Rose-Hulman Undergraduate Mathematics Journal

Volume 15
Issue 1

Article 14

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Rose-Hulman Undergraduate Mathematics Journal

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Volume 15, No. 1, Spring 2014

Sponsored by

Rose-Hulman Institute of Technology
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Rose-Hulman Undergraduate Mathematics Journal
Volume 15, No. 1, Spring 2014

The Volume of n-balls

Jake Gipple

Abstract. In this short paper, we compute the volume of n-dimensional balls in $\mathbb{R}^n$. The computations rely on techniques from multivariable calculus and a few properties of the gamma function.

Acknowledgements: I would like to thank my mentor, Professor Mike Munn, for his guidance throughout this project , as well as my family for their lasting support.

1 Introduction

For a natural number $n \geq 1$, an $(n-1)$-dimensional sphere of radius r is the set of all points in $\mathbb{R}^n$ which are a fixed distance r from a given center point. Taking the center point to be the origin, we denote by $\mathbb{S}^{n-1}(r)$ the $(n-1)$-sphere of radius r in $\mathbb{R}^n$; that is

$$\mathbb{S}^{n-1}(r) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = r^2\}.$$

When $n = 1$, the 0-sphere consists of just the two points on the real line $\mathbb{R}^1$ located at r and $-r$. For $n = 2$, $\mathbb{S}^1(r)$ is the subset of the plane given by

$$\mathbb{S}^1(r) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = r^2\}.$$

Graphically, $\mathbb{S}^1(r)$ is simply a circle of radius r centered at the origin. Note that the interior is not included. Taking $n = 3$, it follows that $\mathbb{S}^2(r)$ is a subset of $\mathbb{R}^3$ given by

$$\mathbb{S}^2(r) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = r^2\},$$

which describes a sphere of radius r centered at the origin, again not including the interior. In higher dimensions when $n \geq 4$, for example a 3-dimensional sphere, $\mathbb{S}^n(r)$ is more difficult to visualize. However, using our intuition of lower dimensional spheres described above, we can get some idea of a description for higher dimensional spheres.

Here is one way to help visualize the 3-sphere. If you take a 0-sphere, which is the endpoints of a line segment living in $\mathbb{R}^1$, and rotate it about it's center point (or the z-axis extending out into $\mathbb{R}^3$), you will sweep out a 1-sphere in $\mathbb{R}^2$, a circle. Similarly, if you take a 1-sphere, a circle in $\mathbb{R}^2$, and rotate every point about any axis going through its center point and lying in the plane of $\mathbb{R}^2$, you will sweep out a 2-sphere in $\mathbb{R}^3$. Finally, if you take a 2-sphere, a sphere in $\mathbb{R}^3$, and rotate every point about any axis going through the center point and lying in $\mathbb{R}^3$, you will sweep out a 3-dimensional sphere in $\mathbb{R}^4$. This is difficult to see, but we can carefully define these rotations using some reduction techniques and group actions of the special orthogonal group on subsets in Euclidean space.

Recall that orthogonal matrices represent linear transformations which preserve the dot product of vectors. They represent isometries of Euclidean space and denote rotations or reflections. We denote the group of orthogonal matrices in $\mathbb{R}^n$ by

$$O(n) = \{A : A^T A = I\}.$$

By definition, orthogonal matrices have determinant ± 1. The matrices in $O(n)$ with determinant $+1$ represent the rotations. These are called special orthogonal matrices and are denoted by

$$SO(n) = \{A : A^T A = I; \det A = 1\}.$$

Consider the following rotation given as a square matrix in $SO(n+1)$.

$$A_j = \begin{bmatrix} I_{j-1} & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & I_{n-j} \end{bmatrix} \quad \text{for } 1 \leq j \leq n,$$

where $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a 2×2 (counter-clockwise) rotation matrix, I_k is the $k \times k$ identity matrix, and j specifies where the rotation matrix is placed. Note that A_j rotates elements in the $x_j x_{j+1}$-plane but leaves all other dimensions fixed. For instance A_1 is the $(n+1) \times (n+1)$-matrix

$$A_1 = \begin{bmatrix} R & 0 \\ 0 & I_{n-1} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to see that for the general case A_j, the determinant will always be 1 and since $A_j^T = A_j^{-1}$, it is true that A_j is in the special orthogonal group.

To help us see how these matrices A_j generate spheres in $\mathbb{R}^{n+1}$, we look at the case with $n = 3$ to find a parameterization of a 3-sphere in $\mathbb{R}^4$. We will start with the point $P = (1, 0, 0, 0)$ in $\mathbb{R}^4$ and inductively apply our rotations. Applying the rotation A_1 to P for all values of θ in $0 \leq \theta < 2\pi$. We obtain

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = (\cos\theta, \sin\theta, 0, 0).$$

This gives a familiar parameterization of the circle $S^1 \subset \mathbb{R}^4$ lying in the x_1x_2-plane. We can then apply the rotation A_2 to our circle in the x_1x_2-plane to obtain a two-dimensional sphere living in $x_1x_2x_3$-space.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \\ 0 \end{bmatrix} = (\cos\theta, \cos\phi\sin\theta, \sin\phi\sin\theta, 0),$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Notice now that the parameterization resembles spherical coordinates. Continuing in the same manner, letting the new variable ψ range between 0 and 2π, and letting ϕ, θ range now from 0 to π, we arrive at a parameterization of the 3-sphere in $\mathbb{R}^4$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\psi & -\sin\psi \\ 0 & 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \cos\theta \\ \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ 0 \end{bmatrix} = (\cos\theta, \cos\phi\sin\theta, \cos\psi\sin\phi\sin\theta, \sin\psi\sin\phi\sin\theta).$$

Thus we can see that rotations in higher dimensions can be realized as the action of a linear transformation in which there is one free parameter. This parameter does a rotation in two dimensions and leaves all other dimensions fixed.

Continuing this construction in higher dimensions, we can produce a parametrization of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ given by

$$x_1 = \cos\theta_1 \tag{1}$$
$$x_2 = \sin\theta_1 \cos\theta_2 \tag{2}$$
$$x_3 = \sin\theta_1 \sin\theta_2 \cos\theta_3 \tag{3}$$
$$\vdots \tag{4}$$
$$x_{n-1} = \sin\theta_1 \cdots \sin\theta_{n-2} \cos\theta_{n-1} \tag{5}$$
$$x_n = \sin\theta_1 \cdots \sin\theta_{n-2} \sin\theta_{n-1}, \tag{6}$$

where $0 \leq \theta_{n-1} < 2\pi$ and $0 \leq \theta_i \leq \pi$, for $i = 1, 2, \ldots, n-2$. We will see later how these coordinates can be used to simplify our computations.

In this paper, we are concerned with the volume of subsets of $\mathbb{R}^n$ bounded by $\mathbb{S}^{n-1}(r)$, i.e. n-dimensional balls in $\mathbb{R}^n$. More precisely, define

$$\mathbb{B}^n(r) = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq r^2\}.$$

We will derive a well-known formula [1] to compute the volume of $\mathbb{B}^n(r)$ for any natural number n. To simplify our computations, we begin by computing the volume of a unit n-ball; i.e. $\mathbb{B}^n(1)$. Throughout this paper, we will denote $V(n) = \mathrm{Vol}\,(\mathbb{B}^n(1))$, the volume of the unit n-ball. We begin by proving some computational lemmas which will be useful later.

2 Lemmas

As we will see, the volume of n-balls is closely related to the gamma function. In this section we compute various quantities related to the gamma function which will aid our computations later in the paper.

Lemma 2.1. $\displaystyle\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

Proof. Setting $I = \displaystyle\int_0^\infty e^{-x^2} dx$, note that by Fubini's Theorem,

$$I^2 = \int_0^\infty e^{-x^2}\, dx \int_0^\infty e^{-y^2}\, dy = \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2}\, dxdy = \int_0^\infty \int_0^\infty e^{-x^2+-y^2}\, dxdy.$$

By converting to polar coordinates, taking $x = r\cos\theta$ and $y = r\sin\theta$, we have by Change of Variables with $0 \leq r < \infty$, and $0 \leq \theta \leq \dfrac{\pi}{2}$

$$I^2 = \int_0^{\frac{\pi}{2}} \int_0^\infty re^{-r^2} drd\theta.$$

Furthermore, using integration by substitution with $u = -r^2$, we have

$$I^2 = \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{-1}{2} e^u = \frac{\pi}{4}.$$

Thus, taking the square root on both sides we have $I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$ and we are done. Note that a rotation similar to those described in the introduction was used to calculate I^2. This underlying theme of rotations will keep surfacing in the computations throughout the paper. $\square$

The gamma function was first studied in the mid 18th century by Euler and Stirling. It has since been used in many different areas of mathematics such as complex analysis, probability, statistics, and combinatorics. For our purposes, we will define the gamma function as $\Gamma(s)$, for $s > 0$, by

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt. \tag{7}$$

Lemma 2.2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof. Plugging $\frac{1}{2}$ into the gamma function, we have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} t^{\frac{-1}{2}} e^{-t} dt.$$

Using substitution with $u = \sqrt{t}$, so $du = \frac{1}{2} t^{\frac{-1}{2}} dt$ we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-u^2} du.$$

Using Lemma 2.1, and substituting in for the integral, we have

$$\Gamma\left(\frac{1}{2}\right) = 2\left(\frac{\sqrt{\pi}}{2}\right).$$

Finishing up, we have that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, and we are done. $\square$

Lemma 2.3. $\Gamma(s+1) = s \cdot \Gamma(s)$.

Proof. Plugging $s+1$ into the gamma function, we have

$$\Gamma(s+1) = \int_0^{\infty} t^s e^{-t} dt.$$

Using integration by parts with $u = t^s$ and $dv = e^{-t}$, we have

$$\Gamma(s+1) = -e^{-t}t^s\Big|_0^\infty - (-s)\int_0^\infty t^{s-1}e^{-t}dt.$$

Notice that $-e^{-t}t^s\big|_0^\infty$ goes to zero because $\lim_{t\to\infty} -e^{-t}t^s = \lim_{t\to\infty}\dfrac{t^s}{-e^t} = 0$, because excessive applications of L'Hopitals Rule will show that the denominator gets exponentially large. And by using equation (7), we are left with

$$\Gamma(s+1) = s\cdot\Gamma(s),$$

and we are done. $\square$

3 Recursion formula for $V(n)$

In this section, we develop a recursion formula for the volume of the unit n-ball. Denoting the volume $\mathrm{Vol}(\mathbb{B}^n(1))$ as $V(n)$, we can write

$$V(n) = \int\cdots\int_{x_1^2+x_2^2+\cdots+x_n^2\le 1} dx_1\,dx_2\cdots dx_n. \tag{8}$$

We have the following recursion formula for $V(n)$ for all $n \ge 2$.

Proposition 3.1. $V(n) = V(n-2)\dfrac{2\pi}{n}$.

Proof. We begin with the understanding that

$$V(n) = \int\cdots\int_{x_1^2+x_2^2+\cdots+x_n^2\le 1} dx_1\,dx_2\cdots dx_n.$$

Breaking this up into an iterated integral of an $(n-2)$-dimension integral and a 2-dimensional integral, we have

$$V(n) = \iint_{x_1^2+x_2^2\le 1}\left(\int\cdots\int_{x_3^2+\cdots+x_n^2\le 1-x_1^2-x_2^2} dx_3\,dx_4\cdots dx_n\right)dx_1\,dx_2,$$

which can also be written as

$$V(n) = \iint_{x_1^2+x_2^2\le 1}\mathrm{Vol}\left(\mathbb{B}^{n-2}\left(\sqrt{1-x_1^2-x_2^2}\right)\right)dx_1dx_2. \tag{9}$$

We will now simplify the above equation by using the following

Claim: For $r > 0$, Vol$(\mathbb{B}^n(r)) = r^n \cdot V(n)$.

Proof of Claim. Recall that in any dimension, the volume of a ball of radius r can be written as

$$\text{Vol}(\mathbb{B}^n(r)) = \underset{x_1^2+\cdots+x_n^2 \le r^2}{\int \cdots \int} (1)\, dx_1 dx_2 dx ... dx_n.$$

Using the change of variables formula, taking $x_1 = ru_1, x_2 = ru_2,$ and $x_n = ru_n$, we see that the Jacobian is given by $\det \begin{bmatrix} r & 0 & \cdots & \\ 0 & r & 0 & \cdots \\ \vdots & 0 & \ddots & \\ & \vdots & & r \end{bmatrix} = r^n.$

Thus, we now have

$$\text{Vol}(\mathbb{B}^n(r)) = \underset{u_1^2+\cdots+u_n^2 \le 1}{\int \cdots \int} (1)|r^n|\, du_1 du_2 du ... du_n.$$

Therefore, by equation (8), and since $r > 0$, we have justified the claim and indeed

$$\text{Vol}(\mathbb{B}^n(r)) = r^n \cdot V(n). \tag{10}$$

By utilizing (10), we can now simplify the integrand of (9). Namely,

$$\begin{aligned} V(n) &= \iint\limits_{x_1^2+x_2^2 \le 1} \text{Vol}\left(\mathbb{B}^{n-2}\left(\sqrt{1-x_1^2-x_2^2}\right)\right) dx_1 dx_2 \\ &= \iint\limits_{x_1^2+x_2^2 \le 1} \left(\sqrt{1-x_1^2-x_2^2}\right)^{n-2} V(n-2)\, dx_1\, dx_2 \\ &= V(n-2) \iint\limits_{x_1^2+x_2^2 \le 1} \left(\sqrt{1-x_1^2-x_2^2}\right)^{n-2} dx_1\, dx_2. \end{aligned}$$

By converting to polar coordinates, taking $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ with $0 \le r \le 1$, and $0 \le \theta \le 2\pi$, we can rewrite the double integral above to yield

$$V(n) = V(n-2) \int_0^{2\pi} \int_0^1 \left(1-r^2\right)^{\frac{n-2}{2}} r\, dr\, d\theta.$$

Now using substitution with $u = 1 - r^2$ we have

$$V(n) = V(n-2)\frac{1}{2} \int_0^{2\pi} \int_0^1 u^{\frac{n-2}{2}}\, du\, d\theta.$$

Treating n as a constant, and evaluating the integral, we find that

$$V(n) = V(n-2) \cdot \frac{1}{2} \int_0^{2\pi} d\theta \cdot \int_0^1 u^{\frac{n-2}{2}} = V(n-2) \cdot \frac{1}{2} \cdot 2\pi \cdot \frac{2}{n} u^{\frac{n}{2}} \Big|_0^1 = V(n-2) \frac{2\pi}{n}.$$

□

3.1 The Recursion Formula for $V(n)$ via Spherical Coordinates.

It is worth noting how the computation made above (performed using rectangular coordinates) can also be done using the coordinate system developed in the introduction; i.e. "spherical" or hyperspherical coordinates.

Recalling (1), it is possible to parameterize the n-dimensional ball $\mathbb{B}^n(1) \subset \mathbb{R}^n$ by

$$\begin{aligned} x_1 &= r \cos\theta_1 \\ x_2 &= r \sin\theta_1 \cos\theta_2 \\ x_3 &= r \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ &\vdots \\ x_{n-1} &= r \sin\theta_1 \cdots \sin\theta_{n-2} \cos\theta_{n-1} \\ x_n &= r \sin\theta_1 \cdots \sin\theta_{n-2} \sin\theta_{n-1}, \end{aligned}$$

taking

$$\begin{aligned} &0 \le r \le 1, \\ &0 \le \theta_i \le \pi, \text{ for } i = 1, 2, \ldots, n-2 \\ &0 \le \theta_{n-1} < 2\pi. \end{aligned}$$

It then follows from the Change of Variables formula that the rectangular volume element $dV = dx_1 dx_2 \cdots dx_n$ can be written in spherical coordinates as

$$\begin{aligned} dV &= \left| \det \left(\frac{\partial x_i}{\partial (r, \theta_j)} \right) \right|_{\substack{1 \le i \le n \\ 1 \le j \le n-1}} dr d\theta_1 \cdots \theta_{n-1} \\ &= r^{n-1} \sin^{n-2}(\theta_1) \sin^{n-3}(\theta_2) \cdots \sin(\theta_{n-2})\ dr d\theta_1 \cdots \theta_{n-1}. \end{aligned}$$

Thus,

$$\begin{aligned} V(n) = \mathrm{Vol}\left(\mathbb{B}^n(1)\right) &= \underset{x_1^2 + x_2^2 + \cdots + x_n^2 \le 1}{\int \cdots \int} dx_1\, dx_2 \cdots\, dx_n, \\ &= \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^1 r^{n-1} \sin^{n-2}(\theta_1) \cdots \sin(\theta_{n-2})\ dr d\theta_1 \cdots d\theta_{n-2} d\theta_{n-1}, \\ &= \int_0^{2\pi} d\theta_{n-1} \int_0^1 r^{n-1} dr \int_0^{\pi} \sin^{n-2}\theta\ d\theta \cdots \int_0^{\pi} \sin\theta\ d\theta, \\ &= \frac{2\pi}{n} \int_0^{\pi} \sin^{n-2}\theta\ d\theta \cdots \int_0^{\pi} \sin\theta\ d\theta. \end{aligned}$$

In the third line we used Fubini's Theorem and dropped the index dependence of the θ_i's after splitting the intreated integral into a product of integrals. In the last line, we evaluated the first two integrals to arrive at $\frac{2\pi}{n}$.

Going forward, keep in mind the useful relation

$$\int_0^\pi \sin^{n-2}\theta \; d\theta \cdots \int_0^\pi \sin\theta \; d\theta = \frac{n}{2\pi} V(n).$$

In particular (and relevant to our later calculations), since $n-4 = (n-2)-2$,

$$\int_0^\pi \sin^{n-4}\theta \; d\theta \cdots \int_0^\pi \sin\theta \; d\theta = \frac{n-2}{2\pi} V(n-2). \tag{11}$$

In order to justify the final recursion formula, the following integral formula for $\int \sin^m \theta \; d\theta$ will be helpful to us. For any integer $m \geq 2$, we have

$$\begin{aligned} \int_0^\pi \sin^m\theta \; d\theta &= -\left.\frac{\sin^{m-1}\theta\cos\theta}{m}\right|_{\theta=0}^{\theta=\pi} + \int_0^\pi \sin^{m-2}\theta d\theta \\ &= \int_0^\pi \sin^{m-2}\theta d\theta. \end{aligned}$$

Note that when m is even, say $m = 2k$, then

$$\int_0^\pi \sin^{2k}\theta \; d\theta = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \pi. \tag{12}$$

Similarly, when m is odd, say $m = 2k+1$, then

$$\int_0^\pi \sin^{2k+1}\theta \; d\theta = \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 2. \tag{13}$$

We are now in a position to prove the recursion formula. Combining these facts above, and assuming without loss of generality that n is even, we get

$$\begin{aligned} V(n) &= \frac{2\pi}{n} \int_0^\pi \sin^{n-2}\theta \; d\theta \int_0^\pi \sin^{n-3}\theta \; d\theta \int_0^\pi \sin^{n-4}\theta \; d\theta \cdots \int_0^\pi \sin\theta \; d\theta, \\ &= \frac{2\pi}{n} \int_0^\pi \sin^{n-2}\theta \; d\theta \int_0^\pi \sin^{n-3}\theta \; d\theta \cdot \frac{n-2}{2\pi} V(n-2), \quad \text{by (11)}, \\ &= \frac{2\pi}{n} \left(\frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \pi \right) \cdot \left(\frac{n-4}{n-3} \cdots \frac{2}{3} \cdot 2 \right) \cdot \frac{n-2}{2\pi} V(n-2), \quad \text{by (12) and (13)}, \\ &= \frac{2\pi}{n} \left(\frac{n-2}{n-2} \cdot \frac{n-3}{n-3} \cdot \frac{n-4}{n-4} \cdots \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{2\pi}{2\pi} \right) \cdot V(n-2), \quad \text{by rearranging terms} \\ &= \frac{2\pi}{n} V(n-2), \end{aligned}$$

which is precisely the conclusion of Proposition 3.1 above.

4 Computing $V(n)$ as a sequence

Consider the sequence defined by

$$f(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{1}{2}n+1\right)}, \text{ for } n \text{ being any natural number.}$$

Proposition 4.1. *The sequence $f(n)$ satisfies the same recursion formula as $V(n)$. Namely,*

$$f(n) = f(n-2)\frac{2\pi}{n}.$$

Proof. Plugging $\frac{\pi^{n/2}}{\Gamma\left(\frac{1}{2}n+1\right)}$ into both sides of the recursion formula, we have

$$\frac{\pi^{n/2}}{\Gamma\left(\frac{1}{2}n+1\right)} = \frac{2\pi}{n}\frac{\pi^{(n-2)/2}}{\Gamma\left(\frac{1}{2}(n-2)+1\right)}.$$

Further simplifying the right side, we have

$$\frac{\pi^{n/2}}{\Gamma\left(\frac{1}{2}n+1\right)} = \frac{2\pi^{\frac{n}{2}-1}}{\Gamma(\dfrac{n}{2})},$$

and simplifying the left side using Lemma 2.3, we have

$$\frac{\pi^{n/2}}{\frac{1}{2}n\left(\Gamma\left(\frac{1}{2}n+1\right)\right)} = \frac{2\pi^{\frac{n}{2}-1}}{\Gamma(\dfrac{n}{2})}.$$

Further simplifying of the left side gives us

$$\frac{2\pi^{\frac{n}{2}-1}}{\Gamma(\dfrac{n}{2})} = \frac{2\pi^{\frac{n}{2}-1}}{\Gamma(\dfrac{n}{2})}.$$

Thus proving that the initially given sequence $f(n)$ satisfies the same recursion formula as $V(n)$. $\square$

We can now combine what we have done to compute the volume of an n-dimensional ball.

Proposition 4.2. *For any natural number $n \geq 1$ and any real number $r > 0$,*

$$Vol(\mathbb{B}^n(r)) = \frac{\pi^{n/2}}{\Gamma\left(\frac{1}{2}n+1\right)}r^n.$$

Proof. To begin, note that

$$V(1) = \mathrm{Vol}\left(\mathbb{B}^1(1)\right) = 2,$$

and

$$f(1) = \frac{\pi^{1/2}}{\Gamma\left(\frac{1}{2}+1\right)} = \frac{\pi^{1/2}}{\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)} = \frac{\pi^{1/2}}{\frac{1}{2}\pi^{1/2}} = 2.$$

Since both $V(n)$ and $f(n)$ satisfy the same recursion formula (as verified in Proposition 4.1 above), we have that $V(n) = f(n)$ for all $n \geq 1$. Thus,

$$V(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{1}{2}n+1\right)}, \text{ for } n \text{ being any natural number.}$$

Furthermore, by equation (10), we have, for any natural number $n \geq 1$,

$$\mathrm{Vol}\left(\mathbb{B}^n(r)\right) = r^n \cdot V(n) = r^n \cdot f(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{1}{2}n+1\right)} r^n.$$

□

5 Calculations

Here we list calculated values for $V(n)$ which give volume of the interior of a unit-sphere in dimensions $n = 1$ through $n = 10$. Note: to calculate $\mathrm{Vol}\left(\mathbb{B}^n(r)\right)$, a ball of radius r, we need only add an r^n to the calculations below.

n	$V(n) = \mathrm{Vol}\left(\mathbb{B}^n(1)\right)$
1	$2 = 2$
2	$\pi \approx 3.14$
3	$\frac{4}{3}\pi \approx 4.19$
4	$\frac{1}{2}\pi^2 \approx 4.93$
5	$\frac{8}{15}\pi^2 \approx 5.26$
6	$\frac{1}{6}\pi^3 \approx 5.16$
7	$\frac{16}{105}\pi^3 \approx 4.72$
8	$\frac{1}{24}\pi^4 \approx 4.10$
9	$\frac{32}{945}\pi^4 \approx 3.29$
10	$\frac{1}{120}\pi^5 \approx 2.54$

It is interesting to note that the volume begins to decrease after $n = 5$.

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