Mathematical Sciences Technical Reports (MSTR)

Mathematics

12-26-2002

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Rebecca Lehman
Princeton University

Chad White

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Rebecca Lehman and Chad White

Adviser: S. Allen Broughton

Mathematical Sciences Technical Report Series
MSTR 02-02

December 26, 2002

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Rebecca Lehman*   Chad White*

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Abstract

A useful way to investigate closed geodesics on a kaleidoscopically tiled surface is to look at the billiard path described by a closed geodesic on a single tile. When looking at billiard paths it is possible to ignore surfaces and restrict ourselves to the tiling of the hyperbolic plane. We classify the smallest billiard paths by wordlength and parity. We also demonstrate the existence of orientable paths and investigate conjectures about the billiard spectrum for the $(2, 3, 7)$ tiling.

Contents

1   Introduction

The $(2, 3, 7)$ tiling is the complete covering of the hyperbolic plane by non-overlapping triangles with angles $\pi/2, \pi/3$, and $\pi/7$. A finite portion of the tiling is pictured in Figure 1. This is a kaleidoscopic, geodesic tiling. The tiling is *kaleidoscopic* because the reflection in any edge leaves the tiling invariant. The tiling is *geodesic* because the geodesic fixed by the reflection in any edge is entirely composed of edges. Such a tiling exists for every hyperbolic triangle whose angles are π/l, π/m and π/n for integers $l, m, n \geq 2$. While our methods are essentially general, we chose to focus on the $(2, 3, 7)$ tiling, which has long been the tiling of choice because it is the minimal kaleidoscopic, geodesic tiling (in the sense of the tiles having minimal area). As such, it is the universal (tiling) cover for surfaces, called Hurwitz surfaces, which have the maximal degree of symmetry, Klein's Quartic Curve being a well-known example.

Knowledge of the lengths of the translations – the universal length spectrum – in the tiling group of the universal cover of a surface would permit some understanding of the closed geodesics of any surface that bears a $(2, 3, 7)$ tiling. The purpose of this paper is to present a way of constructing and classifying short translations of a kaleidoscopic, geodesic tiling, specifically the $(2, 3, 7)$ tiling. Our investigation was motivated by two questions: (The terms will be defined in subsequent sections.)

- What is the initial segment of the length spectrum of the $(2, 3, 7)$ tiling?

- What are the salient geometric features of short translations in the $(2, 3, 7)$ tiling?

We shall address these questions by means of the equivalence between conjugacy classes of translations and hyperbolic billiards. Hyperbolic billiards are defined as follows:

Definition 1 *A closed hyperbolic billiard path is a finite (cyclic) sequence of geodesic segments inside a hyperbolic triangle such that the end each segment meets the beginning of the next segment on the boundary of the triangle. Each segment pair satisfies the law of reflection at the point where they meet the boundary of the triangle. Special rules apply if the segments intersect in a vertex (see section 2.3.1)*

A billiard may be thought of as an ideal billiard ball moving according to "hyperbolic laws" (following hyperbolic lines) on a triangular billiard table, or alternatively, light travelling along hyperbolic straight lines and reflecting from a hyperbolic kaleidoscope constructed in the shape of the triangle. These interpretations motivate the definitions of the terms "billiard path" and "kaleidoscopic triangle." Examples of billiards are given in Figures 5 and 6.
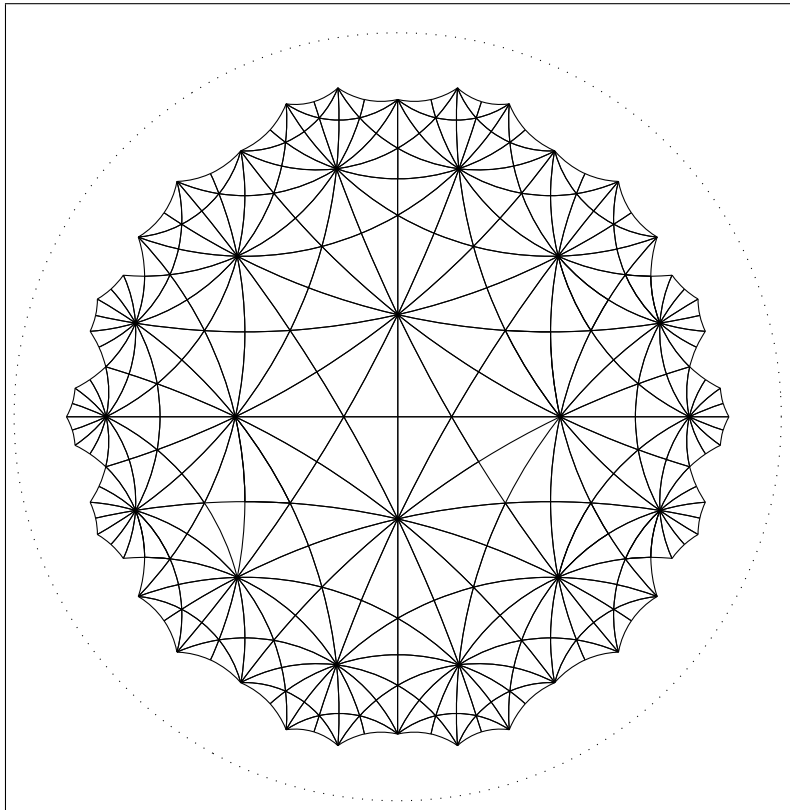

Figure 1: The 2-3-7 tiling

Outline of the paper: In Section 2, we provide a brief outline of hyperbolic geometry and the theory of tiling groups. We discuss the connections between the geometry of the hyperbolic plane or the tiled surface, the algebra of the

tiling group, and the geometry of the billiard path. In Section 3, we present an algorithm for generating translations and show how geometric and algebraic considerations can be used to improve this algorithm. In Sections 4 and 5 we examine some important properties of the billiards generated by our algorithm, and show why the $(2, 3, 7)$ tiling is unusual. In Section 6 we present our unsolved conjectures and questions for future work.

2 Background

2.1 Hyperbolic Geometry

In order to tile surfaces of genus 2 or greater, we must abandon Euclidean geometry in favor of the hyperbolic geometric model. Because of the symmetry and visual niceties it provides, we use the open unit disk model $\mathbb{H}$ for hyperbolic geometry as in Figures 1 and 2. It may be helpful to review some fundamental properties of the hyperbolic plane and differences from Euclidean geometry. We refer to Beardon's book [1] for all our background.

Hyperbolic lines are the intersection of the open unit disc with circles perpendicularly intersecting the boundary of the unit disc. The centers of these circles must lie outside the disk and their radii are positive real numbers. Hyperbolic lines passing through the origin are simply Euclidean lines, i.e., diameters of the unit disk, considered to have infinite radius. Geodesics are straight lines in a space, so a hyperbolic line is also called a geodesic. We can easily see that the edges if tiles in the $(2, 3, 7)$ tiling (Figure 1) form hyperbolic lines.

Distance in hyperbolic geometry contains the pivotal difference between hyperbolic and Euclidean geometry. The distance between two points $z, w \in \mathbb{H}$ is measured along the unique geodesic connecting the points, and is given by:

$$\rho(z, w) = log \left(\frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|} \right).$$

Intuitively, hyperbolic distance "looks" like Euclidean distance near the origin, but space appears to shrink closer to the boundary of the disk. Pictorially, a triangle near the center of the disk will look much larger than that same triangle translated closer to the boundary, but the triangles have the same angle measures, hence equal area and side lengths (see Figure 1).

In our kaleidoscopic, geodesic, hyperbolic tiling there are two types of conformal transformations: translations and rotations. We concern ourselves primar-

ily with translations. A translation T is uniquely determined by its two fixed points on the boundary of the hyperbolic plane and the length and direction of motion. Points in the plane are moved along the family of circles intersecting both fixed points. These circles, called *hypercycles* or *equidistant curves*, and the fixed points are pictured in Figure 2. One of the curves is perpendicular to the boundary and is therefore a geodesic. This geodesic is called the axis of translation, and is denoted L_T. The axis of translation is important because all points on the axis are moved the same distance. I.e., $\rho(T(z), z) = t$, for a fixed t independent of where z is on L_T. Therefore, iterates of the translation T move $z \in L_T$ equal amounts. The quantity t is called the translation distance of T and is denoted by $l_t(T)$. Points that are off-axis move along the equidistant curves, but in this case $\rho(T(z), z) > t$ (more on this in subsection 3.2). This is obviously unlike Euclidean geometry because translations move off-axis points farther than points on the axis, whereas translations in Euclidean geometry move every point in the plane the same distance.
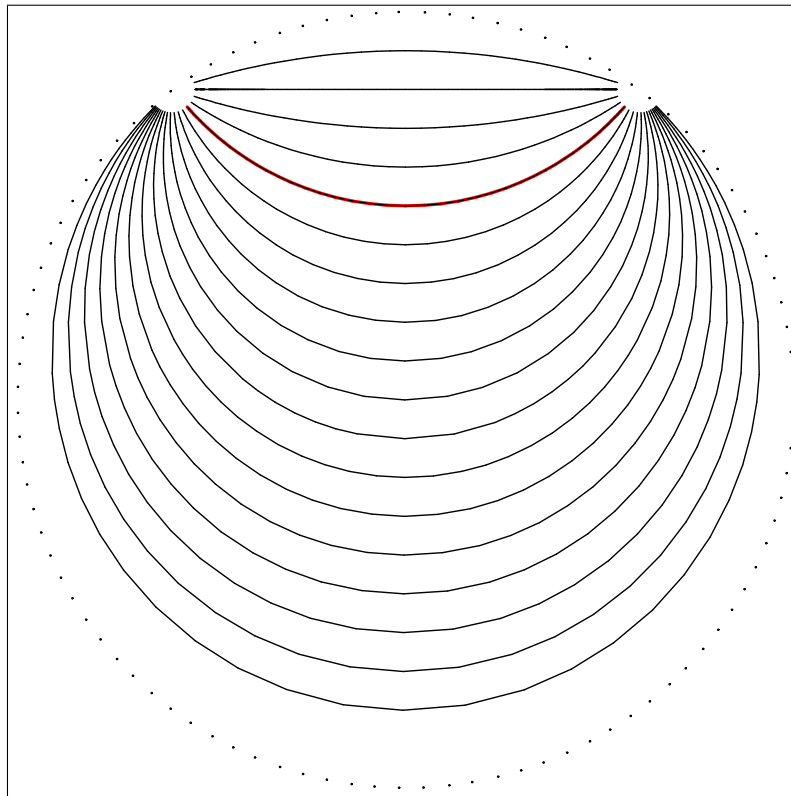

Figure 2: Hyperbolic Translation on the Hyperbolic Plane

Another transformation on the plane is a glide reflection. A glide reflection,

or simply a glide, is a non-conformal transformation which translates points along an axis, then reflects the plane over that axis of translation. Note that the square of a glide is a translation.

Here are some useful trigonometric properties of the hyperbolic plane, also from [1]:

- Area of a Triangle
$$A = \pi - \alpha - \beta - \gamma,$$
  where α, β, γ are the angles of the triangle.

- Law of Sines
$$\frac{\sin(\alpha)}{\sinh(a)} = \frac{\sin(\beta)}{\sinh(b)} = \frac{\sin(\gamma)}{\sinh(c)},$$
  where a is the length of the side opposite α, etc.

- First Law of Cosines
$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma).$$

- Second Law of Cosines
$$\cosh(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)}.$$

2.2   The Tiling Groups

In order to exploit the symmetry properties of the tiling in the hyperbolic plane, we make use of group theory. For this subsection and most of the remaining portions of this section we will consider a general (l, m, n) tiling of the hyperbolic plane by hyperbolic triangles with vertex angles π/l, π/m, and π/n.

2.2.1   Notation: The Master Tile and its Reflections

We choose a single triangle to be called "master tile" and label its vertices R, P, and Q (see Figure 3). We denote the opposite edges with the lower-case letters r, p and q, which we will also use to refer to the reflections over those edges. We classify all edges in the tiling as "r-edges, p-edges, or q-edges" according to which edges correspond to them in the master tile. In this particular tiling, the edges are easily distinguishable by length. Note that the product of two reflections is the rotation about their common vertex.

2.2.2   Three Important Groups

The edges of an (l, m, n) tiling $\mathcal{T}$ generate three important groups, each of which possesses important representation properties for the study of translations.
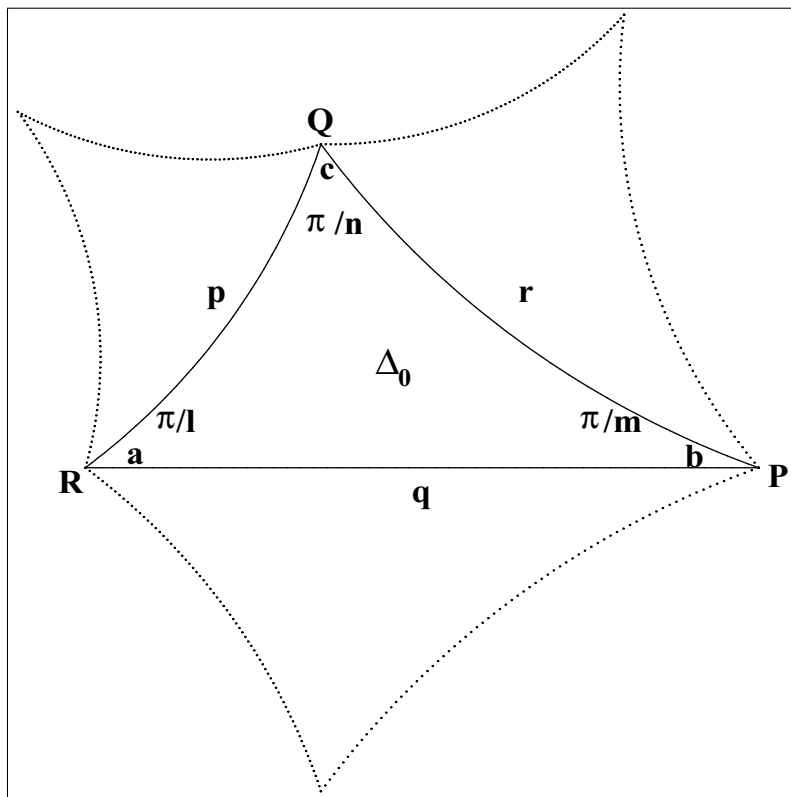
Figure 3: The Master Tile and Generating Refelections

Definition 2 *The* path group $\mathcal{F}_{p,q,r} = \langle p, q, r \rangle$ *is the free group on the letters p, q, and r. It represents the group of all possible paths from the master tile to the interior of some other triangle in the hyperbolic plane, naturally only defined up to choice of an initial master tile. The word corresponding to a given path is constructed by writing down the labels of the edges crossed by the path in the order that it crosses them, to the right of the existing word. Multiplication in this group is simply concatenation.*

Remark 3 *The path group is best understood in terms of the dual tiling $\mathcal{T}'$ of our tiling $\mathcal{T}$. The vertices of $\mathcal{T}'$ are the incenters of the tiles of $\mathcal{T}$. The edges of $\mathcal{T}'$ are formed by connecting two vertices of $\mathcal{T}'$ by a hyperbolic line segment if the two corresponding tiles of $\mathcal{T}$ meet in an edge. The common edge in $\mathcal{T}$ perpendicularly bisects the connecting segment in $\mathcal{T}'$. We may color the edges of $\mathcal{T}'$ with colors p, q and r depending the type of the unique edge from $\mathcal{T}$ that it meets. The tiles of $\mathcal{T}'$ are the closures of the components obtained by removing the vertices and edges of $\mathcal{T}'$ from $\mathbb{H}$. The tiles are regular 2l-gons, 2m-gons,*

or $2n$-gons, with a corresponding vertex from $\mathcal{T}$ at their center. Let Ω be the infinite graph formed from the vertices and edges of $\mathcal{T}'$. The elements $\mathcal{F}_{p,q,r}$ are in 1-1 correspondence with the paths or walks in Ω starting in Δ_0. Two paths are multiplied by moving the second to the end of the first and concatenating.

The free group also carries a well-ordering.

Definition 4 *The* ShortLex *ordering on the free group is the ordering which defines a word to be earlier than another if it has shorter wordlength, or if it has equal wordlength and earlier in the dictionary order.*

Definition 5 *The* tiling group Λ^* *is the quotient of the free group by the relations*

$$p^2 = q^2 = r^2 = 1 \tag{1}$$
$$(pq)^l = (qr)^m = (rp)^n = 1 \tag{2}$$

The group Λ^* represents a group of isometries of the (l, m, n) tiling, where p, q and r also represent the reflections over the p, q and r edges, respectively. More precisely, $\Lambda^* = \langle p, q, r \rangle$, the group generated by the reflections in the sides of the master tile. The group is independent of which tile is chosen as the master tile. The translations and glide reflections are exactly the elements of infinite order in this group.

The first three relations in (1) result from the fact p, q and r are reflections. The remaining relations in (2) are derived as follows. The composition of reflections pq, qr and rp are counter-clockwise rotations about R, P and Q through angles of $2\pi/l$, $2\pi/m$, and $2\pi/n$ respectively. Thus they have the indicated orders. There are no other relations, because $\mathbb{H}$ is simply-connected.

Definition 6 *The* conformal tiling group Λ *is the subgroup of Λ^* consisting of products of an even number of reflections.*

The subgroup Λ is generated by the rotations $a = pq$, $b = qr$ and $c = pr$ with the relations

$$a^l = b^m = c^n = 1 \tag{3}$$
$$abc = 1 \tag{4}$$

The relation (4) comes from $abc = pqqrrp = 1$.

The conformal tiling group represents all rotations and translations in the tiling. The generators a, b and c are rotations of $2\pi/l$, $2\pi/m$ and $2\pi/n$ about the R, P and Q vertices respectively.

Definition 7 *The automorphism θ of Λ is conjugation by q in Λ^*.*

Note that $\theta(a) = qpqq = qp = a^{-1}$ and $\theta(b) = qqrq = rq = b^{-1}$.

2.2.3 The Rewriting System

It so happens that the group Λ^* has an automatic structure as a subgroup of the free group, which means that there is a set of reduction rules such that any word viewed as an element of Λ^* can be reduced to a canonical form by replacing substrings with equivalent substrings where each stage of the reduction makes the word earlier in the ShortLex ordering. This allows us to represent any element of Λ^* uniquely as a word in the generators.

The rewriting system can be derived from the initial generators by the Knuth-Bendix Procedure [5]. It is constructed and implemented in the Magma script billiardsearch.mgm, available at [11].

Definition 8 *The wordlength $l_w(g)$ of a group element in $g \in \Lambda^*$ is said to be the minimal wordlength of any word in $\mathcal{F}(p, q, r)$ which represents that element. Geometrically, $l_w(g)$ is the smallest number of walls that one must cross among all paths from Δ_0 to $g\Delta_0$ that do not pass through any vertices.*
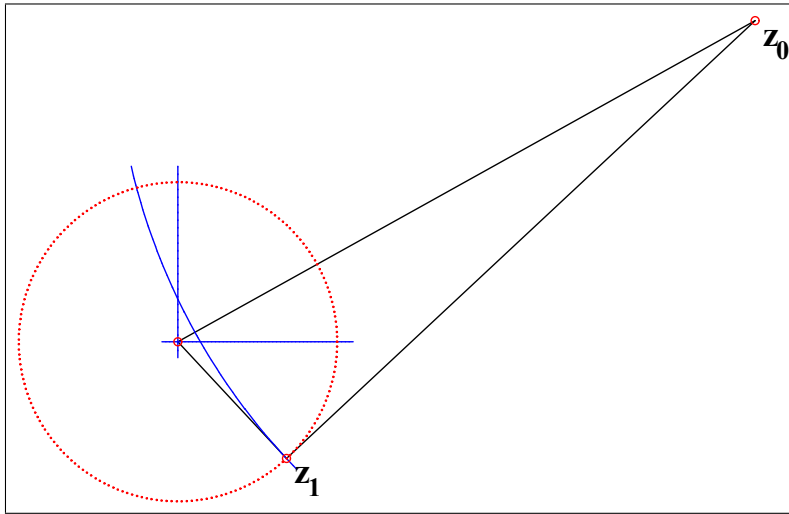
Figure 4: The 2-3-7 master tile

2.2.4 The Fractional Linear Transformation Representation

The conformal group Λ has a matrix representation as a subgroup of $PSL_2(\mathbb{C})$, the group of 2x2 complex matrices of determinant 1 under the identification $M = -M$ for all matrices M. Following Derby-Talbot [4], we construct the representation explicitly for the $(2, 3, 7)$ tiling with master tile pictured in Figure 4, by considering the generators of Λ as fractional linear transformations of the

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hyperbolic disk:

$$
\begin{aligned}
a : z &\mapsto -z \\
b : z &\mapsto \frac{\bar{z_0}z - 1}{z - z_0} \\
c : z &\mapsto \frac{-z_0 z - 1}{-z - \bar{z_0}}.
\end{aligned}
\tag{5}
$$

The point z_0 is the center of the circle in $\mathbb{C}$ corresponding to the hypotenuse of the master tile, depicted in Figure 4. This matrix representation is helpful because it gives us an algebraic formula for the translation distance a translation moves a point on its axis (see [4]):

$$
l_t(M) = 2ln(|tr(M)| + \sqrt{|tr(M)|^2 - 4}).
\tag{6}
$$

2.3   The Hyperbolic Billiard Table

Hyperbolic geometry allows us to describe a triangle up to congruence given only the three angles at the vertices. We restrict ourselves to triangles having vertices of angle measures π/l, π/m and π/n, where l, m and n are integers greater than 1. A triangle with angle measures π/l, π/m and π/n is called an (l, m, n) triangle. The parity of the l-vertex is defined to be the parity of l, and likewise for m and n.

The tool we propose to use in this paper incorporates a hyperbolic billiard table. For our purposes, a hyperbolic billiard table is an (l, m, n) triangle whose edges are geodesics in hyperbolic space. A billiard ball on a hyperbolic table obeys a set of rules similar to the laws of physics applied to a ball on an ordinary Euclidean billiard table.

Henceforth, a billiard path described by a ball bouncing around the billiard table will be referred to simply as the "path." We will concentrate on closed primitive paths, that is, paths ending where they first return to their initial position and direction. We will essentially ignore those paths which are powers of shorter primitive paths.

2.3.1   Laws of Reflection

The following are the laws of reflection for a billiard ball on a hyperbolic billiard table:

1. A billiard ball travels along a geodesic.

2. The angle of incidence is the angle between the wall and the incoming path of the ball. The angle of reflection is the angle between the wall and the outgoing path of the ball. When a ball bounces off a wall of the table, an edge of the triangle, the angle of incidence equals the angle of reflection. This rule is equivalent to letting the path extend in a straight hyperbolic line across a triangle edge, then reflecting the path over the triangle edge the path intersects, back into the original triangle.

3. A path intersecting a vertex of the triangle obeys different laws depending on the parity of the vertex.

   (a) If the path intersects an odd vertex, then the path reflects over the angle bisector of the vertex. This rule is motivated as follows: a path entering a vertex is equivalent to letting the path extend in a straight hyperbolic line though the vertex and then reflecting the path, in sequence, over every line in the tiling intersecting the vertex in question, back into the original triangle.

   (b) If the path intersects an even vertex then the path makes a turn of π radians, effectively bouncing straight back out of the even vertex. The motivation here is the same as for the odd vertex rule.

The vertex rules are what one would obtain by continuity, by following paths that just miss the vertex. Examples of billiards passing through odd and even vertices are given in Figures 5 and 6 respectively.
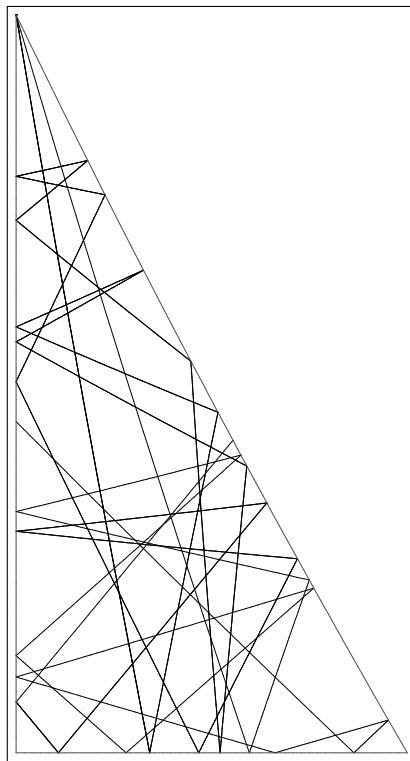

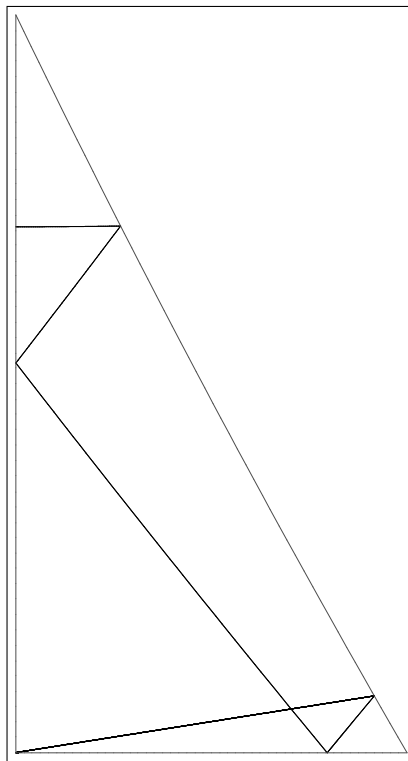Figure 5: Reflection in an odd vertex

Figure 6: Reflection in an even vertex

A path that reflects an even number of times before repeating itself is called an even path. Even paths correspond to translations. A path that reflects an odd number of times before repeating itself is called an odd path. Odd paths correspond to glide reflections. It is important to recall the motivation in rule number 3, that is, a path entering a vertex is reflected over every line in the tiling intersecting the vertex. This implies that in calculating parity, a vertex of order n corresponds to n "bounces" or reflections, as we would conclude following the continuity model.

2.3.2 Billiard Paths and the Tiling Group

Before progressing further, we need to make an important connection between geodesics in hyperbolic space and billiard paths. Let L be a geodesic in hyperbolic space overlaid by a tiling of triangles. Allowing the sides of the triangles in the tiling to segment L, we get an infinite set of triangles containing pieces of L. Two examples of this segmentation of a ray are shown in Figures 7 and 8. Using appropriate successions of reflections, reflect each triangle containing a segment of L back to the master tile, forming the corresponding billiard path B.
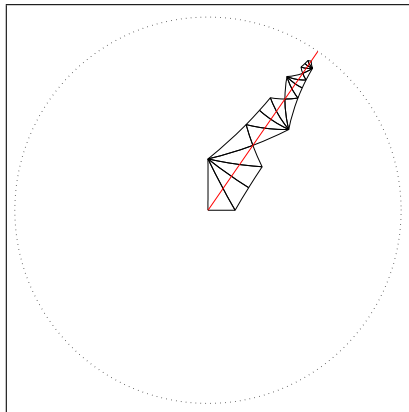
Figure 7: Tiles along a geodesic.   Figure 8: Tiles along a geodesic.
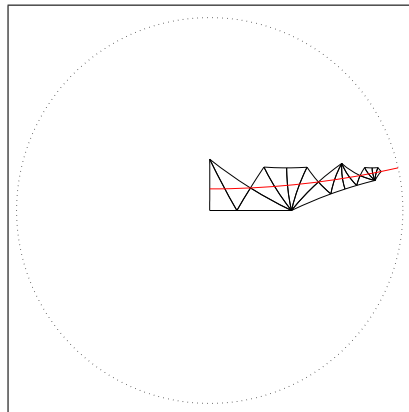
More formally, the billiard path of a given hyperbolic line is its image under projection to the quotient space of the tiling under Λ^*, that is, the space consisting of the orbits of points under Λ^*. To represent the quotient space, we choose the unique element of each orbit which lies inside the master tile. For most geodesics the master tile will fill with an infinite number of segments. However, some billiards will be finite, yielding a closed, repeating path. It is these billiard paths that will be the most interesting for us, and we shall restrict our attention to them.

For both of the geodesics pictured in Figures 7 and 8 the resulting billiard is that pictured in Figure 10. The billiard in Figure 9 is the shortest of all nondegenerate billiards. There is one degenerate "perimeter" billiard that traces out the perimeter of the master tile, starting at the vertex of order 2.
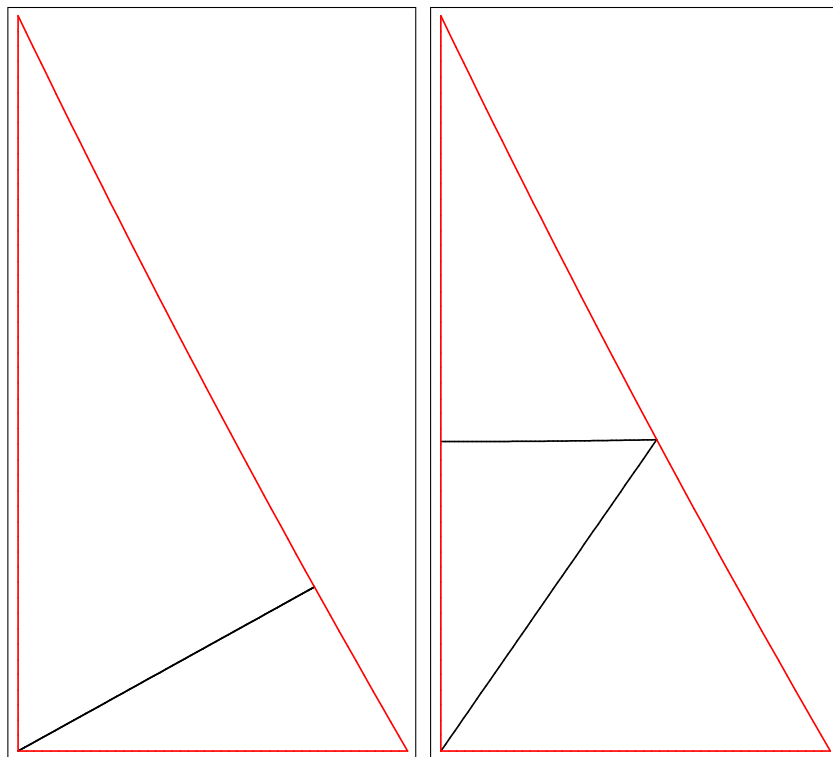
Figure 9: Billiard from *cba*   Figure 10: Billiard from *cbababba*

We can see that reflecting the segments of L back onto the master tile preserves, indeed motivates, the rules of a hyperbolic billiard table, especially at the vertices. A billiard ball travels on a geodesic. When a ball bounces off a wall, it is reflected over that edge of the triangle. When a ball enters a corner of a table, its path is reflected over each edge of each triangle in the tiling meeting at that vertex. We will apply this construction to the axis of translation of group elements, and call the resulting billiard path the path of the element.

Now we will examine our path B. Choose a point X and a direction on B. We can construct a word w corresponding to B in the free group on the generators of Λ^*, by forming a string of p,q and r reflections. Starting at point X and proceeding in the chosen direction, right compose the letter representing each edge encountered along the path onto w. When we have traced the path exactly once, that is, we are back at point X facing the initial direction, the

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word w completely and uniquely describes B.

Lets formalize this construction.

Construction 9 *Let L be any hyperbolic line meeting the master tile in a segment. Let $\ldots, \Delta_{-1}, \Delta_0, \Delta_1, \ldots$ be the sequence, in order, of triangles for which $L_i = L \cap \Delta_i$ is an interval of positive length. Then $\ldots, L_{-1}, L_0, L_1, \ldots$ is a sequence of hyperbolic segments, which when laid out in order, form the line L. The Δ_i are unique unless L is one of the lines of the tiling resulting in a perimeter billiard. In Figures 7 and 8 we would remove those triangles that meet the line in just a vertex, and of course extend the segment in both directions.*

Now define $g_i \in \Lambda^*$ by $\Delta_i = g_i \Delta_0$ and define $B_i = g_i^{-1} L_i \subset \Delta_0$. Then, the sequence $B = \ldots, B_{-1}, B_0, B_1, \ldots$ satisfies the requirements to be a billiard. Observe that $g_0 = 1$ and the segment $L_0 = B_0$ both determine the line and the billiard. The elements $w_i = g_i g_{i-1}^{-1}$ are computed as follows:

1. *If $\Delta_i \cap \Delta_{i-1}$ is an edge of type $s \in \{p, q, r\}$ then $w_i = s$.*

2. *If $\Delta_i \cap \Delta_{i-1}$ is a vertex of odd order k, then $w_i = stst \cdots s = tsts \cdots t$ (k factors) where $s, t \in \{p, q, r\}$ are the two edge types meeting at the vertex.*

3. *If $\Delta_i \cap \Delta_{i-1}$ is a vertex of even order k, then $w_i = stst \cdots t = tsts \cdots s$ (k factors) where $s, t \in \{p, q, r\}$ are the two edge types meeting at the vertex.*

Observe that the g_i's could be obtained by concatenating, in order, all the edge types we encounter as we move along the tiles pictured in Figures 7 and 8, always crossing from one tile to the next through a wall. Also observe that the g_i's can be constructed from the geometry of the billiard alone.

Remark 10 *Observe that the billiard is closed (finite), if and only if the billiard is periodic, i.e., $B_{i+n} = B_i$, for some positive n, taking orientation into account. If n is chosen as small as possible, then we will have taken exactly one complete tour of the billiard, travelling the path twice in case of a reversing billiard. Thus the billiard path $B_0, \ldots, B_{n-1}$ is primitive. For this n*

$$g_n L_0 = g_n B_0 = g_n B_n = L_n,$$

i.e., $g = g_n$ maps L_0 to L_n in an orientation-preserving manner, implying that g is a glide or translation and L is its axis.

As we move along the complete tour of the billiard the number of bounces is defined to be the number of wall crossed. This is a trivial notion for walls, but for vertices the number of bounces is the order of the vertex. Observe that for a closed primitive billiard, if the number of bounces is N, then the element g_n can be written as a product of N reflections in $\{p, q, r\}$.

Remark 11 *It is not difficult to see that odd paths are described by words with odd wordlength, so these words are contained in Λ^*, but not in Λ. These odd words describe glide reflections, translations that are followed by a reflection. Glide reflections are obviously anti-conformal. Since it is necessary to calculate translation distances using the matrix representation of the conformal group Λ, we sometimes consider odd paths as translations which are the squares of glides.*

Remark 12 *The element g coming from a primitive billiard is a primitive element of Λ^*, i.e., if $g = h^e$ for $h \in \Lambda^*$ then $e = \pm 1$. In fact, the subgroup $\Lambda_L^* = \{g \in \Lambda^* : gL = L\}$ is isomorphic to $\mathbb{Z}$ or the infinite dihedral group D_∞. In the first case $\Lambda_L^* = \langle g \rangle$, in the second case $\langle g \rangle$ has index 2 in Λ_L^*.*

This construction of translation or glide from a billiard is dependent on a random choice of the point X. Suppose we choose a point X' on B which is on a different segment than X, and we choose the same direction as previously. Following the same procedure as before, we produce a word w' which is a cyclic permutation of w. The words w and w' are not equal, but describe the same path. As group elements, they are conjugates and the words are cyclic permutations of each other. For example, *pqpr* and its cyclic permutations $qprp = p^{-1}(pqpr)p$, $prpq = (pq)^{-1}(pqpr)pq, \ldots$ all describe the same billiard.

We have seen how a primitive, closed billiard generates a primitive glide or translation. What about the reverse? Pick a primitive translation or glide $g \in \Lambda^*$. Let L be its axis. If L does not pass through Δ_0 then pick an $h \in \Lambda^*$ so that $hL \cap \Delta_0$ in a segment. Then hgh^{-1} has hL as its axis. Now the word w generated by $hL \cap \Delta_0$ generates the subgroup of glides and translations that map hL to itself. Thus $hgh^{-1} = w^{\pm 1}$. It is not hard to show that if another conjugating element h' had been chosen then the corresponding w' would have to be one of the cyclic permutations chosen above. Thus any two conjugate primitive elements produce the same billiard path, suitably renumbered, perhaps with an orientation reversal.

Proposition 13 *There is a 1-1 correspondence between primitive closed billiard paths B and conjugacy classes of primitive glides and translations. (See proposition 24 for questions on orientability.) Moreover, for any glide or translation g whose axis meets the master tile in a segment, the wordlength $l_w(g)$ equals the number of bounces of the billiard path B.*

2.4 Historical Motivation: Surfaces and their Tilings

The billiard path problem arose out of problems involving hyperbolic tilings on surfaces. Recent work has been completed concerning small, closed geodesics on the $(2, 3, 7)$ tiling of Klein's quartic curve [4] and systole lengths on other $(2, 3, 7)$ tileable hyperbolic surfaces [7].

The hyperbolic plane $\mathbb{H}$ is the universal cover for Hurwitz surfaces. Thus for any such surface S, there is a covering map $\phi : \mathbb{H} \to S$, which maps the tiling of $\mathbb{H}$ to a tiling by the same triangle on S, and thereby preserves the local geometry. This map is simply the projection into the quotient space under the subgroup Γ of Λ^* consisting of those group operations which are the identity on the surface.

It may be shown that each closed geodesic of S is the image of the axis of a translation g in Γ. The geodesic lifts to its pre-image in $\mathbb{H}$, a hyperbolic line which is the axis of g acting as an element of Λ^*. If g is primitive in Γ, then the segment between any two nearest g-equivalent points on the axis maps to

the geodesic on S in a 1-1 fashion, with finitely many exceptions. Moreover, since the map $\eta : \Lambda^* \rightarrow \Lambda^*/\Gamma$ has a finite image (there are only finitely many triangles on a finite surface), the image of any translation g' in Λ^*/Γ is of finite order. Therefore, some power of g' is in Γ.

The systole, the shortest geodesic of a surface can be computed by finding the shortest element in Γ. This can be done by compiling a table of the shortest primitive elements of Λ, then finding the primitive $g \in \Lambda$ for which the product $o(\eta(g))l_t(g)$, equal to the length of the generated geodesic, is as small as possible. The first step of this process, finding the initial primitive translations, is the time-consuming part, but it can be done entirely in the universal cover of the surface, that is, the tiling on $\mathbb{H}$. This allows us greater generality since the lengths of elements of Λ^* are surface-independent and can then be projected down to any quotient surface bearing the given tiling. For instance, given the $(2,3,7)$ translations of length up to L, one can find the geodesics up to L of any hyperbolic surface bearing a $(2,3,7)$ tiling. Working in the universal cover also gives us cleaner mathematics, as the universal tiling group happens to be an infinite group with an easy matrix representation and finitely many rules of reduction.

3   Translation-producing Algorithms

3.1   Initial Algorithm (geometric focus)

In their analysis of surface geodesics in [4], [7], Ryan Derby-Talbot and Kevin Woods present algorithms for constructing $(2,3,7)$ translations of short hyperbolic length. In this section we follow their presentation.

The essential idea of this algorithm is that since every translation of length L is conjugate to one that moves the master tile a distance L into the first quadrant, it suffices to generate all translations that satisfy these conditions, which can be done inductively.

Begin with a list containing the identity. At each step, we multiply the elements of the list by A, B, C, B^{-1} and C^{-1} on the right. For each constructed word, add it to the list if and only if it is not equal to an element already constructed, and it takes the origin to a point within the first quadrant sector bounded by the circle of radius L centered at the origin. Repeat this process until no new words are generated.

The resulting list will contain all translations of length less than L, as well as a number of rotations. To eliminate rotations, compute the absolute value of the trace of each element. If it is less than 2, the element is a rotation. Otherwise it is a translation whose length can be computed from formula (6).

Using this algorithm the first 8 conjugacy classes in the $(2,3,7)$ tiling were generated (lengths up to 3.63). Unfortunately, it also generated many useless rotations and a multiplicity of elements from each conjugacy class. Therefore, it quickly became computationally unmanageable. What we need is an efficient way to generate a single representative element from each conjugacy class, with-

out repetition. This we do not know how to do, but we have found several ways to reduce redundancy and construct a better algorithm.

3.2 Prerequisite Notions for an Improved Algorithm

Definition 14 *The* canonical representative *of a conjugacy class C is defined as the word in the free group on p, q, and r which evaluates to an element of C in Λ^* and is earliest under the ShortLex ordering. Note that the canonical element is of minimal wordlength.*

An ideal algorithm would produce only the canonical representative of each conjugacy class. We do not know a simple algebraic test for whether an element is the canonical representative or not. Our algorithm will produce the set of elements which share certain characteristic properties of the canonical representative.

Definition 15 *The path representative of a conjugacy class is the earliest word obtained by writing out the reflections of the (directed) billiard of a conjugacy class. All other such words are cyclic permutations of the path representative.*

The path representative is not a nice algebraic form: it's usually not even irreducible under the rewrite system, although it is always of minimal wordlength. However, given any element of the class, it is easy to construct its path representative by computing its billiard geometrically (this can be done to sufficient precision by Maple) and examining the sequence of reflections.

This means that given an algorithm that produces a small set of elements including the canonical representative from each conjugacy class, we can determine the conjugacy classes and find the canonical representatives by examining the path representatives of each output element.

Definition 16 *A cyclic reduction of a word w is the ShortLex-earliest word which can be obtained by reducing a cyclic permutation of w. A word which is its own cyclic reduction will be called cyclic-irreducible.*

Proposition 17 *The canonical representative of a conjugacy class is cyclic-irreducible.*

Proof. Conjugacy classes are closed under cyclic reduction, since they are closed under reduction and cyclic permutation. All elements move to ShortLex-earlier elements under cyclic reduction, so the canonical representative can only go to itself. ∎

The following characterizations of minimal wordlength glides and translations are proven in [2].

Proposition 18 *Let $g \in \Lambda^*$. Write $l_w(g)$ to be the wordlength of g. Then,*

$$l_w(g^n) = n l_w(g)$$

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for all n, if and only if g is a translation or glide of minimal wordlength in its conjugacy class. Moreover, $l_w(g)$ equals the number of bounces in the corresponding billiard.

Proposition 19 *Let g be a translation or glide in Λ^*. If the axis of g passes through the master tile then g has minimal wordlength in its conjugacy class.*

The following proposition is a bit more usable in practice.

Proposition 20 *Let $g \in \Lambda^*$ be a translation. Write $l_w(g)$ to be the wordlength of x. Then, there exists some n, which can be determined from $l_t(g)$, the translation length of g, and the action of g on the on the origin, such that if*

$$l_w(x^g) = n l_w(g) \tag{7}$$

then g is a translation of minimal wordlength in its conjugacy class.

Remark 21 *In using this proposition we assume that we can compute the matrix corresponding to g, the translation length $l_t(g)$, and the image of 0 by the formulas (5) and (6). For glides we just work with the square of the transformation.*

Proof. Suppose that g is a translation conjugate to the shorter word h by u, that is, $g = uhu^{-1}$. Then,

$$l_w(g^n) = l_w(uh^nu^{-1}) \leq 2l_w(u) + n l_w(h) < n l_w(g)$$

for all sufficiently large n. Thus g^n reduces to the shorter word uh^nu^{-1}. Now condition (7) fails as soon as

$$2l_w(u) + n l_w(h) < n l_w(g)$$

or

$$\frac{2l_w(u)}{l_w(g) - l_w(h)} < n.$$

Since $l_w(g) - l_w(h) \geq 1$, we need only test for $n > 2l_w(u)$. This requires only knowing a bound on $l_w(u)$.

Now suppose g is any translation. Let us construct u and h so that $g = uhu^{-1}$, h has minimal word length, and $l_w(u)$ is as small as possible. Let z_0 be some point in the master tile (e.g., $z_0 = 0$), let L be the axis of g, let L' be the line passing through z_0 and perpendicular to L, and let z_1 be the point of intersection of L with L'. Let Δ be a tile containing z_1 and meeting L in an interval and let u be such that $\Delta = u\Delta_0$. Set $h = u^{-1}gu$, the axis of h is $u^{-1}L$ which passes through $\Delta_0 = u^{-1}\Delta$. By Proposition 19, h has minimal wordlength. To get an estimate on $l_w(u)$ we will need the following inequalities which may be found in [1] and [3], respectively.

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- Let the distance from a point z to the axis of a translation g be α. If the translation length of g is t, then the distance D that the translation moves z is given by :

$$\cosh D = \cosh(t) \cosh^2(\alpha) - \sinh^2(\alpha) \qquad (8)$$

- Let z be any point in the master tile $g \in \Lambda^*$. There exist $A, B, C > 0$ such that

$$l_w(g) \le A\rho(z, gz) + B \qquad (9)$$
$$\rho(z, gz) \le Cl_w(g) \qquad (10)$$

Knowing t and D for $z_0 = 0$ (t from the trace, D from the action of g on the origin) we can solve for α, how close the axis must be to the origin. (Observe that the equation is quadratic in $e^{2\alpha}$). Now as u maps Δ_0 to Δ and since $z_1 \in \Delta$ then uz_0 is at most the diameter of Δ away from z_1. If we let δ be the diameter Δ then we have $|\alpha - \delta| \le \rho(z_0, uz_0) \le \alpha + \delta$. But then from equation (9) $l_w u \le A(\alpha + \delta) + B$. Pick n to be $A(\alpha + \delta) + B$.

If we needed to know the canonical representative of g and the element by which g is its conjugate, it would be possible (although computationally slow) to construct the conjugate of g by all reduced words of wordlength between $|\alpha - \delta|/C$ and $A(\alpha + \delta) + B$. The ShortLex-earliest of these would be the canonical representative. But we don't actually need to do the exhaustive search. The size of α is enough to find n. ∎

Remark 22 *If we do not want to compute translations lengths but want to eliminate elements which are not glides or translations, we can do that by raising elements to powers. If g is not a translation or glide, then g is a non-identity element of finite order. The orders of the elements of finite order in Λ are 2,3, and 7, so let $n \ge 42$. We have*

$$l_w(g^n) = l_w(g^{42} g^{n-42}) \le l_w(1) + (n - 42)l_w(g) < nl_w(g)$$

The discussion shows that we can actually use $n = 7$ to eliminate elements of finite order.

Our algorithm will use the two properties of reducibility and wordlength to find candidates for the canonical representative. This is not the ideal algorithm as there do exist non-canonical cyclic-irreducible words of minimal wordlength.

3.3 New Inductive Algorithm (algebraic and combinatorial focus)

3.3.1 Word Production

Begin with a list L_{n-1} of reduced words of length $n - 1$. This will have been generated at the previous stage (at $n = 1$, begin with L_0 containing the identity). To construct the list L_n of reduced words of length n, multiply each element by p, q, and r, reduce, and add to the list if they do not appear on this or any previous list. Magma can carry out the reductions automatically.

3.3.2 Test minimality

To get a first simplification L'_n of L_n, raise each of the words in L_n to a high power which is divisible by 42. For simplicity, we used 42 in our computations. This may not have been optimal, but if it is too low, we simply get more redundancy. We can not lose any conjugacy classes this way. Reduce the result, and remove the word from the list if its wordlength shortens. This removes elements not of minimal wordlength. Since every element appears in the initial list, the canonical representative of every word is in the list, so it is possible simply to remove non-canonical words without directly finding their canonical representatives.

3.3.3 Test cyclic-irreducibility

Order L'_n alphabetically (this is ShortLex since all the elements of L_n are of length n). For each element e of L'_n, in order, remove all its cyclic permutations from the list. They are not cyclic-irreducible because e is a ShortLex-earlier permutation of them.

Next, for each element e which has a reducible permutation (it contains a reducible string when viewed as a cycle (mod n), find the cyclic reduction of e. If it comes before e, remove e from the list.

3.3.4 Output

Convert the words in p,q, and r to words in A,B, and B^2, squaring the odd ones. Compute their traces and fixed points, and construct billiards if needed.

This algorithm was implemented using the computer scripts billiardsearch.mgm and billiardpics.mws. Along with first script are data files containing lists of minimal words from conjugacy classes. The second script also computes the lengths, draws a picture of the billiard, and computes the word corresponding to the billiard. [11].

3.4 Billiard Path Conjugacy Classes

Using the algorithm described above, we were able to classify all billiard paths of wordlength up to 50. Pictures of billiards with length under 30 are at the web site [11]. The billiard paths are in exact correspondence with the set of conjugacy classes of infinite order in Λ^*, up to taking powers of an element and inversion.

Section 7 contains a partial table of the number of non-trivial translations and glides of wordlength n for n up to 50. A table of the shortest and other significant billiard paths can be found in Section 8. The classes are listed in ascending order according to their wordlengths. Along with the pictures of the paths and their wordlengths, we have listed the number of bounces and the canonical representative of one of the conjugacy classes described by the path.

4 Reversing and Non-Reversing Paths (RP's and NRP's)

4.1 Definitions and Notes

Definition 23 *A billiard path is reversing, or non-orientable, if it retraces itself in the opposite direction: if it bounces from point X to Y and back again from Y to X only along the same path.*

A primitive reversing element must reverse itself exactly twice, since a closed path cannot have a beginning without an end. A non-reversing path is called orientable. Figures 12 and 13 present examples of orientable and non-orientable billiards. As the $(2, 3, 7)$ triangle contains no short orientable paths (see next section), we use the $(5, 5, 5)$ triangle.

A reversing path determines a single conjugacy class: all the elements one can produce from the path are conjugate by cyclic permutation. The inverse of such an element is its conjugate by some permutation, the permutation that takes the path to the next path starting from the same point. A non-reversing path requires an orientation: given a starting point, one can read the path "clockwise" or "counter-clockwise" and get two distinct paths which are not cyclic permutations, but inverses of each other. So the non-reversing paths correspond exactly to the group elements which are not conjugate to their own inverses.
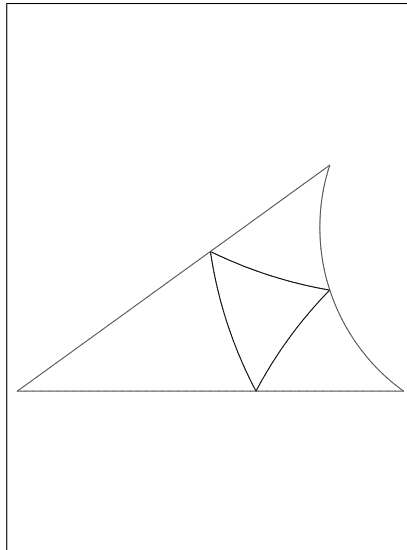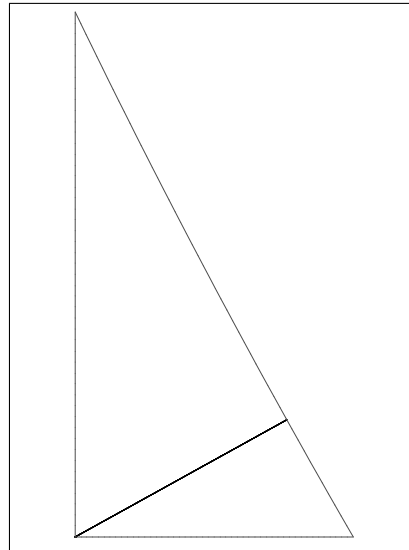

Figure 12: A non reversing 5-5-5 billiard.

Figure 13: A reversing 2-3-7 billiard.

According to the path production rules, there are three types of reversals corresponding to the three types of bounces. When a path meets an edge, the angle of incidence is equal to the angle of reflection, so the path reverses itself there if and only if the angle is right. When a path enters an even vertex, it rotates 180° and returns along itself: all paths through the 2-vertex are reversing. When a path enters an odd vertex, it reflects over every edge at the vertex and returns reflected over the angle bisector, so it reverses if and only if it *is* the angle bisector. In the $(2, 3, 7)$ triangle this actually happens at both odd edges: see Appendix 1. Obviously the bisector cases, if they exist are unique.

In the first case the reflection in the wall will conjugate a translation or glide into its inverse. In the second case there be a half turn at the point of order two which will conjugate a translation or glide to its inverse. In the third case there will be a reflecting wall at the vertex which is perpendicular to the billiard.

We may summarize the above in the following proposition.

Proposition 24 *Let B be a primitive closed billiard and C the corresponding conjugacy class of primitive glides or translations in Λ^*. Then B is orientable if and only if the class C contains its own inverses.*

The following proposition proves a peculiarity observed about primitive glides.

Proposition 25 *If the billiard of a primitive glide is non-orientable, then it passes through the origin and a perpendicular edge.*

Proof. We use a parity analysis. A reversing word is of form $wxw^{-1}y$ where w is a general element of Λ^*, and x and y are either p, q, r, or pq, since these are all the reversals except the two bisectors of B and C. Reversing at B or C, we get $x = qrq$ or $prprprp$, respectively, so we can take $w' = wq$ or $wprp$, and $x' = r$. Since w and w^{-1} are of equal parity, the parity of $wxw^{-1}y$ is odd if and only if x and y are of opposite parity. This forces exactly one of them to be pq. So the glide must reverse at the origin once and once somewhere else. The unique billiards corresponding to the bisectors of P and Q are known and even, so the only possible odd paths reverse once at the origin and once at a perpendicular edge. ∎

4.2   Wordlength 3

Proposition 26 *A tiling has a non-reversing path of wordlength 3 if and only if the tiling does not have a 2-vertex.*

Proof. Recall that a non-reversing path is orientable, that is, the non-reversing paths correspond to those glides and translations which are not conjugate to their inverses, as discussed in subsection 4.1.
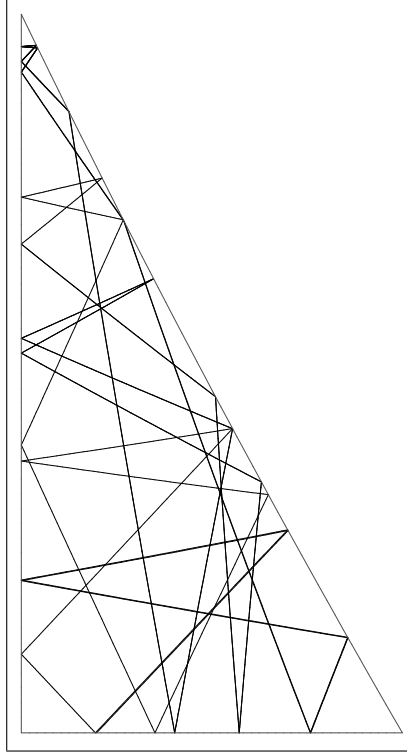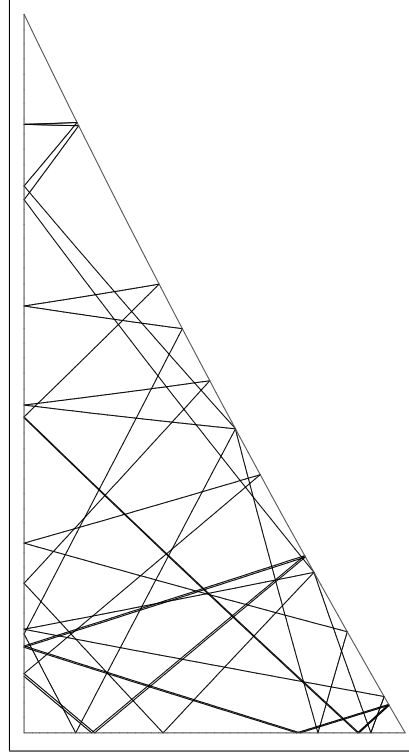
The only possible paths of wordlength 3 are those involving each reflection once, since the others are conjugate to the generating reflections. By renaming appropriately, we can assume that the path is pqr. Suppose the path pqr is conjugate to its inverse, where the inverse of pqr is rqp and the conjugates of the inverse are qpr and prq.

If $pqr = qpr$ then $pq = qp$ which means that $a^2 = 1$, therefore $ord(a) = 2$ and the tiling contains a 2-vertex. Conversely, if the tiling contains a 2-vertex at R then $ord(a) = 2$, so $pq = qp$, which implies that $pqr = qpr$.

If $pqr = prq$ then $qr = rq$ and $b^2 = 1$, therefore $ord(b) = 2$ and the tiling contains a 2-vertex. Conversely, if the tiling contains a 2-vertex at P then $ord(b) = 2$, so $qr = rq$, which implies that $pqr = prq$.

If $pqr = rqp$ then $qrpq = pr$ and $\theta(c) = c^{-1}$. Since

$$ba = \theta(b^{-1}a^{-1}) = \theta(c) = c^{-1} = ab,$$

we conclude that ba commutes with ab. Since any translation can be written using only powers of a and b, if

$$x = a^{i_1} b^{j_1} a^{i_2} b^{j_2} \cdots,$$

where i_1, j_1, i_2, j_2,... are integers, then

$$x = a^\alpha b^\beta,$$

where $\alpha = i_1 + i_2 + \ldots$ and $\beta = j_1 + j_2 + \ldots$. As $\Lambda = \langle a, b, \rangle$ then $\Lambda = Z_l \times Z_m$, a finite group, contradicting the fact that Λ has elements of infinite order.

If the tiling contains a 2-vertex at Q then $ord(c) = 2$, so $rp = pr$, which means that $qrp = qpr$. Since pqr is conjugate to qrp and rqp is conjugate to qpr, we conclude that pqr and rqp are conjugate. ∎

Figures 12 and 13 showed the 3-paths in the $(5, 5, 5)$ (orientable) and $(2, 3, 7)$ (non-orientable) cases.

4.3   Existence of NRP's in $(2, 3, 7)$

Given the obvious aberration in (2,m,n) tilings stated in Proposition 26, we began investigating the existence of NRP's in the $(2, 3, 7)$ tiling. They do exist, but not in the initial portion of the spectrum. Our search algorithm produced the following minimal-wordlength examples of non-reversing paths: Even:

Wordlength 30, translation length 4.8418,
$w = pqrpqrpqrprpqrpqrpqrpqrprpqrpr$
See Figure 14.

Odd: Wordlength 33, glide length 5.3822,
$w = pqrpqrpqrpqrpqrprpqrpqrpqrprpqrprpqrpqrpqrpqrpqrprpqrpqrpqrprpqrpr$
See Figure 15.

Figure 14: Even non-reversing path

Figure 15: Odd non-reversing path.

5 Translations of form $x\theta(x)$

Note 27 *For all x in Λ,*

$$x\theta(x) = xqxq = (xq)^2$$

Proposition 28 *Words of the form $x\theta(x)$ are translations for any x in Λ.*

Proof. The reflection q corresponds to reflection over the x-axis, or complex conjugation. So, $x\theta(x) = xqxq = x\bar{x}$, when considered as matrices. Let

$$x = \left[\begin{array}{cc} X & Y \\ \bar{Y} & \bar{X} \end{array} \right].$$

Then

$$x\theta(x) = \left[\begin{array}{cc} X\bar{X} + Y^2 & X(Y + \bar{Y}) \\ \bar{X}(Y + \bar{Y}) & \bar{Y}^2 + \bar{X}X \end{array} \right]$$

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This has trace
$$2X\bar{X} + Y^2 + Y\bar{Y}$$
Since x has determinant 1, $X\bar{X} - Y\bar{Y} = 1$, so the trace is
$$2 + Y^2 + Y\bar{Y} + \bar{Y}^2 = 2 + 4\Re(Y)^2 \geq 2$$
If $Re(Y) = 0$ then $Y + \bar{Y} = 0$ and $Y\bar{Y} = -Y^2 = \bar{Y}^2$ and hence $x\theta(x)$ is the identity, a trivial translation. Otherwise, the trace of $x\theta(x)$ exceeds 2 and so $x\theta(x)$ must be a translation of positive translation length. ∎

Note 29 *The above does not hold for all x in Λ^*. For example, let $x = r$, then $x\theta(x) = rqrq = b^{-2} = b$, a rotation.*

Proposition 30 *The translation corresponding to an odd-bouncing path can be expressed as $x\theta(x)$ for some x in Λ.*

   Proof. The set of elements in Λ^* not in Λ, the reflections and glides, have odd p, q, r wordlength. Any glide or reflection h may be written in the form xq for some x in Λ, by taking $x = hq$. Any primitive translation $g \in \Lambda$ coming from an odd-bouncing path is a square of a glide or reflection in Λ^*, as we need to make two circuits of the billiard. But then $g = (xq)^2 = x\theta(x)$. ∎

Note 31 *Although every odd path can be represented in Λ by some element of form $x\theta(x)$, it is not true that every word corresponding to an odd path is of this form. For instance, $rprpqrpq$ is conjugate to $rpqrpq = c\theta(c)$, but $rprpqrpq$ cannot be written as $x\theta(x)$.*

Note 32 *It appears that for sufficiently long paths (and for powers of shorter paths), x can always be taken to be a translation. We do not know whether this is true in general.*

6 Further Questions

6.1 Other Triangles

Take odd integers m and n not equal to 3 and 7. What are the low-length conjugacy classes of the $(2, m, n)$ tiling group? Do the angle bisectors exist? Do non-reversing paths exist? If so, what are the minimal wordlengths at which they appear (for instance, does there exist a triangle with a 5-bouncing NRP)?

6.2 Inductive θ Generation

Let $t_0 = identity$ and $t_n = x_n t_{n-1}\theta(x_n)$ for $n = 1, 2, 3, \ldots$ and some $x_n \in \Lambda$. Now $t_1 = x_1\theta(x_1)$ is a translation, but what happens for $n > 1$? An attempt at a proof of the same form as for $n = 1$ did not yield comprehensible results. At low wordlengths, t_n appears to be a translation. Is it always a translation? Does this inductive procedure produce all translations? Does it produce a specific representative of each conjugacy class?

6.3 $x\theta(x)$ forms

Which elements of a conjugacy class can be written as $x\theta(x)$? Can a power of every class be constructed with x a translation?

6.4 Cyclic Reduction

Cyclic reduction is a stronger reduction scheme than word reduction (equality in Λ^*) but weaker than conjugacy. When will two words have the same cyclic reduction? Does cyclic reduction produce an interesting rewriting system? Can we get to the canonical conjugate by adding additional algebraic/combinatorial rules to the cyclic reduction procedure?

6.5 Canonical Representatives

Given some word representing a translation or glide, find an efficient algorithm to compute the canonical representative of its conjugacy class. It is sufficient to find the canonical representative from the path representative, since it is easy to construct the path representative from any word.

Is there a purely algebraic way to find the path representative (not taking fixed points, intersections, etc.)?

6.6 Length Spectrum

The length spectrum of the $(2, 3, 7)$ tiling is the set of all lengths of translations. Determine its properties as a set of real numbers (e.g. does it grow uniformly as n increases?) We know several non-conjugate paths whose length is the same. Many (but not all) include one orientable and one non-orientable path. What is the multiplicity function on the length spectrum? Is there an easy way to predict when two paths have the same length?

6.7 Wordlength-Path Number Relations

The number of paths of wordlength n stays at 1 until 23, and then seems to be essentially increasing as n grows. How quickly does it grow? Is the number of paths a nice function of the wordlength?

7   Multiplicities

In the table below the partial results for the number of reduced words, geometrically distinct billiard paths, and non-reversing paths (NRP's) is given as a function of the number of bounces.

Bounces	#Reduced words	#Paths	#NRP's	Bounces	#Reduced words	#Paths	#NRP's
1	0	0	0	26	3	2	0
2	0	0	0	27	4	3	0
3	1	1	0	28	4	3	0
4	0	0	0	29	4	4	0
5	1	1	0	30	6	3	1
6	0	0	0	31	5	5	0
7	1	1	0	32	6	5	0
8	0	0	0	33	8	5	1
9	1	0	0	34	8	6	1?
10	1	0	0	35	9	6	?
11	1	1	0	36	11	7	?
12	1	0	0	37	11	10	?
13	1	1	0	38	13	11	?
14	1	0	0	39	16	11	?
15	2	0	0	40	18	12	?
16	1	1	0	41	19	15	?
17	1	1	0	42	24	15	?
18	2	1	0	43	25	19	?
19	1	1	0	44	30	20	?
20	2	1	0	45	35	24	?
21	2	0	0	46	38	?	?
22	2	1	0	47	44	?	?
23	2	2	0	48	53	?	?
24	3	2	0	49	58	?	?
25	3	1	0	50	69	?	?

8 Some Important Paths (Figures)

8.1 Initial Classes

Here are the billiards of 20 bounces or less. Figures the classes with 30 or less bounces may be found at classes may be found at [11].
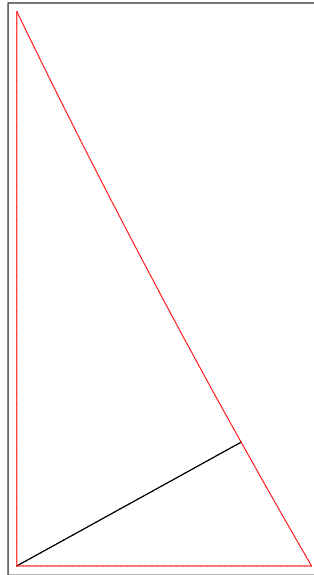
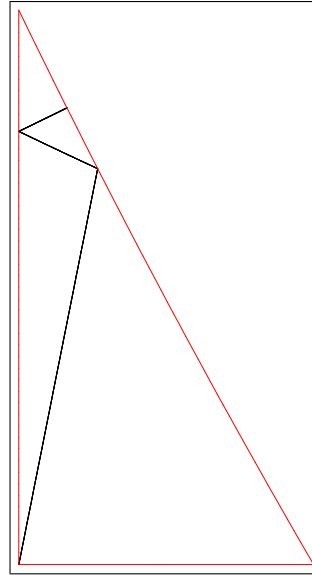$g = pqr, l_w(g) = 3,$
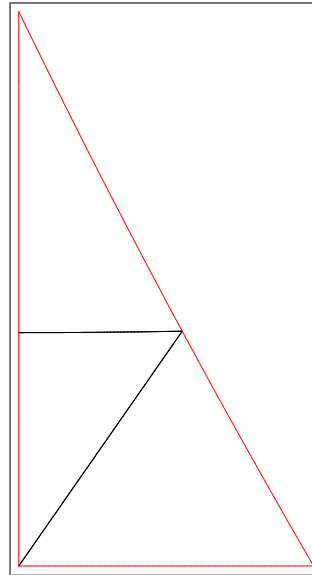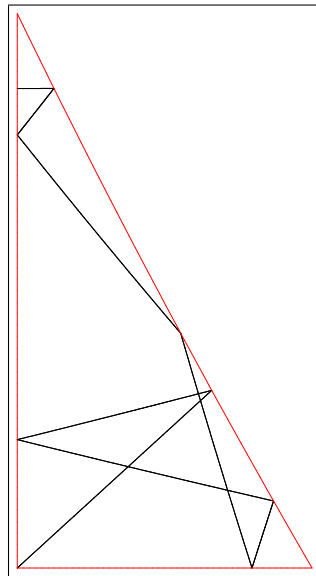

$g = pqrpr, l_w(g) = 3$
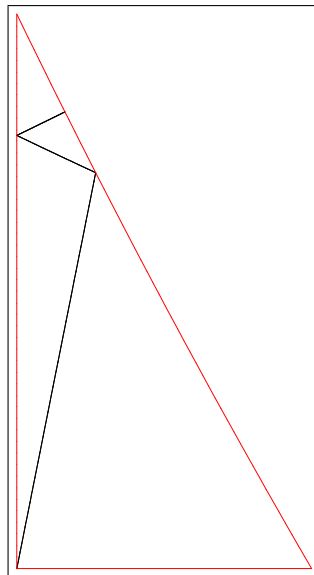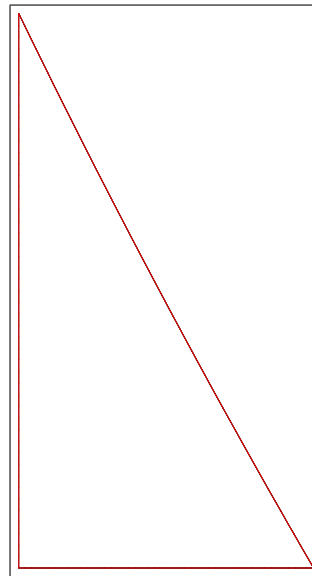

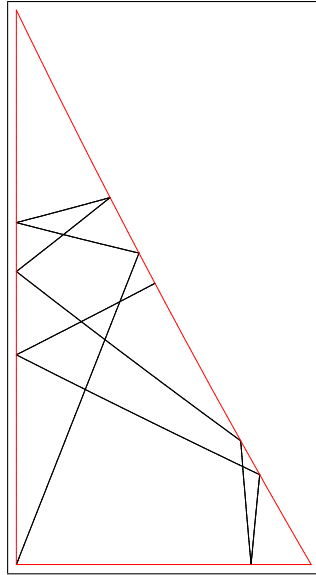$g = pqrprpr, l_w(g) = 7,$
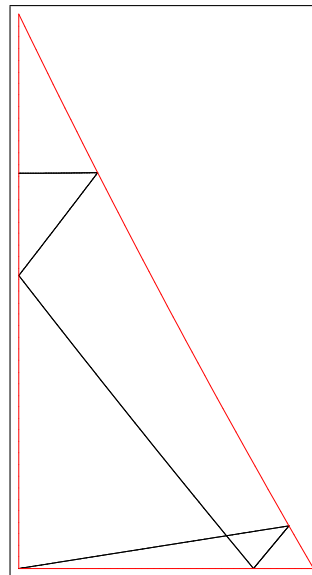
$g = pqrpqrpqrpr, l_w(g) = 11$
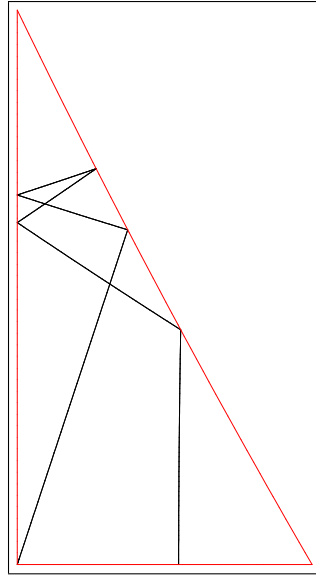
$g = pqrpqrpqrpqrpr,,$
$l_w(g) = 14$

$g = pqrpqrpqrprpqrpr,$
$l_w(g) = 16$

$g = pqrpqrpqrpqrpqrpr,,$
$l_w(g) = 17,$

$g = pqrpqrprpqrprpqrpr$
$l_w(g) = 18$
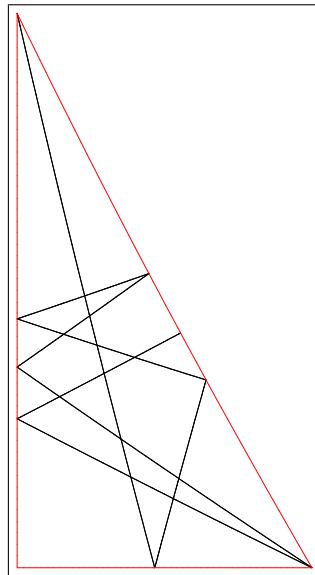exceptional perimeter billiard

$g = pqrpqrpqrpqrprpqrpr,,$
$l_w(g) = 19$
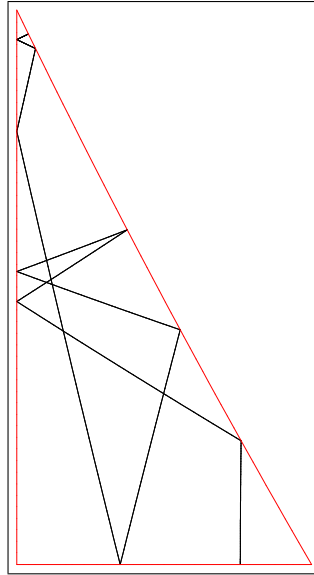
$g = pqrpqrpqrpqrpqrpqrpr,$
$l_w(g) = 20$

8.2 Billiards of Odd Angle Bisectors
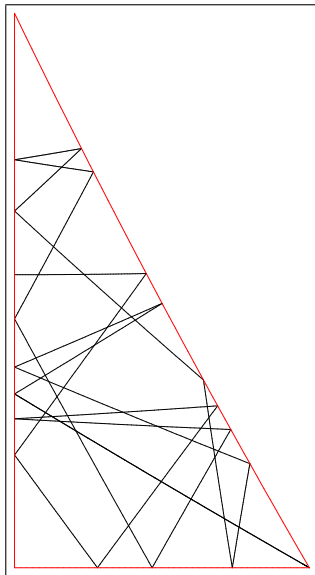
$g = qrqprprqrprprpqrprqprp,,$
$l_w(g) = 22$
vertex of order 3

$g = qrprpqrqprpqrqprprqrprprpr,$
$l_w(g) = 26$
vertex of order 7

8.3   Distinct Billiards with the Same Number of Bounces

Up to 22 bounces there are no examples of geometrically distinct billiards with the same number of bounces. For 23 bounces there are 2 classes. The billiard lengths are also given.
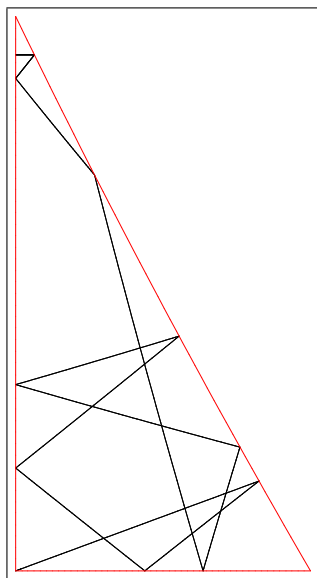
$g = pqrpqrpqrpqrpqrpqrpqrpr,,$
$l_w(g) = 23$
billiard length: 3.679159266
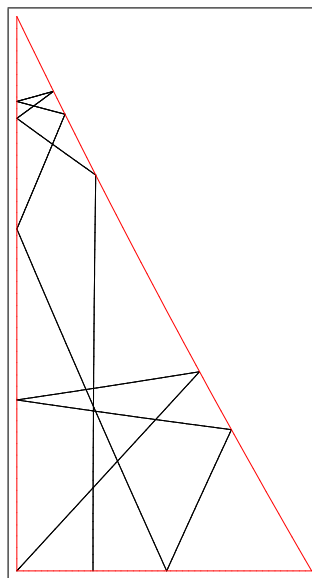
$g = pqrpqrprpqrprpqrprpqrpr,$
$l_w(g) = 23$
billiard length: 3.799186511

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