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A Classification of Quadratic Rook Polynomials

Alicia Velek

Abstract. Rook theory is the study of permutations described using terminology from the game of chess. In rook theory, a generalized board B is any subset of the squares of an $n \times n$ square chessboard for some positive integer n. Rook numbers count the number of ways to place non-attacking rooks on a generalized board, and the rook polynomial is a polynomial that organizes the rook numbers for a board B. In our research, we classified all quadratic polynomials that are the rook polynomial for some generalized board B.

Acknowledgements: Samantha Tabackin

1 Introduction to Rook Theory

Classical rook theory was first introduced in the 1940's by Riordan and Kaplansky, who wished to study permutations with restricted position. In chess, a rook can attack any square in its corresponding row or column of the 8×8 chessboard. Rook theory focuses on the placement of non-attacking rooks in a more general situation.

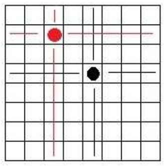

Figure 1: A chess board with two non-attacking rooks.

Definition: We define a *board* to be a square $n \times n$ chessboard for any $n \in \mathbb{N}$. A *generalized board* is any subset of squares of the board. Thus, a generalized board can be any arrangement of squares that is completely contained inside a board, or it can be the board itself.

We use ordered pairs in the first quadrant of $\mathbb{R}^2$ to label squares on our $n \times n$ board. For example, $(1, 1)$ denotes the square in the bottom left corner of the board, $(2, 3)$ denotes the square in the second column from the left and third row from the bottom, and (n, n) denotes the square in the top right corner of the board.

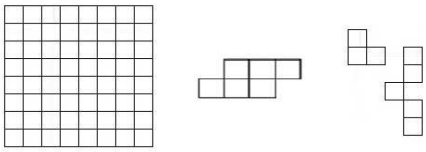

Figure 2: A board and two generalized boards.

Definition: A *rook placement* is an arrangement of some number of non-attacking rooks on some board. Note that since rooks attack squares in their row and column, a rook placement cannot have more than one rook in a given row or column.

A placement of n rooks on an $n \times n$ square board can be associated to a permutation $\sigma = \sigma_1 \ldots \sigma_n$ of $1, 2, ..., n$ by saying the placement P_σ has a rook on the square (i, j) of the board iff $\sigma_i = j$.

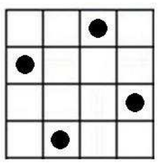

Figure 3: Example of a permutation.

For example, the board pictured here corresponds to the permutation $\sigma = 3142$.

In this way, we see that any theorem about rook placements is a generalization of a theorem about permutations.

Definition: The k^{th} *rook number*, $r_k(B)$, counts the number of ways to place k non-attacking rooks on a generalized board B. We will often denote $r_k(B)$ as r_k when B is clear.

Some notes about rook numbers:

1. r_0 is always 1 because there is only one way to place 0 rooks on a generalized board.
2. r_1 is always the number of squares of B because a single rook can be placed in any square of B with no other rook to attack it.
3. Since a rook attacks all squares in its row and column, each rook in a rook placement must be in a different row or column.
4. Once we attain $r_k = 0$, we will always have $r_{k+1}, r_{k+2}, ... = 0$. For example, if we are unable to place 3 non-attacking rooks on a generalized board, we cannot place 4 or more non-attacking rooks on the generalized board.
5. If B is contained in an $n \times n$ square board and $k > n$, then we have that $r_k = 0$. However, note that r_k could be equal to 0 for smaller values of k as well.

Definition: We can construct a polynomial that keeps track of all of the rook numbers of a generalized board at once. Such a polynomial is known as the *rook polynomial in* x and is denoted $R(B, x)$. The r_k's are the coefficients of the x^k terms, as shown:

$$R(B, x) = r_0(B) + r_1(B)x + r_2(B)x^2 + \ldots + r_{n-1}(B)x^{n-1} + r_n(B)x^n$$

Example 1: Consider the generalized board B in Figure 4. By our comments above, it follows that $r_0 = 1$ and $r_1 = 6$. Moreover, there are 9 ways to place 2 non-attacking rooks on B, as shown below; therefore, $r_2 = 9$. Similarly, we see $r_3 = 3$ because there are 3 ways to place 3 non-attacking rooks on B. However, there are only 3 rows in B, so $r_4, r_5, r_6, .. = 0$.

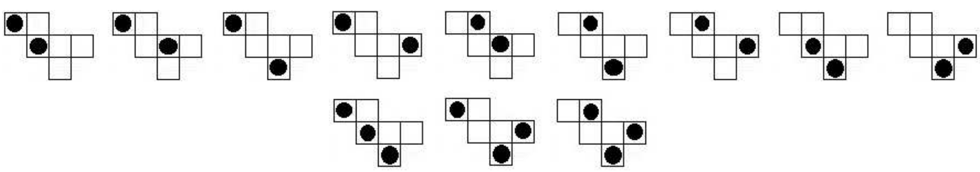

Figure 4: Example of a generalized board.

Thus, after obtaining these rook numbers, the corresponding rook polynomial for this board is: $1 + 6x + 8x^2 + 3x^3$.

Definition: Two boards are *rook equivalent* if they have the same rook polynomial (and hence, the same rook numbers).

Example 2: The board in Figure 5 is rook equivalent to the board in Example 1. Verification is left to the reader.

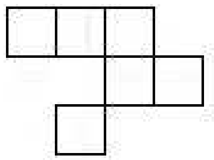

Figure 5: Example of an equivalent board.

Our problem is to classify all quadratic polynomials which are the rook polynomial for some generalized board B. That is, given a first rook number r_1, we can find all corresponding possibilities for r_2 and, thus, find all possible quadratic rook polynomials for a board B. The idea of rook equivalency will prove to be very important in allowing us to simplify the number of boards that we need to consider to address our given problem.

Mathematicians again became interested in the ideas of rook theory in the late 1990's due to its strong applications in enumerative combinatorics. In this paper, this is the relation that we will be focusing on. In Section 2, we provide the necessary background to work with boards that satisfy our desired conditions. We prove several theorems regarding rook equivalency that allow us to minimize the number and types of boards that we need to work with. In Section 3, we focus on the classification of all quadratic polynomials. We provide a theorem that introduces a formula that we can use to solve our problem. We conclude with an example that utilizes our formula and offer ideas for future research and expansion.

2 Boards with $r_3 = 0$

In this section, we will prove several theorems that lead to the classification of all quadratic polynomials that are the rook polynomial of some generalized board B. In order for a board B to have a quadratic rook polynomial, we want $r_3(B) = 0$. Then, by note 4 above, $r_k(B) = 0$ for all $k \geq 3$. Moreover, we also must have $r_2(B) > 0$ to ensure that our polynomial is quadratic and not linear. All boards satisfying these properties are classified by the following theorem.

Theorem 1: Let B be a nonempty board such that $r_3(B) = 0$. Then B satisfies at least one of the following criteria:

1. B is contained in two or fewer rows;

2. B is contained in two or fewer columns;

3. B is the union of one part of the board contained in one row and another part contained in one column. An example of such a board is shown in Figure 6.

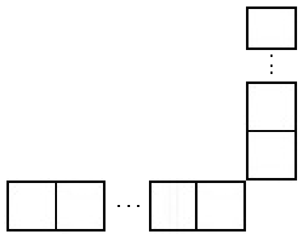

Figure 6: A board meeting criterion 3.

Proof: We will prove by contrapositive. Hence, we aim to show that if a board B does not satisfy any of the properties above, then $r_3 \neq 0$.

Assume B does not satisfy any of the listed criteria. So B has at least three rows, at least three columns, and does not have one part of the board contained in one row and one part contained in one column. These conditions force B to have three squares each in a different row and column from the other squares. Non-attacking rooks can be placed on these squares, so $r_3(B) \neq 0$, as desired. □

Now our goal is to show that for the purposes of our classification of quadratic rook polynomials, we only need to consider boards contained in two rows. We do this in the following two theorems.

Theorem 2: A generalized board contained in two columns is rook equivalent to one contained in two rows.

Proof: If B is a generalized board contained in two columns of an $n \times n$ board, let B' be the board obtained by rotating B 90° clockwise, as shown in Figure 7.

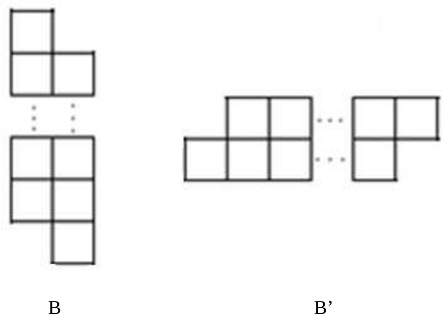

Figure 7: Equivalent generalized boards.

It is clear that B' is contained in two rows, and B and B' have the same rook numbers. Hence, a board contained in two columns is rook equivalent to a board contained in two rows. □

In order to show that a board satisfying criterion 3 of Theorem 1 is rook equivalent to a board contained in two rows, we will use the following theorem.

Theorem 3: Any generalized board with one part contained in one row and another part contained in one column has the same rook polynomial as a board contained within two rows.

Proof: Note that empty rows and columns of the $n \times n$ square board have no effect on rook numbers. Thus, all boards we consider will have all empty rows and columns removed. We have two cases to consider.

Case 1: The squares in the column cannot attack any square in the row. Note that changing the position of any column has no effect on the rook numbers. In this case, the board pictured below on the left is rook equivalent to the board pictured on the right because $r_0(B) = r_0(B') = 1$ and $r_1(B) = r_1(B') = a + b$, or the number of squares of each board. To find r_2 for either board, we place one rook in any of the a squares in the bottom row. Then there are b choices for a second non-attacking rook. Hence, $r_2(B) = r_2(B') = ab$. Last, $r_k(B) = r_k(B') = 0$ for $k \geq 3$.

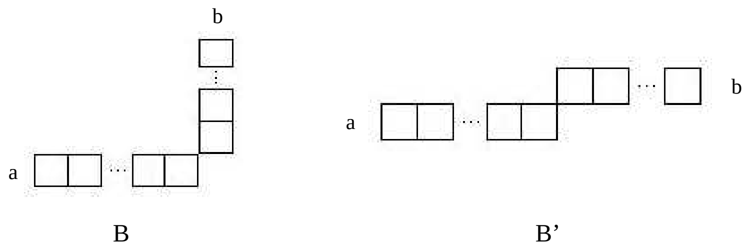

Figure 8: Rook equivalent boards as in case 1.

Case 2: The squares in the column can attack squares in the row. Such a board is pictured on the left in Figure 9. We will show that this board is rook equivalent to the board on the right in Figure 9, which is contained in two rows.

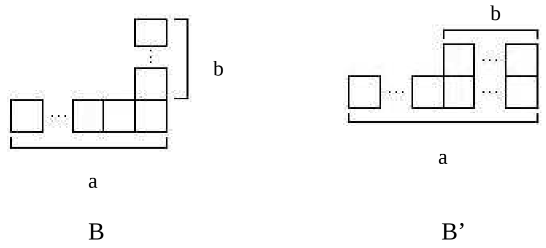

Figure 9: Rook equivalent boards as in case 2.

We see that $r_0(B) = r_0(B') = 1$ and $r_1(B) = r_1(B') = a + b$.

Consider the board on the left in Figure 9. We can place our first rook in any of the first $a - 1$ squares in the bottom row. Then we have b squares to place a second non-attacking rook. Hence, $r_2(B) = (a - 1)b = ab - b$.

Now consider the board on the right in Figure 9. We can see that b is equal to the number of columns where the two rows overlap. We can place our first rook in the first $a - b$ squares in the bottom row, leaving b squares in which to place a second non-attacking rook. We could also place our first rook in one of the b squares in the bottom row, leaving $b - 1$ squares in which to place a second non-attacking rook. Hence, $r_2(B) = (a - b)b + b(b - 1) = ab - b$.

Thus, $r_2(B) = r_2(B') = ab - b$. Also note that $r_k(B) = r_k(B') = 0$ for $k \geq 3$. Hence, the two boards are rook equivalent and have the same rook polynomial, as desired. $\square$

3 Classification of Quadratic Rook Polynomials

Now we move on to the classification of all quadratic polynomials which are the rook polynomial for some generalized board B. Suppose $r_0(B) + r_1(B)x + r_2(B)x^2$ is a quadratic rook polynomial for some board B. Recall that $r_0(B) = 1$ for any board B and $r_1(B)$ is the number of squares of B. Thus, $r_1(B)$ can be any positive integer greater than 1. We must also have $r_2(B) \geq 1$ and $r_k(B) = 0$ for $k \geq 3$. If $r_3(B) \neq 0$, our rook polynomial would be of a higher degree. Hence, the polynomials that we seek have the form $1 + r_1(B)x + r_2(B)x^2$.

Thus, taking into account these equivalences and the requirement of $r_3(B) = 0$, we found that it suffices to consider generalized boards that lie within two rows of a board and have spaces which lie consecutively within each row in order to find the desired rook polynomials.

The above discussion reduces the classification in proving the following theorem.

Theorem 4: Let B be a generalized board contained in two rows whose rook polynomial is quadratic. Given $r_1(B)$, every possible $r_2(B)$ will have the form $r_2(B) = a(r_1(B)-a) - i$, for $1 \leq a \leq \lfloor \frac{r_1}{2} \rfloor$ and $0 \leq i \leq a$.

Proof: Given any generalized board B satisfying the conditions stated in the first paragraph of Section 3 and given its analogous value of $r_1(B)$, we want to find all possible corresponding values of $r_2(B)$ by considering each pair of positive integers a and b of $r_1(B)$ such that $a+b = r_1(B)$. Without loss of generality, let's assume that $a \leq b$. This will imply that $1 \leq a \leq \lfloor \frac{r_1}{2} \rfloor$. Note that $b = r_1 - a$. Here a and b represent the number of squares in the two rows that B occupies, respectively.

Then the a and b squares can be arranged such that a squares lie consecutively in one row and b squares lie consecutively in the next row. We can rearrange the columns of a generalized board B so that i columns containing squares in both rows lies in the center, columns with empty squares in row 1 lie adjacent on the right, and columns with empty squares in row 2 lie adjacent on the left, as illustrated in Figure 10. Such a rearrangement will not change the rook numbers.

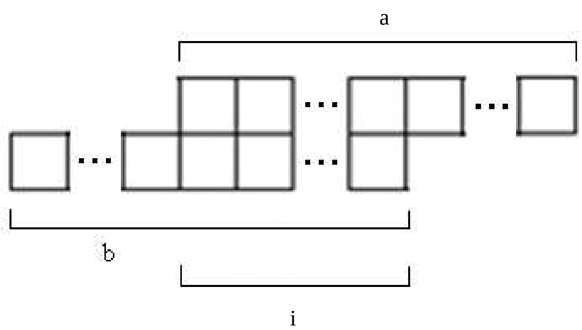

Figure 10: A generalized board B.

By placing one rook in each row, the possible values of $r_2(B)$ can be found for each value $0 \leq i \leq a$. The maximum value of r_2 will be when $i = 0$ (a and b are completely disjoint), or $r_2(B) = ab$. The minimum value of $r_2(B)$ will be when $i = a$ (a and b completely overlap), or $r_2(B) = a(b-1)$.

Every value between $r_2(B) = ab$ and $r_2(B) = a(b-1)$ can be obtained as well. Beginning with a disjoint pair, shift the row containing a squares one square to the left, which decreases the value of r_2 by one. Continuing this process until a and b are overlapping exhausts all possible values of r_2. Hence, for a given a and b, each choice of i will yield a different possible $r_2(B)$ value.

For any particular arrangement, the value of $r_2(B)$ is equal to $(a-i)(b) + (i)(b-1)$, which simplifies to $ab - i$. Recall that $b = r_1(B) - a$.

Thus, given a first rook number $r_1(B)$, every possible $r_2(B)$ will have the form $r_2(B) = a(r_1(B) - a) - i$, for $1 \leq a \leq \lfloor \frac{r_1}{2} \rfloor$ and $0 \leq i \leq a$. $\square$

Example of Finding $r_2(B)$ Values: Suppose we wanted to classify all quadratic rook polynomials with $r_1(B) = 10$. The possible integer pairs for a and b are: 1, 9; 2, 8; 3, 7; 4, 6; 5, 5. Apply the formula $r_2(B) = a(r_1(B) - a) - i$ for each pair.

$(1)(9) - 0 = 9$	$(1)(9) - 1 = 8$	$(2)(8) - 0 = 16$	$(2)(8) - 1 = 15$
$(2)(8) - 2 = 14$	$(3)(7) - 0 = 21$	$(3)(7) - 1 = 20$	$(3)(7) - 2 = 19$
$(3)(7) - 3 = 18$	$(4)(6) - 0 = 24$	$(4)(6) - 1 = 23$	$(4)(6) - 2 = 22$
$(4)(6) - 3 = 21$	$(4)(6) - 4 = 20$	$(5)(5) - 0 = 25$	$(5)(5) - 1 = 24$
$(5)(5) - 2 = 23$	$(5)(5) - 3 = 22$	$(5)(5) - 4 = 21$	$(5)(5) - 5 = 20$

Therefore, a generalized board with 10 squares can obtain every value for $r_2(B)$ between 8 and 25 except for $10, 11, 12, 13$, and 17.

The possible rook polynomials for $r_1(B) = 10$ are as follows:

$1 + 10x + 8x^2$	$1 + 10x + 9x^2$	$1 + 10x + 14x^2$	$1 + 10x + 15x^2$
$1 + 10x + 16x^2$	$1 + 10x + 18x^2$	$1 + 10x + 19x^2$	$1 + 10x + 20x^2$
$1 + 10x + 21x^2$	$1 + 10x + 22x^2$	$1 + 10x + 23x^2$	$1 + 10x + 24x^2$
$1 + 10x + 25x^2$			

This process can be completed for any value of $r_1(B)$.

Conclusion:

The following is a table of all quadratic rook polynomials with $r_1(B) < 10$. Note that we must begin with $r_1(B) = 2$.

$1+2x+x^2$	$1+3x+x^2$	$1+3x+2x^2$	$1+4x+2x^2$
$1+4x+3x^2$	$1+4x+4x^2$	$1+5x+3x^2$	$1+5x+4x^2$
$1+5x+5x^2$	$1+5x+6x^2$	$1+6x+4x^2$	$1+6x+5x^2$
$1+6x+6x^2$	$1+6x+7x^2$	$1+6x+8x^2$	$1+6x+9x^2$
$1+7x+5x^2$	$1+7x+6x^2$	$1+7x+8x^2$	$1+7x+9x^2$
$1+7x+10x^2$	$1+7x+11x^2$	$1+7x+12x^2$	$1+8x+6x^2$
$1+8x+7x^2$	$1+8x+9x^2$	$1+8x+10x^2$	$1+8x+11x^2$
$1+8x+12x^2$	$1+8x+13x^2$	$1+8x+14x^2$	$1+8x+15x^2$
$1+8x+16x^2$	$1+9x+7x^2$	$1+9x+8x^2$	$1+9x+12x^2$
$1+9x+13x^2$	$1+9x+14x^2$	$1+9x+15x^2$	$1+9x+16x^2$
$1+9x+17x^2$	$1+9x+18x^2$	$1+9x+19x^2$	$1+9x+20x^2$

By Theorem 4, we now see that a quadratic polynomial is the rook polynomial of a generalized board iff it has the form:

$$1 \,+\, r_1(B)x \,+\, [a(r_1(B) - a)i]x^2$$

for positive integers $r_1(B)$, a, i, where $r_1 > 1$, $1 \leq a \leq \lfloor \frac{r_1}{2} \rfloor$, and $0 \leq i \leq a$.

Future Research:

One major source of future research is a classification of higher degree polynomials, most notably the cubic case. However, it becomes increasingly more difficult to restrict larger generalized boards in order to obtain their analogous higher degree rook polynomials.

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[2] Irving Kaplansky and John Riordan, *The Problem of the Rooks and Its Applications.* Duke Mathematics Journal, 13 (1946) 259-268.