λ containing Δ as a subgroup of index 2 has the presentation

$$\Lambda = \langle c_0, c_1, c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^k, (c_1c_2)^l, (c_0c_2)^m \rangle$$

and we shall refer to c_0, c_1 and c_2 as the canonical reflections of Λ.

From the presentations it is trivial to check that the assignment $a \mapsto a^{-1}, b \mapsto b^{-1}$ induces an automorphism

$$\varphi : G \to G \tag{3}$$

and so applying the criterion of Singerman [15] X is symmetric. This is equivalent to saying that Γ is a normal subgroup of Λ, and precisely, the symmetries on X are the involutions $x\Gamma \in \Lambda/\Gamma = \widetilde{G}, x \in \Lambda \setminus \Delta$. We shall check later, that the images $\theta(c_i) = \sigma_i, i = 0, 1, 2$ of the c-reflections under the canonical projection $\theta : \Lambda \to \Lambda/\Gamma$ are not conjugate in $\widetilde{G}$. Since all reflections in Λ are conjugate to one of the c-reflections and each symmetry σ in X with non-empty set of fixed points X_σ is the image $\sigma = \theta(c)$ of some reflection $c \in \Lambda$ (see [4]), classifying all symmetries σ with X_σ non-empty reduces to computing the species $\mathrm{sp}(X, \sigma_i), i = 0, 1, 2$. In this section we calculate $k(X, \sigma_i) = k_i$. If we represent the subgroup $\langle \sigma_i \rangle$ as Λ_i/Γ for some group $\Lambda_i \subset \Lambda$ it follows that k_i is the number of boundary components of the Klein surface $X/\langle \sigma_i \rangle = H/\Lambda_i$. In other words, k_i is the number of conjugacy classes of reflections in Λ_i. If C_i is the preimage under θ of the centralizer $C(\widetilde{G}, \sigma_i)$ of σ_i in $\widetilde{G}$, the set of all reflections of Λ_i is

$$R_i = \{c_i^w = wc_iw^{-1} \ : \ w \in C_i\}.$$

Now given $u, v \in C_i$, the reflections c_i^u and c_i^v are conjugate in Λ_i if and only if $u^{-1}\gamma v \in C(\Lambda, c_i)$ for some $\gamma \in \Lambda_i$. On the other hand, $u^{-1}\gamma v = (u^{-1}v)(v^{-1}\gamma v)$. So as C_i normalizes Λ_i, we see that $u^{-1}\gamma v \in C(\Lambda, c_i)$ if and only if $u^{-1}v \in C(\Lambda, c_i)\Lambda_i = C(\Lambda, c_i)\Gamma$. Hence we get

$$k_i = [C_i : C(\Lambda, c_i)\Gamma] = [C_i/\Gamma : C(\Lambda, c_i)\Gamma/\Gamma] = [C(\widetilde{G}, \sigma_i)) : \theta(C(\Lambda, c_i))]. \quad (4)$$

In particular, for $i = 0$ it is easy to prove that

$$C(\Lambda, c_0) = \langle c_0 \rangle \oplus \left(\langle (c_0c_1)^{k/2} \rangle * \langle (c_0c_2)^{m/2} \rangle \right) \quad (5)$$

and so

$$k_0 = \frac{\#C(\widetilde{G}, \sigma_0)}{4\#((\sigma_0\sigma_1)^{k/2}(\sigma_0\sigma_2)^{m/2})} \quad (6)$$

and similar formulae can be found for k_1 and k_2 (see [4] and [15]).

Let us attack the explicit computations in case $X = X_g$. Here we have

$$\theta : \Lambda \to \Lambda/\Gamma = \mathrm{Aut}^+(X) \times_\varphi Z_2 = \langle a, b \rangle \times_\varphi \langle t \rangle \quad (7)$$

defined by

$$\theta(c_0) = at, \theta(c_1) = t, \theta(c_2) = tb,$$

where φ is defined in (3).

To apply (6) we must check that $\sigma_0, \sigma_1, \sigma_2$ are not conjugate in $\widetilde{G}$. In fact, each element $x \in \widetilde{G}$ can be written in a unique way as $x = a^r b^s t^\mu$ with $0 \le r \le 2g+1, 0 \le s \le 3, 0 \le \mu \le 1$. Hence, $xtx^{-1} = a^{2r}b^{2s}t$ as b^2 commutes with a. So $\sigma_0 = at$ and $\sigma_1 = t$ are not conjugate. Analogously we check the other two cases. Let us calculate the order of $C(\widetilde{G},\sigma_0)$. The element $x\in\widetilde{G}$ above commutes with σ_0 if and only if either

$$a=a^r b^s a b^s a^r, \mu=0 \ \text{ or } \ a=a^r b^s a^{-1} b^s a^r, \mu=1.$$

In both cases we have eight solutions. In the first case:

$$s=0,2\,;\ r=0,g+1 \ \text{ and } \ s=1,3\,;\ r=1,g+2.$$

whilst in the the second one:

$$s=0,2\,;\ r=1,g+2 \ \text{ and } \ s=1,3\,;\ r=0,g+1.$$

Consequently, $\#C(\widetilde{G},\sigma_0)=16$. On the other hand, $(\sigma_0\sigma_1)^{k/2}(\sigma_0\sigma_2)^{m/2}=a^{g+2}b$ has order 2 for odd g and order 4 for even g. Hence from (6), $k_0=2$ for odd g and $k_0=1$ for even g. Similarly we prove that $k_1=1$ and $k_2=g+1$.

We still have to determine how many conjugacy classes of symmetries without fixed points exist. Such symmetries are elements of order 2, $y=a^r b^s t\in\widetilde{G}$ which are not conjugate to t, at, tb and it is easy to see that there is exactly one class of such elements for even g and two classes if g is odd.

Concluding, for odd g, X_g has five classes of symmetries: two without fixed points, and three $\sigma_0,\sigma_1,\sigma_2$ with2, 1 and $g+1$ ovals, respectively. For even g there are four conjugacy classes of symmetries: one without fixed points, two with one oval and one more with $g+1$ ovals.

The analysis of the symmetries of Y_g for $g\equiv 3 \bmod 4$ follows the same technical lines. There are three classes of symmetries $\sigma_0,\sigma_1,\sigma_2$ with fixed points having $2,1$ and $(g+1)/4$ ovals and if $g\equiv 3 \bmod 8$ there is exactly one class more without fixed points.

3. Separability character of the symmetries.

Let σ be a symmetry on $X=X_g$ or Y_g. If X_σ is empty then σ is non-separating. Thus, in what follows we are going to compute $\varepsilon(X,\sigma_i), i=0,1,2$ with the notation of the preceding sections. It follows from (1) that $\varepsilon=0$ if $k=g+1$, and $\varepsilon=1$ when $k=1$ and g is odd. Furthermore since $(g+1)/4$ is odd for $g\equiv 3 \bmod 8$, we conclude that $\varepsilon=1$ in the case $k=\frac{g+1}{4}$, $X=Y_g$, $g\equiv 3 \bmod 8$. Consequently

$$\mathrm{sp}(X,\sigma)=\begin{cases} +(g+1) & \text{if } (X,\sigma)=(X_g,\sigma_2),\\ -1 & \text{if } \sigma=\sigma_1 \text{ except if } X=X_g \text{ and } g \text{ is even},\\ -(g+1)/4 & \text{if } (X,\sigma)=(Y_g,\sigma_2) \text{ and } g\equiv 3 \bmod 8.\end{cases} \tag{8}$$

On the other hand, surfaces admitting a symmetry σ_2 with species $+(g+1)$ have been studied by Natanzon in [14] and by Bujalance and Costa in [3] and it follows directly from Theorem 3.3 in the latter paper that:

$$\mathrm{sp}(X_g,\sigma_0) = +2 \ \text{ if } g \text{ is odd.} \tag{9}$$

To calculate $\varepsilon(X,\sigma)$ in the remaining cases we use two distinct strategies. First we prove:

$$\mathrm{sp}(X,\sigma) = \begin{cases} -1 & \text{if } (X,\sigma) = (X_g,\sigma_1) \text{ and } g \text{ is even,} \\ -(g+1)/4 & \text{if } (X,\sigma) = (Y_g,\sigma_2) \text{ and } g \equiv 7 \bmod 8. \end{cases} \tag{10}$$

For this purpose we need the following result which combines Theorem 1 in [11] and Lemma 2.8 in [2]:

Proposition 3.1 *Let $G = \mathrm{Aut}^+(X), X = X_g$ or Y_g, with the presentation of the introduction . Let σ, τ be two commuting symmetries on X. Let us define for each $x \in G$ the integer δ_x to be either 1 if $h = \sigma\tau$ is conjugate in G to some power of x and 0 otherwise. Then, if σ is separating, the following inequality holds:*

$$\#C(G,h)\left(\frac{\delta_a}{2g+2} + \frac{\delta_b}{4} + \frac{\delta_{ab}}{2}\right) \leq 2k(X,\sigma) \tag{11}$$

Now if we apply this with $X = X_g$, $\sigma = \sigma_1, \tau = \sigma_2\sigma_1\sigma_2^{-1}$, then $h = b^2$ has order 2, $\delta_a = \delta_{ab} = 0, \delta_b = 1$ and $\#C(G,h) = 8(g+1)$. Hence, if σ_1 is separating we get from (11) $g + 1 \leq k(X_g,\sigma_1) \leq 1$, a contradiction. Analogously, if $X = Y_g, \sigma = \sigma_2$, and $\tau = \sigma_1\sigma_2\sigma_1^{-1}$ we get again $h = b^2$ has order 2, $\delta_a = \delta_{ab} = 0, \delta_b = 1$ and now $\#C(G,h) = 4(g+1)$. Thus if σ_2 is separating, $(g+1)/4 = k(Y_g,\sigma_2) \geq (g+1)/2$, absurd. This proves (10).

To finish we prove the following

$$\mathrm{sp}(X_g,\sigma_0) = +1 \text{ if } g \text{ is even ; } \mathrm{sp}(Y_g,\sigma_0) = -2. \tag{12}$$

In this case we apply a different approach due to Hoare and Singerman [6]. Let X be either X_g or Y_g and with the notation in Section 2 let Γ be a Fuchsian surface group uniformizing X. Let us consider the subgroup $K = \langle\sigma_0,\sigma_1\rangle$ of $\tilde{G} = \mathrm{Aut}^{\pm}(X)$, which is isomorphic to the dihedral group D_{2g+2} of $4g+4$ elements. Then K can be written as Γ'/Γ for some NEC-group Γ' whose signature

$$\sigma(\Gamma') = (0; +; [-]; \{(2g+2, 2g+2, g+1)\})$$

can be easily computed by the Riemann-Hurwitz formula (see e.g [13] pg 438). Let ρ_0, ρ_1, ρ_2 be a set of canonical reflections generating Γ' such that the natural projection $\Phi : \Gamma' \to K$ maps $\rho_j \mapsto \sigma_j, j = 0, 1$. It is easy to see that

$$\Phi(\rho_2) = \begin{cases} \sigma_0(\sigma_1\sigma_0)^{2g} & \text{if } \ X = X_g, \\ \sigma_0(\sigma_1\sigma_0)^{g-1} & \text{if } \ X = Y_g. \end{cases}$$

Following the quoted result in [6], the symmetry σ_0 is separating if and only if the Schreier graph $\mathcal{S}$ of the set of cosets $K/\langle\sigma_0\rangle$ corresponding to the system of generators $\{\Phi(\rho_j) \ : \ j = 0, 1, 2\}$ is bipartite. Assume now $X = X_g$. The $2g+2$ classes in $K/\langle\sigma_0\rangle$ are denoted by $K/\langle\sigma_0\rangle = \{[i] \ : \ 0 \leq i \leq 2g+1\}$, where $[i] = \{(\sigma_0\sigma_1)^i, (\sigma_0\sigma_1)^i\sigma_0\}$. To produce the graph $\mathcal{S}$ it is necessary to know how each $\Phi(\rho_j), j = 0, 1, 2$, links two vertices of $K/\langle\sigma_0\rangle$. But it is easy to check that the action of $\Phi(\rho_j)$ on the vertices is given by

$$\Phi(\rho_j)([i]) = [2g+2-i-j], 0 \leq i \leq 2g+1, 0 \leq j \leq 2$$

Then the Schreier graph $\mathcal{S}$ (with the loops deleted) admits a bipartition

$$V_1 = \{[i] \ : \ 0 \leq i \leq g\} \ ; \ V_2 = \{[i] \ : \ g+1 \leq i \leq 2g+1\}$$

of the set of vertices of $\mathcal{S}$, and so, σ_0 is separating in this case. In a similar way it is proved that σ_0 is not separating if $X = Y_g$. We summarize our results in the following Theorem:

Theorem 3.2 *The symmetry types of the Accola-Maclachlan surface and the Kulkarni surface are as in the following Table:*

X	g	σ_0	σ_1	σ_2
X_g	odd	+2	-1	$+(g+1)$
X_g	even	+1	-1	$+(g+1)$
Y_g	$\equiv$ 7 mod 8	-2	-1	$-(g+1)/4$
Y_g	$\equiv$ 3 mod 8	-2	-1	$-(g+1)/4$

4. Real plane curves representing the real forms.

Let $N = g+1$ and consider the complex plane curve:

$$C = \{(z, w) \in \mathbf{C}^2 : w^2 = z^{2N} - 1\}.$$

The desingularized projective complexification of this curve is the Accola-Maclachlan surface X_g. As noted previously there are at most three conjugacy classes of symmetries with ovals and in particular for both X_g and Y_g there are exactly three classes. Thus there are exactly three inequivalent real forms corresponding to non-empty curves. To find real plane curves representing the real forms of X_g, we shall find three polynomials $Q_i \in \mathbf{R}[z,w], i = 1,2,3$ such that if

$$C_i = \{(z,w) \in \mathbf{C}^2 : Q_i(z,w) = 0\},$$

with real part

$$C_i(\mathbf{R}) = \{(z,w) \in \mathbf{R}^2 : Q_i(z,w) = 0\},$$

then the following are true:

1. Each C_i is $\mathbf{C}$-birationally equivalent to C.
2. The real parts $C_i(\mathbf{R})$ and $C_j(\mathbf{R})$ are not $\mathbf{R}$-birationally equivalent if $i \neq j$.

Let us denote by X_i the projective smooth (real) model of $C_i(\mathbf{R})$, i.e, a smooth, projective real algebraic curve, $\mathbf{R}$-birationally equivalent to $C_i(\mathbf{R})$. In each case we shall see if X_i disconnects its complexification $\tilde{X}_i$ and count the number of components of X_i. (Note we are also assuming that $\tilde{X}_i$ is also a projective, smooth complex curve). We shall compare the results to our previous calculations. Let $r_0(X)$ denote the number of topological components of a real curve X.

Case 4.1: X_1 with 1 oval for N even, 2 ovals for N odd.

This is the simplest case since it suffices to take

$$Q_1 = w^2 - (z^{2N} - 1).$$

To count the number of ovals of X_1 one checks that $P = (0:1:0)$ is the only point at infinity of C_1. If we set $w = 1$ in the homogeneous equation $w^2t^{2N-2} = z^{2N} - t^{2N}$, we must study the local behaviour at $(z,t) = (0,0)$ of

$$t^{2N-2} = z^{2N} - t^{2N}.$$

From Newton-Puiseux we have the branches $t^{N-1} = \pm z^N$, and from this it follows that X_1 has the required number of ovals.

Case 4.2: X_2 with 1 oval for all N. Choose

$$Q_2 = w^2 + (-1)^N z^{2N} - 1.$$

The map

$$\alpha : C \to C_2, \ (z,w) \to (iz, iw)$$

is a birational isomorphism and X_2 is connected.

Case 4.3: C_3 with $r_0(C_3) = N$

Let $\zeta = e^{2\pi i/2N}$ and $\phi : \mathbf{P}_1(\mathbf{C}) \to \mathbf{P}_1(\mathbf{C})$ the map $z \to \dfrac{z-\zeta}{1-\zeta z}$ with inverse $\psi : \mathbf{P}_1(\mathbf{C}) \to \mathbf{P}_1(\mathbf{C})$: $z \to \dfrac{z+\zeta}{1+\zeta z}$. Now each $a_k = \phi(\zeta^k)$ lies in $\mathbf{R} \cup \infty$, $k = 0, \ldots, 2N-1$. In fact, ζ^k lies on the unit circle and for any point w on the unit circle $\bar{w} = w^{-1}$ and hence

$$\overline{\phi(w)} = \frac{\bar{w} - \bar{\zeta}}{1 - \bar{w}\bar{\zeta}} = \frac{w^{-1} - \zeta^{-1}}{1 - w^{-1}\zeta^{-1}} = \frac{w - \zeta}{1 - w\zeta} = \phi(w).$$

Let $P(z) = z^{2N} - 1$, $Q(z) = P(\psi(z))$. Clearly $Q(a_k) = P(\zeta^k) = 0$, and $a_0, \ldots, a_{2N-2}$ are real, whereas $a_{2N-1} = \infty$. Also a_k's are all distinct since ϕ is injective. Now,

$$Q(z) = \Big(\frac{z+\zeta}{1+\zeta z}\Big)^{2N} - 1 = \frac{(z+\zeta)^{2N} - (1+\zeta z)^{2N}}{(1+\zeta z)^{2N}} = \frac{R(z)}{(1+\zeta z)^{2N}},$$

where $R(z) = (z+\zeta)^{2N} - (1+\zeta z)^{2N} \in \mathbf{C}[z]$ is a polynomial of degree $2N-1$. Since the finite zeros of Q, $\{a_0, \ldots, a_{2N-2}\}$, are also roots of R and R has degree $2N-1$, we may factor R:

$$R(z) = \rho^2(z - a_0) \cdots (z - a_{2N-2}),$$

for some $\rho \in \mathbf{C}^*$. Thus the polynomial we seek is

$$Q_3(z,w) = w^2 - (z - a_0) \cdots (z - a_{2N-2}) \in \mathbf{R}[z,w].$$

The map $\alpha : C \to C_3, \ (z,w) \to (\phi(z), \rho^{-1}w(1+\zeta z)^N)$ is a rational mapping for, if $(z,w) \in C$, then upon setting $\phi(z) = u$, i.e., $z = \psi(u)$, we obtain

$$w^2 = z^{2N} - 1 = \psi(u)^{2N} - 1 = P(\psi(u)) = Q(u)$$

$$\frac{R(u)}{(1+\zeta u)^{2N}} = \frac{\rho^2(u - a_0) \cdots (u - a_{2N-2})}{(1+\zeta u)^{2N}},$$

i.e.,

$$(u - a_0) \cdots (u - a_{2N-2}) = [\rho^{-1}(1+\zeta z)^N w]^{\,2}.$$

In fact α is an isomorphism with inverse $\alpha^{-1} : C_3 \to C, \ (u,v) \to (\psi(u), \frac{\rho v}{(1+\zeta u)^N})$. It must be checked that X_3 has N connected components, but this is obvious from the graph of $C_3(\mathbf{R})$.

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