Rotationally Symmetric Rose Links

Amelia Brown
Simpson College, amebrn@yahoo.com

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Amelia Brown

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\textsuperscript{a}Simpson College
Abstract. This paper is an introduction to rose links and some of their properties. We used a series of invariants to distinguish some rose links that are rotationally symmetric. We were able to distinguish all 3-component rose links and narrow the bounds on possible distinct 4 and 5-component rose links to between 2 and 8, and 2 and 16, respectively. An algorithm for drawing rose links and a table of rose links with up to five components are included.

Acknowledgements: The initial cursory observations on rose links were made during the 2011 Dr. Albert H. and Greta A. Bryan Summer Research Program at Simpson College in conjunction with fellow students Michael Comer and Jes Toyne, with Dr. William Schellhorn serving as research advisor. The research presented in this paper, including developing the definitions and related concepts about rose links and applying invariants, was conducted as a part of the author’s senior research project at Simpson College, again under the advisement of Dr. Schellhorn. The author would like to thank Dr. Schellhorn for his help and support, especially for his assistance with the definitions involving polygonal diagrams.
Figure 1: The unknot is the simplest knot.

1 Introduction

The study of knots began in the 1880s when chemists believed that the universe was pervaded by a substance called ether. It was hypothesized that different types of matter were composed of atoms that took the form of knots in the ether. Different knots would then correspond to different types of matter [1]. Lord Kelvin and others began attempting to tabulate knots, forming what they believed would be a table of elements. By the late nineteenth century it had been shown that ether did not exist and scientists abandoned the study of knots. By that time mathematicians had taken an interest in the subject and knot theory was born.

Knot theory is a subfield of topology, which is the study of properties of geometric objects that are preserved under deformations [1]. Informally, a knot can be thought of as a knotted rope with the ends joined together. The ends must be joined so that the knot cannot “escape” and become unknotted. More formally, a knot is a closed curve in 3-dimensional space that does not intersect itself [1]. A 3-dimensional knot can be represented by 2-dimensional diagrams, called knot diagrams when crossing arrangements are depicted and knot projections when crossing arrangements are not depicted. The most basic knot is called the trivial knot or unknot, which is shown in Figure 1.

A link is a disjoint union of a finite number of knots, where each knot is called a component of the link [2]. As with knots, links are 3-dimensional and can be represented by 2-dimensional diagrams, called link diagrams when crossing arrangements are depicted and link projections when crossing arrangements are not depicted. Note that one component links are knots.

While there are many distinct knots and links, rose links are a special class of links comprised solely of unknotted components. Two links are distinct if no diagram of one can be transformed into a diagram of the other using a sequence of Reidemeister moves and planar isotopies. Due to the difficulty in proving that two links are the same, invariants are used to show that two links are distinct. Using invariants we were able to distinguish all 3-component rotationally symmetric rose links and narrow the bounds on possible distinct 4 and 5-component rotationally symmetric rose links to between 2 and 8, and 2 and 16, respectively. Reidemeister moves and link invariants are discussed in Section 3. Section 2 includes the definition of rose links and a drawing algorithm for obtaining rose projections. The linking number and the pairwise linking number sum invariants are discussed in Section 4. The HOMFLY polynomial invariant is introduced in Section 5. Topics for future research are outlined in Section 6. An appendix depicting the 3, 4, and 5-component rotationally
Figure 2: Rose projections with 2, 3, 4, and 5 components.

Figure 3: From left to right, a 3-sided regular polygon ($P_0$), an inscribed 3-sided regular polygon ($P_1$), and a second inscribed 3-sided regular polygon ($P_2$).

symmetric rose links is also included.

2 Rose Projections and Rose Links

An $n$-component rose link is a link comprised of $n$ unknotted components that can be arranged to yield an $n$-component rose projection. In other words, an $n$-component rose link has a link diagram that becomes an $n$-component rose projection when its crossing arrangements are not depicted. Examples of rose projections are shown in Figure 2.

Polygonal diagrams depict link diagrams and projections using line segments instead of smooth curves. There is a simple algorithm for constructing a polygonal diagram for the $n$-component rose projection when $n \geq 3$. Begin by drawing an $n$-sided regular polygon, call it $P_0$. Next inscribe a second $n$-sided regular polygon, $P_1$, inside the first, such that the vertices of $P_1$ lie at the midpoints of $P_0$. Continue to inscribe $n$-sided regular polygons until there are $n$ total polygons, $P_0$ through $P_{n-1}$. Figure 3 illustrates the drawing algorithm for the 3-component rose projection.

The polygonal diagram for an $n$-component rose projection can be easily converted to a “smooth” link projection by rounding out the edges as shown in Figure 4. The outer vertices of the polygonal diagram are the vertices of the outermost regular polygon, $P_0$, and the inner vertices are the vertices of the inscribed polygons $P_1$ through $P_{n-1}$, as shown in Figure 5. The $n$ outer vertices do not correspond to intersections in the resulting rose projection,
Figure 4: Conversion from a polygonal diagram to a smooth diagram for a 3-component rose projection.

Figure 5: Outer vertices are located on the outermost polygon, while inner vertices occur where inscribed polygons meet.

but intersections do occur where the \( n \cdot (n - 1) \) inner vertices were located in the polygonal diagram.

Rose projections have interesting symmetry. Given the polygonal diagram for the \( n \)-component rose projection, let \( C \) be the point that is the common center of the \( n \) polygons \( P_0 \) through \( P_{n-1} \). Let ray \( R_0 \) be the ray emanating from \( C \) that passes through a chosen outer vertex. Let ray \( R_i \) for \( i \in \{1, 2, \ldots, 2n-1\} \) be the ray emanating from \( C \) that is rotated \( i \cdot \frac{360}{2n} = i \cdot \frac{180}{n} \) degrees from \( R_0 \) in some direction, say counterclockwise. Figures 6 and 7 illustrate these rays in the 3 and 4-component cases, respectively, for a chosen outer vertex. Notice that the rays \( R_0 \) through \( R_{n-1} \) alternate passing through the vertices of \( P_0 \) and the midpoints of the sides of \( P_0 \) (the vertices of \( P_1 \)) in a counterclockwise order. The polygons and rays are constructed so that the rays with even indices pass through the vertices of the polygons with even indices and the rays with odd indices pass through the vertices of the polygons with odd indices. The ray \( R_i \) for \( i \in \{0, 1, \ldots, n - 1\} \) is opposite to the ray \( R_{i+n} \) because \( R_{i+n} \) lies \( n \cdot \frac{180}{n} = 180 \) degrees from \( R_i \). These \( n \) pairs of rays form the \( n \) axes of symmetry for the rose projection, as shown in Figure 8 for the 3 and 4-component rose projections.

The ray \( R_0 \) can be associated with a component of the rose projection in the following manner. In the polygonal diagram, the component is the \((2n-1)\)-sided polygon with vertices \( R_i \cap P_i \) for \( i \in \{0, 1, \ldots, n - 1\} \) and \( R_i \cap P_{2n-i} \) for \( i \in \{n + 1, n + 2, \ldots, 2n - 1\} \). The
Figure 6: Rays of the 3-component polygonal diagram.

Figure 7: Rays of the 4-component polygonal diagram.

Figure 8: The axes of symmetry of the 3 and 4-component rose projections.
component associated with $R_0$ in Figures 6 and 7 are the polygons with sides that are darkened for emphasis. Notice that the line of symmetry determined by $R_0$ bisects its associated component, with $n - 1$ inner vertices on each side of the axis in the polygonal diagram and therefore $n - 1$ intersections on each side of the axis in the “smooth” rose projection. Recall that there are $n$ components in the rose projection, and each one is associated with a choice for the ray $R_0$ (or alternatively, with each outer vertex of the polygonal diagram).

Several properties of the components of a rose projection will be important when we discuss invariants for rose links. Given the polygonal diagram for the $n$-component rose projection, choose an outer vertex for the ray $R_0$ to pass through. When $n$ is odd, the opposite ray $R_n$ does not pass through an outer vertex because the regular polygon $P_0$ has an odd number of sides. Therefore each of the $n$ lines of symmetry bisects a single component of such a rose projection (refer to Figures 6 and 8 for the $n = 3$ case). Moreover, when $n$ is odd, any two components in the rose projection intersect in exactly two points: at the vertices of two polygons $P_i$ and $P_j$ with $i + j = n$. When $n$ is even, the opposite ray $R_n$ passes through an outer vertex because the regular polygon $P_0$ has an even number of sides. Therefore $\frac{n}{2}$ lines of symmetry bisect two components of such a rose projection and the remaining $\frac{n}{2}$ do not bisect any components (refer to Figures 7 and 8 for the $n = 4$ case). Moreover, when $n$ is even, any two components in the rose projection intersect in exactly two points in the same manner as the odd case.

**Theorem 2.1.** The $n$-component rose projection has $n(n - 1)$ intersections.

**Proof.** Applying the drawing algorithm for the $n$-component rose projection yields a polygonal diagram with $n$ total polygons. The outer vertices of the polygonal diagram do not correspond to intersections in the smooth diagram of the rose projection. The inner vertices, however, do correspond to intersections in the resulting smooth diagram. From the drawing algorithm, there are a total of $n - 1$ regular polygons inscribed in the outer regular polygon. Since inner vertices only occur where an inscribed polygon meets another, we are only concerned with the vertices of these $n - 1$ polygons. Each inscribed polygon has $n$ vertices. We can then conclude that there are $n(n - 1)$ intersections in the $n$-component rose projection. \[\square\]

**Corollary 2.2.** There is a maximum of $2^{n(n-1)}$ distinct rose links with $n$ components.

**Proof.** At each intersection of the $n$-component rose projection, there are two possible crossing arrangements, either over or under. It follows from Theorem 2.1, with two possible crossing arrangements at each intersection and $n(n - 1)$ total intersections, that there is a maximum of $2^{n(n-1)}$ distinct rose links with $n$ components. \[\square\]

It should be noted that while $2^{n(n-1)}$ is the maximum number of rose links with $n$ components, some rose link diagrams represent the same link. In other words, with allowable moves we could transform some link diagrams into others. The number $2^{n(n-1)}$ is an upper bound on the number of possible $n$-component rose links. In this paper we will focus only on rotationally symmetric rose links, which limits the number of rose links studied.
Definition. An $n$-component rose link diagram is rotationally symmetric if the pattern of over and under crossings is the same for every component, such that when the diagram is rotated $\frac{360}{n}$ degrees it looks the same. A rose link is called rotationally symmetric if it has a rotationally symmetric diagram.

A 3-component rotationally symmetric rose link diagram is shown in Figure 9. To classify the possible rose link diagrams with $n$ components that are rotationally symmetric, we need only focus on half the crossings on a given component. Each component has an axis of symmetry that bisects it. There are $n - 1$ crossings of the component on either side of the axis, each with two possible crossing arrangements, either over or under. The crossing arrangements on the component will be switched on the other side of the axis of symmetry, with over crossings becoming under crossings, and vice versa. When there is an odd number of components, all of the $n$ axes will be used in the process of creating a rotationally symmetric rose link diagram. When there is an even number of components, half of the axes bisect two components each, so only $\frac{n}{2}$ axes are used when creating a rotationally symmetric rose link diagram. Refer to Figure 8.

While there may be $2^{n(n-1)}$ distinct rose links with $n$ components, focusing only on half of the crossings on a given component yields $2^{n-1}$ possible rose links that are rotationally symmetric. Again this is an upper bound, so there are between 1 and $2^{n-1}$ possible rotationally symmetric rose links with $n$ components.

3 Link Invariants

We can narrow the range of possible rotationally symmetric rose links using link invariants. Link invariants are used to distinguish one link from another and must be preserved under the Reidemeister moves.

Definition. The Reidemeister moves and planar isotopies are the only moves on link diagrams that are needed to represent all the ways of deforming the link in 3 dimensions while preserving the link type [8]. The Reidemeister moves represent changes of subdiagrams in a larger link diagram. There are two versions of each move. The first move adds or removes a twist, either adding or removing a crossing in the link diagram; the Reidemeister I move is depicted in Figure 10. The second move adds or removes two crossings; the Reidemeister II
move is depicted in Figure 11. The Reidemeister III move slides a strand from one side of a crossing to the other, as shown in Figure 12. A link can be considered equivalent to any of its diagrams up to Reidemeister moves and planar isotopies. Therefore we will not emphasize the distinction between a 3-dimensional link and its 2-dimensional diagrams throughout the rest of this paper.

A basic invariant applied to rotationally symmetric rose links is the number of components in the link, which is preserved under the Reidemeister moves [6]. For example, this invariant distinguishes the 2-component rose links from all other \( n \)-component links for \( n > 2 \). The only links that yield 2-component rose projections are known as the Hopf link and the 2-component unlink (shown in Figure 13), which can be distinguished from one another using the linking number invariant discussed in Section 4. This unlink will be discussed again in Section 5 on the HOMFLY polynomial. For the main work of this paper, we focused our attention on 3, 4, and 5-component rose links. While the number of components is a useful invariant, to distinguish a given 3-component rose link from another 3-component rose link we have to introduce stronger invariants. Before we introduce such invariants, we will first introduce a naming convention for the rotationally symmetric rose links for ease of reference.

As defined earlier, a rose link diagram is rotationally symmetric when the crossing arrangements are the same on each component. This leads to a method of naming the rose
link diagrams based on their crossing arrangements. We focused only on the crossings on one side of a component to determine the possible crossing arrangements of the $2^{n-1}$ rotationally symmetric rose link diagrams. The naming convention places the crossing arrangement patterns into alphabetical order and then assigns them a number from 1 to $2^{n-2}$. Here we divide $2^{n-1}$ by two to account for the mirror images of the link diagrams, where all crossings are changed from over to under and vice versa. Mirror images are assigned the same number but marked with an asterisk to denote that it is a mirror image. We use $R$ to indicate that the diagram is a rotationally symmetric rose link with a superscript indicating the number of components in the link and a subscript which comes from the alphabetical numbering system previously described. We will use the 3-component rotationally symmetric rose links as an example to illustrate the naming convention.

There are four 3-component rotationally symmetric rose link diagrams. With respect to any given component, the crossing arrangements for their diagrams are shown in Table 1. We use U to denote an under crossing and O to denote an over crossing, as shown in Figure 14. Notice that the crossing arrangements are listed in alphabetical order. We then assign the numbers 1 and 2 to the first two arrangements. The remaining two arrangements are the mirror images of the first two, which is indicated by the asterisk. Thus the 3-component rotationally symmetric rose link diagrams are denoted by $R_1^3, R_1^{3*}, R_2^3$, and $R_2^{3*}$.

### 4 Pairwise Linking Number Sum

The pairwise linking number sum is an invariant that builds on the linking number invariant. The linking number of a 2-component link is a measure of how many times one component
wraps around the other. Each component is assigned an orientation, which is indicated by an arrow on the diagram. We then focus on the crossings between the components. Crossings between a component and itself do not contribute to the linking number. There are two possible crossing relationships when the components are oriented, $+1$ and $-1$, which are shown in Figure 15. Between the components the crossing values are added together and the sum is then divided by two. Reversing the orientation of one component will negate the linking number, but the absolute value of the linking number remains unchanged. A proof of the following theorem appears in [1].

**Theorem 4.1.** The absolute value of the linking number is a 2-component link invariant.

We can extend the idea of the linking number to rose links by using a pairwise linking number sum (PLNS). Under this extended invariant, each component is given an orientation and the components are labeled. We then consider the absolute value of the linking number for each pair of components in the link. Recall from the definition of a rose link projection that two components intersect in exactly two points. If one component of a rose link is non-split linked to another, the crossing relationships at the two corresponding crossings can only be either both $+1$ or $-1$. Since both have the same crossing relationship, adding them together and dividing by two will yield either $+1$ or $-1$. Since we are only concerned with whether or not the pair is linked, we take the absolute value. Therefore the pairwise linking number for any two non-split linked components will always be 1 in the case of rose links. If two components are not linked, the two points of intersection will have negating crossing relationships, yielding a pairwise linking number of 0. We then sum the absolute value of the linking numbers to get the PLNS value.

**Corollary 4.2.** The PLNS is a rose link invariant.
Figure 16: An oriented $R_1^{3^*}$ link.

Table 2: Linking numbers for the oriented $R_1^{3^*}$ link; this link has pairwise linking number sum 3.

<table>
<thead>
<tr>
<th>Components</th>
<th>Pairwise Linking Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>A,B</td>
<td>1</td>
</tr>
<tr>
<td>A,C</td>
<td>1</td>
</tr>
<tr>
<td>B,C</td>
<td>1</td>
</tr>
</tbody>
</table>

Proof. The absolute value of the linking number is a 2-component link invariant by Theorem 4.1. This implies that the absolute value of the pairwise linking number for any two components of a rose link is preserved by the Reidemeister moves, so the sum of all the pairwise linking numbers must also be preserved. Thus the PLNS is a rose link invariant.

Considering the 3-component rotationally symmetric rose links, there are three components and we are focusing on the relationships between two components at a time, for a total of $3C_2 = 3$ pairs of components. In general there would be $nC_2$ for $n$-component rotationally symmetric rose links, where $nC_2$ denotes the number of combinations of two objects selected from $n$ different objects. Figure 16 depicts an oriented and labeled $R_1^{3^*}$ link. Table 2 is a summary of the results for the chosen orientation and labeling in Figure 16. We first look at the relationship between the A and B components. Oriented as they are, there are two +1 crossings. Adding them together and dividing by two yields +1, which is the linking number between A and B. The linking number of +1 tells us that the two components are linked and the A,B combination has a pairwise linking number of 1. The same relationship holds for the other two combinations. The PLNS of $R_1^{3^*}$ is the sum of these pairwise linking numbers, which is 3. The $R_1^{3}$ link also has a PLNS of 3 because when oriented in a similar manner all pairs of components have linking number $-1$.

Three is the maximum possible PLNS for any 3-component rose link and indicates that each component is linked to every other component. In general, the maximum value of the PLNS invariant will be $nC_2$ for $n$-component rose links. A PLNS of 0 in the case of rotationally symmetric rose links indicates that none of the components are linked pairwise, but they still may be intertwined in such a way that the entire link cannot be unlinked. For example, links $R_2^3$ and $R_2^{3^*}$ both have a PLNS of 0. The link $R_2^{3^*}$ is also known as
the Borromean rings, which have the property that if any one component is removed, the remaining components form an unlink with two components, as is illustrated in Figure 17. The link $R^3_2$ is the mirror image of the Borromean rings and has the same property. Since the PLNS values of $R^3_1$ and $R^3_1^*$ differ from those of $R^3_2$ and $R^3_2^*$, we can conclude that there are at least two distinct 3-component rotationally symmetric rose links. Thus, by way of the PLNS invariant, we now have that there are between two and four distinct 3-component rotationally symmetric rose links, because $2^{3-1} = 4$ is the maximum possible number of 3-component rotationally symmetric rose links.

In the 4-component case, there are a total of $2^3 = 8$ possible rotationally symmetric rose links. Refer to the appendix for diagrams of these links. For each link, there are $4C_2 = 6$ pairs of components. The PLNS values for the 4-component rose links are listed in Table 3. The PLNS only takes two values, 2 and 6, when applied to the 4-component rose links. Six is the maximum value for the PLNS in the 4-component case, which indicates all four components are linked pairwise, and there are four rose links that have this property. The value of two indicates that there are two pairs of linked components, say component A is linked to component C and component B is linked to component D. Because the link is rotationally symmetric, the pairs of linked components must be located directly across from one another in the rose projection. The case where a component is linked to a neighboring component and no others is not possible because the link would fail to be rotationally symmetric. When applied to the 4-component rotationally symmetric rose links, the PLNS yields that there are between 2 and 8 distinct links.

In the 5-component case, there are $2^4 = 16$ possible rotationally symmetric rose links, with $5C_2 = 10$ pairs of components. The diagrams of the 5-component rotationally symmetric rose links, excluding mirror images, are included in the appendix. The PLNS values for the 5-component links are listed in Table 4. In the 5-component case there are three possible values for the PLNS: 0, 5, and 10. Ten, being the maximum, again indicates that all components are linked pairwise and there are four rose links that have this property. With a PLNS value of 0, there are four rose links in which none of the components are linked pairwise, but all happen to be intertwined in such a way that it appears they cannot be unlinked, though we do not have proof of this observation. The PLNS value of 5 is interesting in that a given component is linked with the two components across from it in the rose projection, but not to the two components neighboring it. Since there are three distinct values of the PLNS when applied to the 5-component rotationally symmetric rose links, it follows that there are
Table 3: Pairwise linking number sums for the 4-component rotationally symmetric rose links.

<table>
<thead>
<tr>
<th>Rose Links</th>
<th>PLNS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1^4$, $R_3^4$, $R_4^4$, $R_1^3$</td>
<td>6</td>
</tr>
<tr>
<td>$R_2^4$, $R_1^4$, $R_2^3$, $R_2^2$</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 4: Pairwise linking number sums for the 5-component rotationally symmetric rose links.

<table>
<thead>
<tr>
<th>Rose Links</th>
<th>PLNS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1^5$, $R_7^5$, $R_2^5$, $R_1^4$</td>
<td>10</td>
</tr>
<tr>
<td>$R_2^5$, $R_3^5$, $R_5^5$, $R_8^5$, $R_2^3$, $R_2^2$, $R_2^4$</td>
<td>5</td>
</tr>
<tr>
<td>$R_4^5$, $R_6^5$, $R_4^3$, $R_6^4$</td>
<td>0</td>
</tr>
</tbody>
</table>

between 3 and 16 such links.

To summarize, when the pairwise linking number sum invariant was applied to the 3, 4, and 5-component rotationally symmetric rose links, we narrowed the bounds of distinct links to between 2 and 4, between 2 and 8, and between 3 and 16, respectively.

5 The HOMFLY Polynomial

The HOMFLY polynomial was another link invariant we applied to the rotationally symmetric rose links. The polynomial was discovered around 1984 by Hoste, Ocneu, Millett, Freyd, Lickorish, and Yetter; the first letters of each of their last names make up the name HOMFLY. Two other mathematicians, Przytycki and Traczyk, also discovered the polynomial, but unfortunately their work arrived later than the others. They are occasionally credited for their work with the letters PT being added to the end of HOMFLY [1]. The original announcement and proof that the HOMFLY polynomial is a link invariant can be found in [5] and [7], respectively. The HOMFLY polynomial is a generalization of the Jones and Alexander polynomials, both of which are carried in the HOMFLY polynomial and can be found through simple substitutions [2]. The HOMFLY polynomial, $P$, is a two-variable polynomial defined as follows. First, the polynomial of the unknot is 1, that is $P(\text{unknot}) = 1$. Second, the skein relationship

$$lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0.$$ 

holds, where $L_+$, $L_-$, and $L_0$ correspond to the skein relationship diagrams shown in Figure 18 and $l$ and $m$ are variables.

The process of calculating the HOMFLY polynomial requires orienting each of the components of a link. We then focus our attention on a single crossing and determine if it is an $L_+$, $L_-$, or $L_0$ crossing. The entire link diagram is preserved except for the selected crossing. The goal is for two simpler links to result when the selected crossing is changed to
Figure 18: The skein relationship diagrams for the HOMFLY polynomial are, from left to right, \( L_+ \), \( L_- \), and \( L_0 \).

the remaining two crossings. The process of picking a crossing and changing it to the other two crossings creates a resolving tree with the entire link eventually simplifying to unknots. The HOMFLY polynomial \( P \) comes from solving the skein relationship for each particular crossing and multiplying each branch by its respective coefficients.

We will illustrate the process using a Hopf link. First we will solve the skein relationship for each of the polynomials \( P(L_+) \), \( P(L_-) \), and \( P(L_0) \), with the results being:

\[
\begin{align*}
P(L_+) &= -l^{-2}P(L_-) - l^{-1}mP(L_0) \\
P(L_-) &= -l^2P(L_+) - lmP(L_0) \\
P(L_0) &= -lm^{-1}P(L_+) - l^{-1}m^{-1}P(L_-).
\end{align*}
\]

The link in Figure 19 is an oriented Hopf link; notice that it is also the 2-component rotationally symmetric rose link \( R_1^2 \). We will focus on the top crossing, which is an \( L_- \) crossing, to begin calculating the HOMFLY polynomial for this link. The resolving tree starts with the creation of an \( L_+ \) and \( L_0 \) crossing as shown in Figure 20. Each branch in the resolving tree is multiplied by the coefficients from the corresponding skein relationship. That is, solving the skein relationship for \( P(L_-) \), the original crossing, results in coefficients of \(-l^2\) on the \( L_+ \) crossing and \(-lm\) on the \( L_0 \) crossing. Thus we multiply every subsequent simpler link on the left branch resulting from the \( L_+ \) crossing by \(-lm^{-1}\) and every subsequent simpler link on the right branch resulting from the \( L_0 \) crossing by \(-lm\). Notice that the resulting link on the bottom right is an unknot, so we do not need to continue resolving on that branch. The resulting link on the left branch is a 2-component unlink. Since the two components are not linked to one another, we can manipulate the diagram with Reidemeister moves to obtain the top diagram in Figure 21. Note there is an \( L_0 \) crossing between the two components. Creating two new branches, multiply the left branch resulting from changing the crossing to an \( L_+ \) by \(-lm^{-1}\) and the right branch resulting from the \( L_- \) by \(-l^{-1}m^{-1}\). Notice in Figure 21 that the two resulting links are unknots, so the resolving tree for this oriented Hopf link is complete. The polynomial of this oriented Hopf link is therefore

\[
P(\text{Hopf link}) = -l^2P(\text{unlink}) - lmP(\text{unknot})
\]

\[
= -l^2(-lm^{-1}P(\text{unknot}) - l^{-1}m^{-1}P(\text{unknot})) - lm
\]

\[
= -l^2(-lm^{-1} - l^{-1}m^{-1}) - lm
\]

\[
= l^3m^{-1} + lm^{-1} - lm
\]
because $P(\text{unknot}) = 1$. In a more complicated link, every subsequent simpler link must be multiplied by the coefficient of the crossing at that branch and then added to the other branches.

A major issue arises when applying the HOMFLY polynomial to links with more than one component because the links must first be oriented to calculate the polynomial. There are two possible ways to orient each component in the link, which could result in different HOMFLY polynomials. Some of these polynomials may repeat, but to gain this knowledge we must calculate the polynomial for every possible orientation of the link. Thus for $n$-component links, we must calculate $2^n$ HOMFLY polynomials. One of the properties of the HOMFLY polynomial is that if we reverse the orientation on all of the components in the link, we will arrive at the exact same HOMFLY polynomial [1]. Based on this property, we can reduce the number of HOMFLY polynomials we need to calculate down to $2^{n-1}$.

In the case of the rotationally symmetric rose links, we were only able to calculate the HOMFLY polynomials for the 3-component links. Because there may be up to $2^{n-1}$ rotationally symmetric rose links, and for each we must calculate $2^{n-1}$ HOMFLY polynomials, the 4 and 5-component cases proved to be too time intensive.

Figure 22 depicts oriented versions of the links $R_3^1$ and $R_1^3$. Figure 23 depicts oriented versions of the links $R_2^3$ and $R_3^2$. Notice that in both cases the links are mirror images of each other, but we have maintained the same orientation on each. An additional property of the HOMFLY polynomial is that if we have the HOMFLY polynomial of one link and we substitute $l^{-1}$ for $l$, the result is the HOMFLY polynomial of the mirror image of the link...
Figure 21: The $L_0$ crossing resolved to an $L_+$ and $L_-$ crossing.

Figure 22: Oriented $R_1^3$ and $R_1^{3*}$ links.

Figure 23: Oriented $R_2^3$ and $R_2^{3*}$ links.
with the same orientation [9]. Below are the HOMFLY polynomials that we calculated for each of the four 3-component rotationally symmetric rose links in Figures 22 and 23:

\[
P(R_1^3) = 3l^6 + 3l^4 - l^8m^{-2} - 2l^6m^{-2} - l^4m^2 - l^6m^2 - 4l^4m^2 + l^4m^4
\]

\[
P(R_1^{3*}) = 3l^{-6} + 3l^{-4} - l^{-8}m^{-2} - 2l^{-6}m^{-2} - l^{-4}m^2 - l^{-6}m^2 - 4l^{-4}m^2 + l^{-4}m^4
\]

\[
P(R_2^3) = P(R_2^{3*}) = -l^2m^2 + l^2m^{-2} - l^{-2}m^2 + l^{-2}m^{-2} + m^2 - 2m^2 + m^{-2}
\]

Notice that in the case of \(R_3^3\) and \(R_1^{3*}\), the coefficients in each polynomial are the same, as are the exponents on the variable \(l\). The exponents on the variable \(m\) are negated from one to the other, as they should be according to the mirror image property of substituting \(l^{-1}\) for \(l\). In the case of \(R_2^3\) and \(R_2^{3*}\), the HOMFLY polynomial calculated for each was exactly the same. If we insert \(l^{-1}\) for \(l\) in this case, it will result in the same polynomial because the exponents on the \(l\) variable are “palindromic”. Based on these results we are led to suspect that \(R_3^3\) is equivalent to its mirror image \(R_2^{3*}\), and that \(R_2^3\) and \(R_3^{3*}\) are distinct from one another as well as from \(R_2^3\). However, we must consider the orientation of the components, and therefore based on this one set of results we cannot conclusively say that there are in fact three distinct 3-component rotationally symmetric rose links.

In [4], the \(R_1^3\) and \(R_2^3\) links were listed as six-crossing links and not in the context of rose links. Doll and Hoste calculated the various HOMFLY polynomials for each link under all of the possible orientations and showed that there are two distinct polynomials for the \(R_1^3\) link based on orientation and only one for \(R_2^3\) based on orientation [4]. With a series of substitutions we were able to confirm the above equations match the results of Doll and Hoste (their HOMFLY polynomials used different versions of the variables). Thus, we can conclude that \(R_2^3\) and its mirror image are distinct from \(R_1^3\) and its mirror image because the single HOMFLY polynomial for \(R_2^3\) and \(R_2^{3*}\) is different from each of the four possible HOMFLY polynomials for \(R_1^3\) and \(R_1^{3*}\). Additionally, if we insert \(l^{-1}\) for \(l\) in the polynomials, the resulting HOMFLY polynomial for \(R_3^3\) is distinct from its mirror image under both possible polynomials. The polynomial of \(R_2^3\) matched the polynomial of its mirror image, and we were able to find a sequence of Reidemeister moves from one to the other. Thus, we can conclude that \(R_2^3\) and \(R_2^{3*}\) are equivalent links. In summary, based on our results and those of Doll and Hoste, we can conclude that there are exactly three distinct 3-component rotationally symmetric rose links: \(R_1^3\), \(R_1^{3*}\), and \(R_2^3 = R_2^{3*}\).

6 Future Work

There is much more to be investigated regarding rose links. For example, further work could be attempted to calculate the various HOMFLY polynomials for the 4 and 5-component rotationally symmetric rose links. There are several other link invariants that may be useful in distinguishing these links.

We attempted to distinguish between rose links by applying mod \(p\) coloring invariants, associating a coloring matrix with each link diagram from which a determinant and \(p\) value
can be calculated. For more information about these coloring invariants, refer to [3]. However, in the existing literature about mod $p$ coloring invariants, they are rarely applied to links.

Based on our preliminary research we found that for the rotationally symmetric rose links, in the 3-component case the links $R^3_1$ and $R^{3*}_1$ both had determinant 4, and $R^3_2$ and $R^{3*}_2$ both had determinant 16. The prime factorization of the determinant yields a $p$ value of 2, therefore the mod $p$ coloring invariant did not distinguish any of the rotationally symmetric 3-component rose links from one another. In the 4-component case the resulting determinants were 0, 64, and 384. Looking at the prime factorizations of each, the links $R^4_2$, $R^{4*}_1$, and $R^{4*}_1$ were only colorable mod 2; the links $R^4_3$ and $R^{4*}_3$ were colorable mod 3; and the links $R^4_1$ and $R^{4*}_1$ yielded determinants of 0. Based on these results we may be able to narrow the bound of possible distinct rotationally symmetric 4-component links to between 3 and 8. Lastly, in the 5-component case the prime factorization of the determinants yielded $p$ values of 0, 2, 3, 11, and 19. All but $R^5_3$, which was the only link to have a determinant of 0, were able to be colored mod 2. The links $R^5_1, R^{5*}_1, R^5_2, R^{5*}_2, R^5_4, R^{5*}_4, R^5_8$, and $R^{5*}_8$ are only colorable mod 2; the links $R^5_3, R^{5*}_3, R^5_5$, and $R^{5*}_5$ were colorable mod 3; the links $R^5_6$ and $R^{5*}_6$ were colorable mod 11; and the links $R^5_7$ and $R^{5*}_7$ were colorable mod 19. The results of mod $p$ coloring invariants in the 5-component rotationally symmetric case may be able to narrow the bounds of distinct links to between 5 and 16.

Unfortunately we have yet to find a useful interpretation of 0 and 2 for the value of $p$ in the existing literature. Therefore, erring on the side of caution, we have excluded the results of the mod $p$ coloring invariant from our resultant bounds.

In studying the rotationally symmetric rose links, other interesting questions arose:

- Under the PLNS invariant, the value of 0 occurred only in the 3 and 5-component cases. Recall that the 0 value indicates that none of the components were linked pairwise with the others, but they appear to be intertwined in such a way that they could not be unlinked. We were unable to prove this observation, but is this something that only occurs in the case of an odd number of components?

- In both the 4 and 5-component cases, there were rotationally symmetric rose links in which each component was linked to all components across from it, but not to the
components neighboring it. Does this pattern continue as the number of components increases?

- For \( n \)-component rotationally symmetric rose links with \( 2 \leq n \leq 5 \), in the case of \( R_1^n \) and \( R_1^n^\ast \), the removal of one component yields an \((n - 1)\)-component rotationally symmetric rose link. An example is shown in Figure 24. In each case every component in the link was linked to every other component. Does this result hold for all links with 2 or more components?

In this paper we restricted our attention to rotationally symmetric rose links, but as stated previously, there are many links that give a rose projection but may not be rotationally symmetric. We could expand our attention to include all rose links, not just those that are rotationally symmetric.

References


Appendix

Figure 25: Rotationally symmetric rose link diagrams; note that for 4 and 5 components, some links depicted may be equivalent to one another.
Table 5: The labels in this table correspond to the location of the rose link in Figure 25. For instance, \( R_1^3 \) is the link depicted in the top left corner. Mirror images are not included since they are the same diagrams with all crossings changed.

| \( R_1^3 \) | \( R_2^3 \) | \( R_3^4 \) | \( R_4^1 \) | \( R_5^3 \) | \( R_6^3 \) | \( R_7^5 \) | \( R_8^0 \) |