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Abstract. This paper describes, in detail, a process for constructing Kummer K3 surfaces, and other "generalized" Kummer K3 surfaces. In particular, we look at how some well-known geometrical objects such as the platonic solids and regular polygons can inspire the creation of singular surfaces, and we investigate the resolution of those surfaces. Furthermore, we will extend this methodology to examine the singularities of some complex two-dimensional quotient spaces and resolve these singularities to construct a Kummer K3 and other generalized Kummer K3 surfaces.

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1 Introduction

K3 surfaces are an important class of compact complex surfaces that have been studied since the 19th century. Kummer surfaces are particular examples of K3 surfaces that are constructed by taking the quotient of an abelian surface by an involution, and resolving the 16 singular points. Our goal in this paper is to generalize Kummer's construction by taking the quotient of an abelian surface by a finite group G of order greater than two, and resolving the singularities. The resulting surfaces we call "generalized" Kummer surfaces.

The first step is to classify the finite subgroups of $SU(2)$ (the set of complex unitary matrices with determinant one). Later we will show that an element of $SU(2)$ can act on the ring $\mathbb{C}[u, v]$. In fact, for each subgroup, Γ, of $SU(2)$ there exist certain polynomials in $\mathbb{C}[u, v]$ that are invariant under this action of Γ. Moreover, there are precisely three such fundamental invariants, and they satisfy an algebraic relation [4]. If we take f to be the polynomial in $\mathbb{C}[x, y, z]$ that describes this relationship and then consider f as a complex subvariety in $\mathbb{C}^3$, we will find that f has a simple singularity at the origin and is isomorphic to $\mathbb{C}^2/\Gamma$ [4], i.e., the surface defined by the association of the orbits of Γ. The next step is to resolve this singularity through the method of successive blow-ups, and describe the resulting surface and represent its exceptional locus with a Dynkin diagram. Finally, for each finite subgroup Γ we will look for a rank four lattice Λ_Γ in $\mathbb{C}^2$ that is preserved under left multiplication by elements of Γ. For each subgroup, Γ, if such a lattice, Λ_Γ, exists we will find all fixed points of the space $T_\Gamma = \mathbb{C}^2/\Gamma_\Lambda$ under the action of γ for all $\gamma \subset \Gamma$. Then, the final step is to look at the quotient space $T_\Gamma/\Gamma = (\mathbb{C}^2/\Gamma_\Lambda)/\Gamma$ and resolve the resulting singularities (which occur at the fixed points of γ for all $\gamma \subseteq \Gamma$) via successive blow-ups to obtain a generalized Kummer surface.

In section 2, we classify all finite subgroups of $SU(2)$. In section 3, we find polynomials invariant under a certain action of these groups, and describe the ring of all such invariants for each group, presenting the invariants explicitly for a few of the groups. In section 4, we look at the algebraic relationship between the generators of these rings of invariants. In section 5, we describe the singular varieties given by the zero sets of these algebraic relationships. We then describe how to blow-up these singularities. In section 6, we describe how to completely resolve the singularities appearing in these surfaces through the method of successive blow-ups and include the Dynkin diagrams which represent these chains of blow-ups. In sections 7 and 8, we find lattices in $\mathbb{C}^2$ which are invariant under the action of certain finite subgroups of $SU(2)$. Finally, in section 9, we use the results of previous sections to construct generalized Kummer surfaces.

2 Finding the Finite Subgroups

The first task is to find all the finite subgroups of $SU(2)$. To do this, one can begin by classifying all the finite subgroups of $SO(3, \mathbb{R})$ (the set of real orthogonal matrices with determinant one). Next, we make use of the surjective homomorphism $\Phi : SU(2) \to SO(3,\mathbb{R})$ given by:

$$\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} \to \begin{bmatrix} a^2-b^2-c^2+d^2 & 2ab+2cd & -2ac+2bd \\ -2ab+2cd & a^2-b^2+c^2-d^2 & 2ad+2bc \\ 2ac+2bd & -2ad+2bc & a^2+b^2-c^2-d^2 \end{bmatrix}$$

[7], whose kernel consists of $\pm I$, where I represents the identity element of $SU(2)$. Therefore, we can lift any element, $g \in SO(3,\mathbb{R})$ to an element $\tilde{g} \in SU(2)$ uniquely up to ± 1 such that $\Phi(\tilde{g}) = \Phi(-\tilde{g}) = g$. Therefore we can find any subgroup of $SU(2)$ as follows:

Suppose $G = \langle g_1, g_2, \ldots g_n\rangle$ is a finite subgroup of $SO(3,\mathbb{R})$. Then $\tilde{G} = \langle \tilde{g_1}, \tilde{g_2}, \ldots \tilde{g_n}\rangle$ is a finite subgroup of $SU(2)$ such that $\Phi(\tilde{G}) = G$. If $-I \in \tilde{G}$, then $|\ker(\Phi\restriction_{\tilde{G}})| = 2$, and we have $|\tilde{G}| = 2|G|$. However, if $-I \notin \tilde{G}$, then $|\ker(\Phi\restriction_{\tilde{G}})| = 1$, and we have $|\tilde{G}| = |G|$. For certain $G \subset SO(3,\mathbb{R})$, $-I$ always lies in $\tilde{G}$. For other G (in fact only for the cyclic groups of odd order) whether we choose one lifted generator or its opposite will determine whether $-I$ lies in $\tilde{G}$.

Therefore, once we know all the finite subgroups of $SO(3,\mathbb{R})$ we can lift them to $SU(2)$ to find all the subgroups of $SU(2)$. To find the finite subgroups of $SO(3,\mathbb{R})$, recall that this is the group of matrices that act on $\mathbb{R}^3$ by rotation. One can show that any finite subgroup of $SO(3,\mathbb{R})$ must preserve some symmetrical structure of $\mathbb{R}^3$. It is well known that these structures are three platonic solids; the tetrahedron (self-dual), the cube (dual to the octahedron), and the dodecahedron (dual to the icosahedron—platonic solids that are dual are preserved by isomorphic subgroups, so we need not consider the cases of the octahedron or icosahedron), as well as the n-gon and the two-faced n-gon [5]. From each of these structures, we can find a finite subgroup of $SO(3,\mathbb{R})$ that preserves said structure's symmetry.

2.1 The Cyclic Group: Symmetries of the N-gon.

We lift the generator, r of the cyclic subgroup of order n in $SO(3,\mathbb{R})$ to the generator, $\tilde{r}$ of the cyclic subgroup of order $2n$ in $SU(2)$.

$$\begin{bmatrix} cos(2\pi/n) & sin(2\pi/n) & 0 \\ -sin(2\pi/n) & cos(2\pi/n) & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{bmatrix} \tag{1}$$

where δ is a root of unity of order $2n$. If n is odd we can get another subgroup by choosing to lift:

$$r \to -\tilde{r} = \begin{bmatrix} -\delta & 0 \\ 0 & (-\delta)^{-1} \end{bmatrix}$$

Since n is odd and δ is a $(2n)^{th}$ root of unity, some simple calculations show that $-\delta$ is an n^{th} root of unity. Hence for any m, we can find a cyclic subgroup of order m as follows. If m is even let $r_n \in SO(3,\mathbb{R})$ be the generator given in (1) of the cyclic group of $SO(3,\mathbb{R})$ of order n where $n = \frac{m}{2}$. Now lift $r_n \to \widetilde{r_n}$ where $\widetilde{r_n}$ has order $2n = m$ and thus generates a cyclic group of order m in $SU(2)$. If m is odd, let $r_n \in SO(3,\mathbb{R})$ be the generator given in (1) of the cyclic group of $SO(3,\mathbb{R})$ of order n where $n = m$. Then lift $r_n \to -\widetilde{r_n}$. Since $\widetilde{r_n}$ has order $2n = 2m$, $-\widetilde{r_n}$ has order $n = m$, and thus generates a cyclic group of order m in $SU(2)$.

Lemma 1. *Let $m \in N$. Let β be an m^{th} root of unity in $\mathbb{C}$. Then there exists a cyclic group of order m in $SU(2)$ generated by an element, $\tilde{R}$, of the form:*

$$\mathcal{C}_m = \langle \tilde{R} \rangle = \langle \begin{bmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{bmatrix} \rangle$$

where we have used the upper case to differentiate from $\widetilde{r_n}$ (which we will always assume to be of even order, i.e., $2n$ for some $n \in N$).

2.2 The Binary Dihedral Group: Symmetries of the Two-faced N-gon.

Letting the generators of the representation of the dihedral group in $SO(3,\mathbb{R})$ be r, s as usually defined, we again lift r to $\tilde{r}$, and now lift s to $\tilde{s} \in SU(2)$:

$$s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \tilde{s}$$

At this point we again need to ask whether we will get additional subgroups if we replace $\widetilde{r}$ by $\tilde{R}$ of arbitrary order. To answer this, consider the two subgroups:

$$\Gamma_1 = \langle \begin{bmatrix} \alpha_n & 0 \\ 0 & \alpha_n^{-1} \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \rangle \qquad \Gamma_2 = \langle \begin{bmatrix} \alpha_{2n} & 0 \\ 0 & \alpha_{2n}^{-1} \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \rangle$$

where n is an odd integer, α_n an n^{th} root of unity, and α_{2n} is a $(2n)^{th}$ root of unity. At first glance, one might expect that Γ_2 would have twice as many elements as Γ_1. However, note that:

$$\begin{bmatrix} \alpha_n & 0 \\ 0 & \alpha_n^{-1} \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}^2 = \begin{bmatrix} -\alpha_n & 0 \\ 0 & -(\alpha^{-1}) \end{bmatrix} = \begin{bmatrix} (-\alpha_n) & 0 \\ 0 & (-\alpha_n)^{-1} \end{bmatrix}$$

Since n is odd, we have that $(-\alpha_n\ )$ is a $(2n)^{th}$ root of unity. Thus for some $z \in \mathbb{Z}$ we have:

$$\begin{bmatrix}(-\alpha_n) & 0 \\ 0 & (-\alpha_n)^{-1}\end{bmatrix}^z = \begin{bmatrix}\alpha_{2n} & 0 \\ 0 & \alpha_{2n}^{-1}\end{bmatrix}$$

and we see that Γ_1 contains the generators of Γ_2. Conversely it is easy to see that Γ_2 contains the generators of Γ_1, so the two groups are the same! Therefore, any group of the form $\langle \tilde{R}, \tilde{s}\rangle$ can be represented as $\langle \tilde{r}, \tilde{s}\rangle$ for some $\tilde{r}$ where the order of $\tilde{r}$ is even, regardless of whether the order of $\tilde{R}$ is even (in which case let $\tilde{r} = \tilde{R}$) OR the order of $\tilde{R}$ is odd. Therefore we will refer to the binary dihedral group of order $4n$ in $SU(2)$ as $\mathcal{D}_{2n}$ to represent the fact that $\mathcal{D}_{2n}$ is generated by a cyclic element $\tilde{r}$ of order $2n$ (and an element $\tilde{s}$). This gives us the following result:

Lemma 2. *If a subgroup of $SU(2)$ can be generated by two elements, neither of which lies in the span of the other, and if at least one of these elements has order 4, then this subgroup is isomorphic to the group:*

$$\mathcal{D}_{2n} = \langle \tilde{r}, \tilde{s}\rangle = \langle \begin{bmatrix}\delta & 0 \\ 0 & \delta^{-1}\end{bmatrix}, \begin{bmatrix}0 & i \\ i & 0\end{bmatrix}\rangle$$

Where $n \in \mathbb{N}$, and δ is a $(2n)^{th}$ root of unity.

Take careful note of three things:
1) $\mathcal{D}_{2n}$ contains a cyclic group of order $2n$.
2) $\mathcal{D}_{2n}$ itself has order of $4n$.
3) The group $\mathcal{D}_{2n}$ for $n = 1$ is cyclic of order 4, i.e., isomorphic to $\mathcal{C}_4$.

2.3 The Binary Tetrahedral Group: Symmetries of the Tetrahedron.

The largest subgroup of $SO(3,\mathbb{R})$ that preserves the structure of a tetrahedron in $\mathbb{R}^3$ is generated by $\langle c, d\rangle$ (defined below), which lifts uniquely (i.e., the lifted subgroup is the same no matter how we lift the generators) to $\langle \tilde{c}, \tilde{d}\rangle$ (defined below) in $SU(2)$ as follows:

$$c = \frac{1}{2}\begin{bmatrix}-1 & -\sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 2\end{bmatrix} \rightarrow \frac{1}{2}\begin{bmatrix}1 + i\sqrt{3} & 0 \\ 0 & 1 - i\sqrt{3}\end{bmatrix} = \tilde{c}$$

$$d = \frac{1}{3}\begin{bmatrix}-3 & 0 & 0 \\ 0 & 1 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & -1\end{bmatrix} \rightarrow \frac{1}{\sqrt{3}}\begin{bmatrix}i & \sqrt{2} \\ -\sqrt{2} & -i\end{bmatrix} = \tilde{d}$$

Note that $\tilde{c}$ is just $\tilde{R}$ in the case where $m = 2 \times 3 = 6$ as we might expect from the triangular symmetry of the face of a tetrahedron. We can summarize this result as:

Lemma 3. *Besides $\mathcal{C}_{24}$ and $\mathcal{D}_{2(6)}$ the only subgroup of $SU(2)$ with 24 elements is:*

$$\mathcal{T} = \langle \tilde{c}, \tilde{d} \rangle = \langle \frac{1}{2} \begin{bmatrix} 1 + i\sqrt{3} & 0 \\ 0 & 1 - i\sqrt{3} \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} i & \sqrt{2} \\ -\sqrt{2} & -i \end{bmatrix} \rangle$$

2.4 The Binary Octahedral Group: Symmetries of the Cube.

Similarly, the largest subgroup of $SO(3, \mathbb{R})$ that preserves the structure of a cube in $\mathbb{R}^3$ is generated by $\langle a, b \rangle$ (defined below), which lifts to $\langle \tilde{a}, \tilde{b} \rangle$ (defined below) in $SU(2)$ as follows:

$$a = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \frac{1}{\sqrt{2}} \begin{bmatrix} 1 + i & 0 \\ 0 & 1 - i \end{bmatrix} = \tilde{a}$$

$$b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \to \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \tilde{b}$$

Note that $\tilde{a}$ is just $\tilde{R}$ in the case where $m = 2 \times 4 = 8$ as we might expect from the square symmetry of the face of a cube. We may summarize this result as:

Lemma 4. *Besides $\mathcal{C}_{48}$ and $\mathcal{D}_{2(12)}$ the only subgroup of $SU(2)$ with 48 elements is:*

$$\mathcal{O} = \langle \tilde{a}, \tilde{b} \rangle = \langle \frac{1}{\sqrt{2}} \begin{bmatrix} 1 + i & 0 \\ 0 & 1 - i \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \rangle$$

2.5 The Binary Icosahedral Group: Symmetries of The Dodecahedron.

This calculation is cumbersome and, as we will see, is not needed to exclude the possibility of a lattice in $\mathbb{C}^2$ invariant under this subgroup. Nevertheless, we will denote this subgroup $\mathcal{I}$. That is, we let $\mathcal{I}$ be the subgroup of $SU(2)$ lifted from the subgroup of $SO(3, \mathbb{R})$ that acts to give the symmetries of a dodecahedron.

Lemma 5. *Besides $\mathcal{C}_{120}$ and $\mathcal{D}_{2(30)}$ the only subgroup of $SU(2)$ with 120 elements is $\mathcal{I}$ [5].*

Now we have found all the finite subgroups of $SO(3, \mathbb{R})$, and looked at all the possible ways each subgroup could be lifted to $SU(2)$ (in particular the lifting process is unique on the level of groups, except for the cyclic groups of odd order). Since Φ is defined on all of $SU(2)$, this means we have found all of the finite subgroups of $SU(2)$.

Theorem 1. *A subgroup of $SU(2)$ is finite if and only if it is isomorphic to one of the following:*

$$\mathcal{C}_m, \mathcal{D}_{2n}, \mathcal{T}, \mathcal{O}, \mathcal{I}$$

where $m, n \in \mathbb{N}$.

3 Finding the Invariant Polynomials

Consider the action, $\cdot$ of an element, $g \in SU(2)$, on a polynomial $t(u, v) \in \mathbb{C}[u, v]$, defined as follows:

$$g \cdot t(u, v) = t(g^{-1}(u, v))$$

It is easy to see that this is in fact a group action.

For each finite subgroup Γ of $SU(2)$ we seek polynomials in $\mathbb{C}[u, v]$ which are invariant under all elements of Γ. In order to proceed, we will need the following:

Proposition 1. *Let $\Gamma \subset \mathbb{C}[u, v]$ be a finite subgroup. With respect to the action $\cdot$ defined above:*
1) Polynomials invariant under Γ form a subring of $\mathbb{C}[u, v]$.
2) A polynomial is invariant under Γ iff it is invariant under its generators.

Proof. 1) Suppose q, t are invariant under Γ. Let $g \in \Gamma$. Then

$$\begin{aligned} g \cdot qt(u, v) &= qt(g^{-1}(u, v)) \\ &= [q(g^{-1}(u, v))][t(g^{-1}(u, v))] \\ &= [g \cdot q(u, v)][g \cdot t(u, v)] \\ &= q(u, v)t(u, v) \\ &= qt(u, v) \end{aligned}$$

Hence qt is invariant under an arbitrary element of Γ, and therefore invariant under Γ. Further:

$$\begin{aligned} g \cdot (q + t)(u, v) &= (q + t)(g^{-1}(u, v)) \\ &= q(g^{-1}(u, v)) + t(g^{-1}(u, v)) \\ &= g \cdot q(u, v) + g \cdot t(u, v) \\ &= q(u, v) + t(u, v) \\ &= (q + t)(u, v) \end{aligned}$$

And we see that $(q + t)$ is invariant under an arbitrary element of Γ, and therefore invariant under Γ. Thus the polynomials invariant under the action of Γ form a subring.

2) Suppose t is invariant under Γ. Then it is invariant under all elements of Γ, including its generators. Conversely, suppose t is invariant under the generating set, S_G of Γ. Since Γ is finite all its elements have finite order. Therefore the inverse of each element (in particular generator) is a power of that element (generator). Hence if $g \in \Gamma$ we may write $g = g_1 g_2 \cdots g_n$ where $g_1, g_2, \ldots g_n$ are (not necessarily unique) elements of S_G. Then:

$$\begin{aligned} g \cdot (t) &= (g_1 g_2 ... g_n) \cdot (t) \\ &= (g_1 g_2 ... g_{n-1}) \cdot (g_n \cdot t) \\ &= (g_1 g_2 ... g_{n-1}) \cdot (t) \\ & \quad ... \\ &= t \end{aligned}$$

□

Definition 1. *A set of invariant polynomials is called a set of fundamental invariants for a finite subgroup Γ of $SU(2, \mathbb{C})$ if it is a minimal generating set for the subring of polynomials invariant under Γ.*

Proposition 2. *The set of fundamental invariants of a finite subgroup $\Gamma \subset SU(2)$ may be taken to be homogenous.*

Proof. First note that the action $\cdot$ of $g \in SU(2)$ preserves the degree of homogenous polynomials in $\mathbb{C}[u, v]$. Now, suppose $t(u, v)$ is an invariant polynomial of degree n of some subgroup $\Gamma \subset SU(2)$. Then $t = s_1 + s_2 + \cdots s_n$ where s_i is some homogenous polynomial of degree i. Then for all $g \in \Gamma$:

$$\begin{aligned} g \cdot t &= t \\ \rightarrow g \cdot (s_1 + s_2 + \cdots + s_n) &= s_1 + s_2 + \cdots + s_n \\ \rightarrow g \cdot s_1 + g \cdot s_2 + \cdots + g \cdot s_n &= s_1 + s_2 + \cdots + s_n \\ \rightarrow g \cdot s_i &= s_i \text{ for } 1 \leq i \leq n \end{aligned}$$

(Because g preserves the degree of each homogenous polynomial s_i.)

Therefore, each homogenous term of t is Γ-invariant.

Now suppose $t_1, t_2, \ldots, t_m$ is a set of fundamental invariants generating the subring R of polynomials in $\mathbb{C}[u, v]$ invariant under the action of Γ. Let s_{ij} denote the homogenous part of t_i of degree j. Then $\{s_{ij}\} \subset R$ generates each element $t_1, t_2, \ldots, t_m$. That is, $\{s_{ij}\}$ is a generating set for R. Hence, we can take a minimal subset of $\{s_{ij}\}$ that still generates R to be the fundamental invariants of Γ.

□

Definition 2. *Let $\Gamma \in SU(2)$ be a finite subgroup and $t(u,v) \in \mathbb{C}[u,v]$. The Reynolds operator of Γ applied to $t(u,v)$, denoted $R_\Gamma(t)$ is given by:*

$$R_\Gamma(t(u,v)) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} g\cdot t(u,v)$$

Theorem 2 (Noether)**.** *The ring of invariants of Γ is generated algebraically by:*

$$\langle R_\Gamma(u^\alpha v^\beta) : \alpha+\beta \leq |\Gamma|\rangle$$

[1, Ch. 7.3]

Note that, if $-I \in \Gamma$, then, for each $g \in \Gamma$, $-g \in \Gamma$ as well. Moreover, if $\alpha+\beta$ is odd, it follows that $(-g)\cdot u^\alpha v^\beta = -(g\cdot u^\alpha v^\beta)$, so that whenever $-I \in \Gamma$ and $\alpha+\beta$ is odd, $R_\Gamma(u^\alpha v^\beta) = 0$.

In particular, if Γ is cyclic of order $2n$, then $-I \in \Gamma$, and therefore, whenever $\alpha+\beta$ is odd, $R_\Gamma(u^\alpha v^\beta) = 0$. On the other hand, if $\alpha+\beta$ is even, then $(-g)\cdot u^\alpha v^\beta = g\cdot u^\alpha v^\beta$. Thus, if Γ is cyclic of order $2n$ and $\alpha+\beta$ is even:

$$\begin{aligned} R_\Gamma(u^\alpha v^\beta) &= \frac{1}{2n}\sum_{j=1}^{2n} g^j\cdot u^\alpha v^\beta \\ &= \frac{1}{2n}\sum_{j=1}^{n}(g^j\cdot u^\alpha v^\beta + g^{n+j}\cdot u^\alpha v^\beta) \\ &= \frac{1}{2n}\sum_{j=1}^{n}(g^j\cdot u^\alpha v^\beta + (-g^j)\cdot u^\alpha v^\beta) \\ &= \frac{1}{2n}\sum_{j=1}^{n}2(g^j\cdot u^\alpha v^\beta) = \frac{1}{n}\sum_{j=1}^{n} g^j\cdot u^\alpha v^\beta \end{aligned}$$

Suppose Γ is cyclic of order $2n$. Then by the remarks above, the ring of invariants of Γ is generated algebraically by:

$$\langle\left(\frac{1}{n}\sum_{j=1}^{n} g^j\cdot u^\alpha v^\beta\right) : 2|(\alpha+\beta) \leq 2n\rangle = \langle\left(\sum_{j=1}^{n} g^j\cdot u^\alpha v^\beta\right) : 2|(\alpha+\beta) \leq 2n\rangle$$

Example: Invariants of the cyclic group, $\langle\tilde{s}\rangle$, of order 4.

To find the invariant ring of the group $\langle\tilde{s}\rangle$ of order $2n = 4$ we consider values of α and β such that $2|(\alpha+\beta)$ while $(\alpha+\beta) \leq 4$. For each set of values (α,β) we compute $\sum_{j=1}^{n} g^j\cdot u^\alpha v^\beta$. We then look at the ring that is algebraically generated by those elements as (α,β) ranges through all possible values.

(α, β)	$\sum_{j=1}^{n} g^j \cdot u^\alpha v^\beta$
$(2,0)$	$u^2 - v^2$
$(1,1)$	0
$(0,2)$	$v^2 - u^2$
$(4,0)$	$u^4 + v^4$
$(3,1)$	$u^3 v + uv^3$
$(2,2)$	$2u^2v^2$
$(1,3)$	$uv^3 + u^3 v$
$(0,4)$	$u^4 + v^4$

It is easy to see that a minimal generating set for the algebraic span of the polynomials in the right-hand column is: $\langle uv(u^2+v^2), u^2-v^2, u^2v^2\rangle$. Hence we take the fundamental invariants of $\tilde{s}$ to be $uv(u^2+v^2)$, u^2-v^2, and u^2v^2.

For larger groups, one should make use of a computer algebra program to apply Noether's theorem. Suppose Γ has order n. For a given value of (α, β) in Noether's theorem, each term of the Reynolds operator is given by the product: $(gu)^\alpha(gv)^\beta$, which is a product of at most n elements. Moreover, the Reynolds operator must sum n of these products. Hence for a fixed value of (α, β), the Reynolds operator runs in at most $O(n^2)$ time. Finally, the number of possible values of (α, β) in the statement of Noether's theorem is bounded by n^2. Therefore, a Noether's theorem algorithm can be run in $O(n^4)$ time. Since the largest finite subgroup of $SU(2)$ has order 120, this algorithm can quickly compute the ring of invariants for any subgroup of $SU(2)$.

Systematic application of Noether's theorem applied to cyclic groups gives us the following results:

Cyclic Group	Fundamental Invariants	Names
$\langle\tilde{R}\rangle$	uv, u^m, v^m	R_1, R_2, R_3
$\langle\tilde{r}\rangle$	uv, u^{2n}, v^{2n}	r_1, r_2, r_3
$\langle\tilde{s}\rangle$	$u^2-v^2, u^2v^2, u^3v+uv^3$	S_1, S_2, S_3
$\langle\tilde{a}\rangle$	uv, u^8, v^8	A_1, A_2, A_3
$\langle\tilde{b}\rangle$	u^2-v^2, $u^7v+7u^5v^3+7u^3v^5+uv^7$, $u^8+28u^6v^2+70u^4v^4+28u^2v^6+v^8$	B_1, B_2, B_3

Systematic application of Noether's theorem to $\mathcal{C}_m$, $\mathcal{D}_{2n}$, and $\mathcal{O}$ gives us the following results:

Group	Fundamental Invariants	Names
$\mathcal{C}_m = \langle \tilde{R} \rangle$	uv, u^m, v^m	R_1, R_2, R_3
$\mathcal{D}_{2n} = \langle \tilde{r}, \tilde{s} \rangle$	u^2v^2, $u^{2n} + (-1)^n v^{2n}$, $uv(u^{2n} + (-1)^{n-1}v^{2n})$	D_1, D_2, D_3
$\mathcal{O} = \langle \tilde{a}, \tilde{b} \rangle$	$u^8 + 14u^4v^4 + v^8$, $u^{10}v^2 - 2u^6v^6 + u^2v^{10}$, $u^{17}v - 34u^{13}v^5 + 34u^5v^{13} - uv^{17}$	O_1, O_2, O_3

The invariant ring of $\mathcal{T} = \langle \tilde{c}, \tilde{d} \rangle$ is computed in the same way. However, the ring in question is generated by polynomials with complex irrational coefficients and details will not be included in this paper. The invarinats of $\mathcal{I}$ are also ommitted as we have not included explicit generators for this group, as is needed to apply Noether's theorem.

Proposition 3. *Let $\Gamma = \langle \Gamma_1, \Gamma_2, \ldots, \Gamma_n \rangle$. Denote the ring of fundamental invariants of Γ_i by R_i and the ring of fundamental invariants of Γ by R. Then $R = R_1 \cap R_2 \cap \ldots \cap R_n$.*

Proof. Suppose $t \in R$. Then t is invariant under Γ and hence under each generator of Γ by Proposition 1 (2). Thus $t \in R_1 \cap R_2 \cap \ldots \cap R_n$. Suppose $t \in R_1 \cap R_2 \cap \ldots \cap R_n$. Then t is invariant under each generator of Γ, and hence by Proposition 1 (2), t is invariant under Γ. Thus $t \in R$. □

This allows us to develop some relationships amongst the rings of invariants. In particular, we have

$$\langle D_1, D_2, D_3 \rangle = \langle r_1, r_2, r_3 \rangle \cap \langle S_1, S_2, S_3 \rangle$$
$$\langle O_1, O_2, O_3 \rangle = \langle A_1, A_2, A_3 \rangle \cap \langle B_1, B_2, B_3 \rangle.$$

Also, since $\tilde{b}^2 = \tilde{s}$ we have $\langle \tilde{s} \rangle \subset \langle \tilde{b} \rangle$. From this, and the proposition above, it follows that

$$\langle B_1, B_2, B_3 \rangle \subset \langle S_1, S_2, S_3 \rangle.$$

4 Finding the Algebraic Relationship of the Invariants

Denote the set of fundamental invariants of Γ by $(t_1, t_2, \ldots, t_n)$, (where the $t_i \in \mathbb{C}[u, v]$) and let I_Γ denote the set of polynomials in $\mathbb{C}[y_1, y_2, \ldots, y_n]$ which vanish at $(t_1, t_2, \ldots, t_n)$. That is:

$$I_\Gamma = \{p \in \mathbb{C}[y_1, y_2, \ldots, y_n] : p(t_1, t_2, \ldots, t_n) = 0\}$$

Where 0 represents the 0 element of $\mathbb{C}[u, v]$. It is easy to check that I_Γ is an ideal, known as the syzygy ideal of the fundamental invariants of Γ. Next, form another ideal, J_Γ:

$$J_\Gamma = \langle t_1 - y_1, t_2 - y_2, \ldots, t_n - y_n \rangle \subseteq \mathbb{C}[u, v, y_1, y_2, \ldots, y_n]$$

(Of course the notation above refers to the ideal generated by the listed elements, not their algebraic span as it has previously denoted.)

Theorem 3. *Using the notation developed above, we have the equality:*

$$I_\Gamma = J_\Gamma \cap \mathbb{C}[y_1, y_2, \ldots, y_n]$$

[1, Ch. 7.4]

The easiest way to find the intersection above is to express J_Γ with a Groebner basis, using the lexicographic ordering: $u > v > t_1 > \cdots > t_n$.

Example. For simplicity, consider the case of $\mathcal{C}_2$. We take $t_1 = u^2$, $t_2 = v^2$, $t_3 = uv$ to be the fundamental invariants. Then we have that

$$J_{\mathcal{C}_2} = \langle u^2 - y_1, v^2 - y_2, uv - y_3 \rangle,$$

which has Groebner basis:

$$\{u^2 - y_1, uv - y_3, uy_2 - vy_3, uy_3 - vy_1, v^2 - y_2, y_1y_2 - y_3^2\}.$$

Therefore we have $I_{\mathcal{C}_2} = J_{\mathcal{C}_2} \cap \mathbb{C}[y_1, y_2, \ldots, y_n] = \langle y_1y_2 - y_3^2 \rangle$. We compute I_Γ in the same way for the other finite subgroups of $SU(2)$. Again, a computer algebra program is necessary for the larger groups. Since the ring of invariants of each finite subgroup of $SU(2)$ turns out to be generated by three fundamental invariants, we will express I_Γ using the three variables, (X, Y, Z). Moreover, we find that I_Γ is generated by a single element of $\mathbb{C}[X, Y, Z]$ for each Γ.

Using the method above we conclude the following:

Lemma 6. *The fundamental invariants of $\mathcal{C}_m$, written as X, Y, Z, can be expressed in such a way that they satisfy the algebraic relation:*

$$XY + Z^m = 0$$

Moreover, this algebraic relation generates the syzygy ideal of the invariants X, Y, Z.

Remark. *One can see directly that the above relation is contained in the syzygy ideal of the fundamental invariants: $X = -u^m$, $Y = v^m$, $Z = uv$.*

Note that the syzygy ideal of the invariants of a group, Γ, is not uniquely determined by Γ itself. In fact, one needs to specify a set of fundamental invariants of Γ to uniquely determine a syzygy ideal. For example, we computed the syzygy ideal of the invariants $\{u^2, v^2, uv\}$ of $\mathcal{C}_2$ in the example above to be generated by $XY - Z^2$ in the variables (X, Y, Z). However, in lemma 6 (in the particular case of $m = 2$) we use the invariants $\{-u^2, v^2, uv\}$ of $\mathcal{C}_2$ and find the syzygy ideal of these invariants is generated by $XY + Z^2$. We will use the form $XY + Z^m$ throughout the remainder of the paper, as this form will make computations easier.

Lemma 7. *The fundamental invariants of $\mathcal{D}_{2n}$, written as X, Y, Z, can be expressed in such a way that they satisfy the algebraic relation:*

$$X^2 + Y^2 Z + (-1)^{n+1} Z^{n+1} = 0$$

Moreover, this algebraic relation generates the syzygy ideal of the invariants X, Y, Z.

Remark. *One can see directly that the above relation is contained in the syzygy ideal of the fundamental invariants: $X = 2^{n+1} D_3$, $Y = 2^n D_2$, $Z = -4D_1$.*

Lemma 8. *The fundamental invariants of $\mathcal{O}$, written as X, Y, Z, can be expressed in such a way that they satisfy the algebraic relation:*

$$X^3 Y + Y^3 + Z^2 = 0$$

Moreover, this algebraic relation generates the syzygy ideal of the invariants X, Y, Z.

Remark. *One can see directly that the above relation is contained in the syzygy ideal of the fundamental invariants: $X = -12O_1$, $Y = 432O_2$, $Z = 864O_3$.*

Lemma 9. *The fundamental invariants of $\mathcal{T}$, written as X, Y, Z, can be expressed in such a way that they satisfy the algebraic relation:*

$$X^4 + Y^3 + Z^2 = 0$$

Moreover, this algebraic relation generates the syzygy ideal of the invariants X, Y, Z.

Lemma 10. *The fundamental invariants of $\mathcal{I}$, written as X, Y, Z, can be expressed in such a way that they satisfy the algebraic relation:*

$$X^3 + Y^5 + Z^2 = 0$$

Moreover, this algebraic relation generates the syzygy ideal of the invariants X, Y, Z.

5 The Corresponding Singular Varieties

Theorem 4. *When viewed as three-dimensional complex variety, the zero-set of the algebraic relationship of the fundamental invariants of any subgroup Γ of $SU(2)$ gives a complex surface that is smooth everywhere except at the origin, where it is singular. Furthermore, for each such subgroup, Γ, the surface given by this variety is isomorphic to the quotient space $\mathbb{C}^2/\Gamma$, where $\mathbb{C}^2/\Gamma$ refers to the space parametrizing the orbits of the Γ-action on $\mathbb{C}^2$. [4]*

Now, we can resolve the singularities of these surfaces through the method of successive blow-ups. We will begin with an example of how this is done and then introduce a method which generalizes the process. Here is an example of how this is done in the case of $\mathcal{D}_{2n}$ when $n = 2$. We have the surface: $X^2 + Y^2Z - Z^3$. We need to resolve the singularity that occurs at $(0, 0, 0)$ by replacing the origin with a copy of $\mathbb{C}P^2$.

To clarify, when we refer to the "origin" we will always mean the image of the whole two-dimensional projective subspace of $(0, 0, 0) \times [a : b : c] \subset \mathbb{C}^3 \times \mathbb{C}P^2$ (s.t. $a, b, c \in \mathbb{C}$ not all 0), under the projection: $\mathbb{C}^3 \times \mathbb{C}P^2 \to \mathbb{C}^3$. When we refer to the "point (0, 0, 0)" we will mean the image of some ONE-dimensional curve contained in the subspace $(0, 0, 0) \times [a : b : c] \subset \mathbb{C}^3 \times \mathbb{C}P^2$ under the projection map from $\mathbb{C}^3 \times \mathbb{C}P^2 \to \mathbb{C}^3$. Therefore the "origin" will always blow up into a space isomorphic to $\mathbb{C}P^2$ and the "point (0, 0, 0)" will always blow up into a one-dimensional variety of $\mathbb{C}P^2$. Since the blow-up of a singularity of a surface at the origin is a one-dimensional curve contained in $(0, 0, 0) \times [a : b : c]$ we say that the surface contains the point (0,0,0), but not the origin.

Now, we would like to find the preimage of the surface, $X^2 + Y^2Z - Z^3$, under the blow-up map from $\mathbb{C}^3 \times \mathbb{C}P^2 \to \mathbb{C}^3$, by lifting each point (X, Y, Z) to its preimage of the form $(X, Y, Z) \times [a : b : c]$ where

$$aY = bX, \;\; aZ = cX, \;\; bZ = cY$$

To do this we must consider three coordinate charts of $\mathbb{C}P^2$. For the first coordinate chart we suppose that $a \neq 0$. Then we make the substitutions:

$$X = X, \;\; Y = \frac{b}{a}X, \;\;\; Z = \frac{c}{a}X$$

This substitution yields the equation:

$$(X^2)[1 + X(\frac{b}{a})^2\frac{c}{a} + X(\frac{c}{a})^2] = 0$$

Which factors as:

$X^2 = 0$	Exceptional Divisor
$1 + X(\frac{b}{a})^2\frac{c}{a} + X(\frac{c}{a})^2 = 0$	Surface blow-up in the $a \neq 0$ chart.

At the origin, $X = 0$, so, after making this substitution in the expression of the surface blow-up above, we see that in the chart $a \neq 0$ the point $(0, 0, 0)$ is replaced by the projective variety $1 = 0$. In other words, the image of the blow-up of the point $(0, 0, 0)$ does not intersect the chart $a \neq 0$.

This process is repeated for each chart the results are included in the table below. The coordinates referred to below form a set of local affine coordinates that can be used to describe the surface blow-up in each chart. For instance, any point of the set:

$$\{(X, Y, Z) \times [a : b : c] \in \mathbb{C}^3 \times \mathbb{C}P^2 : aY = bX,\ \ aZ = cX,\ \ bZ = cY\}$$

which lies in the $a \neq 0$ chart, i.e., any such point with $a \neq 0$, can be uniquely specified by the values that $(X, \frac{b}{a}, \frac{c}{a})$ take at that point. In other words, we may introduce a set of local affine coordinates, (x', y', z') on the $a \neq 0$ chart where the values of (x', y', z') are given by:

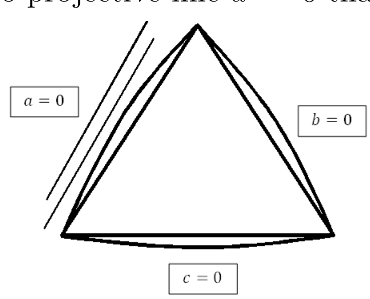
$$x' = X, \quad y' = \frac{b}{a}, \quad z' = \frac{c}{a}$$

For convenience, we say that $(X, \frac{b}{a}, \frac{c}{a})$ form a set of local affine coordinates on the $a \neq 0$ chart and describe the surface blow-up in this chart as a variety in these variables.

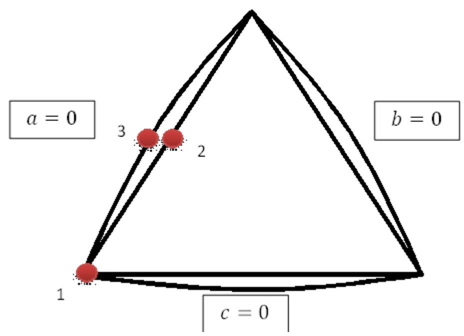
Chart	Coordinates	Ex. Divisor	Surface Blow-up	Image of $(0,0,0)$
$a \neq 0$	$(X, \frac{b}{a}, \frac{c}{a})$	$X^2 = 0$	$1 + X(\frac{b}{a})^2\frac{c}{a} + X(\frac{c}{a})^2 = 0$	$\emptyset$
$b \neq 0$	$(\frac{a}{b}, Y, \frac{c}{b})$	$Y^2 = 0$	$(\frac{a}{b})^2 + Y(\frac{c}{b}) + Y(\frac{c}{b})^3 = 0$	$(\frac{a}{b})^2 = 0 \to a^2 = 0$
$c \neq 0$	$(\frac{a}{b}, \frac{b}{c}, Z)$	$Z^2 = 0$	$(\frac{a}{c})^2 + Y(\frac{b}{c}) + Y(\frac{b}{c})^3 = 0$	$(\frac{a}{c})^2 = 0 \to a^2 = 0$

We see that in the two charts $b \neq 0$ and $c \neq 0$ the point $(0, 0, 0)$ blows up into the set of points in $\mathcal{C}_3 \times \mathbb{C}P^2$ with the first three coordinates equal to $(0, 0, 0)$, and whose last three coordinates satisfy $a^2 = 0$; in other words, points of the form $(0, 0, 0) \times [0 : b : c]$ where not both b and c are 0. Note that this result confirms our first calculation that this variety would not intersect the $a \neq 0$ chart! Since the set of points $[0 : b : c]$ where b and c are not both 0 is isomorphic to $\mathbb{C}P^1$ we will refer to this variety as a copy of $\mathbb{C}P^1$.

Below is a crude representation of $\mathbb{C}P^2$. Each leg of the triangle is formed by a circle that represents the projective line where one of the coordinates, a, b, or c is 0. The vertices of the triangle are formed where two of these circles intersect tangentially in a single point. This point represents where two of the coordinates are 0. The two lines next to the $a = 0$ leg represent the multiplicity two projective line $a^2 = 0$ that we found above.

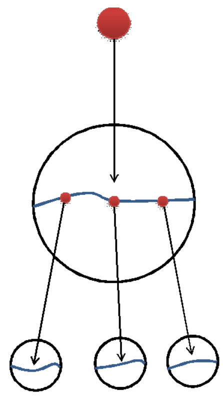


But we are not done yet! The equations for the surface blow-up of the $b \neq 0$ and $c \neq 0$ charts in the table above are both singular. In the $b \neq 0$ chart we find the following singularities: $(0,0,0)$, $(0,0,1)$, and $(0,0,-1)$, which we will denote (1), (2), and (3) respectively. In our local coordinate system for this chart, this implies that we have singularities at the points in $\mathbb{C}P^2$ where: (1) $a = 0 = c$, (2) $a = 0$, $c = 1$, and (3) $a = 0$, $c = -1$. In the $c \neq 0$ chart we find two singularities, but they turn out to be two of the same points of $\mathbb{C}P^2$, namely (2) and (3), that we found to be singular in the $b \neq 0$ chart. Since there are no points at all in the $a \neq 0$ chart these three points are the only singularities of our blown-up surface. Since the singularity (1) lies neither in the $a \neq 0$ chart nor the $c \neq 0$ chart, it must lie where both a and c are 0, which determines it to be the point $[0 : 1 : 0]$ in $\mathbb{C}P^2$. Furthermore, we can work out that point (2) is $[0 : 1 : 1]$ and point (3) is $[0 : 1 : -1]$ when written as points of $\mathbb{C}P^2$. The singularities are shown in red below.



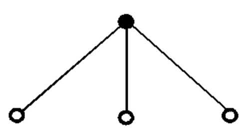
Therefore we need blow each of these points up again. Note that all of the singularities lie in the $b \neq 0$ chart so we can take our coordinates to be $(\frac{a}{b}, Y, \frac{c}{b})$ for all of the blow-ups. Therefore we will go ahead and rename these coordinates as (X, Y, Z), in which case $(\frac{a}{b})^2 + Y(\frac{c}{b}) + Y(\frac{c}{b})^3$ becomes $X^2 + YZ + YZ^3$. Now we repeat the process of blowing this variety up at the points (1), (2), and (3). At (1) $X = 0$, $Y = 0$, and $Z = 0$, so this is simply just a blow-up of the point $(0,0,0)$ at the origin. Introducing new projective coordinates to describe the image of this blow-up, in the same manner as earlier, this time $(0,0,0)$ blows up to the projective variety $a^2 + b = 0$ in $\mathbb{C}P^2$ (more formally it blows up into the direct product of $(0,0,0)$ and this variety as a variety in $\mathcal{C}_3 \times \mathbb{C}P^2$). In order to blow up the point (2) we need to introduce a change of coordinates on the variety $X^2 + YZ + YZ^3$ so that the point (2) becomes the origin. After this, the blow-up process is the same. (3) is blown-up similarly. All together, each of these points blows up into a smooth variety of $\mathbb{C}P^2$ isomorphic to $\mathbb{C}P^1$.

All this can be summarized with the following diagram. Here the large red dot represents our initial singularity at the origin. First the origin is blown up to the large black circle representing $\mathbb{C}P^2$ and the singular point $(0,0,0)$ is mapped to the blue line which represents a copy of $\mathbb{C}P^1$. The blue line contains three small red dots, representing the singularities (1), (2), and (3) each of which is blown up to produce three more blue lines (copies of $\mathbb{C}P^1$) each of which lies in a small black circle (i.e., $\mathbb{C}P^2$).

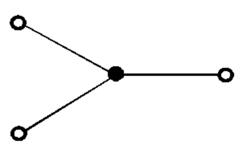


Since each of these three additional copies of $\mathbb{C}P^1$ is a blow up of a point on our original copy of $\mathbb{C}P^1$ they each intersect the original $\mathbb{C}P^1$. And we have the following intersection diagram (where the bold line represents the original copy of $\mathbb{C}P^1$):

If we represent each copy of $\mathbb{C}P^1$ with a dot (where the solid dot represents the original copy of $\mathbb{C}P^1$ we found), and connect the dots iff the corresponding copies of $\mathbb{C}P^1$ intersect each other we have:



Which may be more easily recognized as the Dynkin diagram D_4 in the form:



5.1 Streamlining Computation of Surface Resolutions

The results of the previous section indicate that the singular surfaces found from each of the groups $\mathcal{C}_m$, $\mathcal{D}_{2n}$, $\mathcal{T}$, $\mathcal{O}$, $\mathcal{I}$ can be represented by a polynomial with two or three terms with coefficients of 1 or -1. Because the cases where the polynomial has only two terms and the case where it contains a negative coefficient ($\mathcal{C}_m$, and $\mathcal{D}_{2n}$ for n even respectively) can be handled in a similar manner, we will restrict our attention to computing the blow-ups of surfaces with three terms each of which has coefficient 1. The first step is to represent the variety that describes the surface by a 3×3 matrix as follows:

Each column represents a term of the expression that describes the variety. Within that column the entry in the first row represents the degree of X in that term, the second row represents the degree of Y in that term, etc. For instance, the surface $X^3Y + Y^3 + Z^2 = 0$ is represented as:

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Since we agreed that all coefficients are 1, we have that any such matrix represents to a unique polynomial. Furthermore, we have:

Theorem 5. *Suppose a surface with a singularity at the origin has matrix form:*

$$\begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}$$

Then letting:

$$s_j = \sum_{i=1}^{3} n_{ij}$$

and

$$m = min(s_j)$$

the blow-up of this surface is given, in matrix form, in the local affine coordinates of each chart as:

$$\text{First } (a \neq 0) \text{ chart} \begin{bmatrix} s_1 - m & s_2 - m & s_3 - m \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}$$

$$\text{Second } (b \neq 0) \text{ chart} \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ s_1 - m & s_2 - m & s_3 - m \\ n_{31} & n_{32} & n_{33} \end{bmatrix}$$

$$\text{Third } (c \neq 0) \text{ chart} \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ s_1 - m & s_2 - m & s_3 - m \end{bmatrix}$$

Proof of Theorem. This process exactly parallels the process of blowing up a variety. Just as when we blow up a variety in the i^{th} chart, we make a substitution that replaces each of the other two coordinates with a term that has degree 1 in the i^{th} coordinate and degree 1 in a new local coordinate. Some computation shows that, after this substitution, each term has degree in the i^{th} coordinate equal to its original total degree. But note that the j^{th} terms original total degree is simply the sum of the j^{th} column, which we call s_j. Hence the i^{th} row of the matrix representation becomes $\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}$. At this point we factor out the exceptional divisor, which is exactly the i^{th} coordinate to the power of the highest degree that it appears in all three terms. It is easy to check that this power is precisely the minimum of the s_j which we denoted m. Factoring out the i^{th} coordinate to the power m is equivalent to subtracting m from each entry in the i^{th} row. Thus the i^{th} row becomes $\begin{bmatrix} s_1 - m & s_2 - m & s_3 - m \end{bmatrix}$. Meanwhile, when we replace the other two coordinates by the local coordinates of the i^{th} chart, no entries outside of the i^{th} row change. □

Corollary 1. *The image of the blow-up of the point* $(0, 0, 0)$ *that lies in the* i^{th} *chart is given by the variety that is the sum of the terms represented by the columns of the* i^{th} *chart blow-up matrix that have a 0 in the* i^{th} *row.*

Example: Suppose we want to know the image of blow-up of the point $(0, 0, 0)$ that lies in the 1^{st} chart, for the example given earlier. So we let $i = 1$. Therefore we sum the terms represented by columns that have a 0 in the 1^{st} row for the 1^{st} chart blow-up matrix:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

It is easy to see that only the third column satisfies this requirement, so we have only the term represented by: $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ Recalling that the coordinates in the first chart are given by: $(X, \frac{b}{a}, \frac{c}{a})$, this gives us the equation:

$$X^0(\frac{b}{a})^0(\frac{c}{a})^2 = 0 \iff (\frac{c}{a})^2 = 0 \iff c^2 = 0$$

That is, the point $(0, 0, 0)$ blows up into the (multiplicity two) projective line $c = 0$ in the first chart.

Proof of Corollary. In the i^{th} chart the i^{th} coordinate of the blow-up is the same as the i^{th} coordinate of the original surface. Since the blow-up map from $\mathbb{C}^3$ to $\mathbb{C}^3 \times \mathbb{C}P^2$ is the identity on $\mathbb{C}^3$, we know that the image of $(0, 0, 0)$ in the i^{th} chart must have the i^{th} coordinate equal to 0. Substituting 0 for the i^{th} coordinate in the surface blow-up leaves an expression with only those terms that are degree 0 in the i^{th} coordinate, i.e., only those terms that are represented by a column that has a 0 in the i^{th} row.

□

With these results it is easy to write a program (see Appendix) and use it to completely resolve the singularities of all of the surfaces arising from the groups $\mathcal{C}_m$, $\mathcal{D}_{2n}$, $\mathcal{T}$, $\mathcal{O}$, $\mathcal{I}$, and therefore the singularities of the corresponding quotients spaces $\mathbb{C}^2/\mathcal{C}_m$, $\mathbb{C}^2/\mathcal{D}_{2n}$, $\mathbb{C}^2/\mathcal{T}$, $\mathbb{C}^2/\mathcal{O}$, $\mathbb{C}^2/\mathcal{I}$ in a matter of minutes.

6 Blow-up Results

At this point, it is convenient to introduce some notation to refer to the type of singularities of the surfaces arising from the groups $\mathcal{C}_m$, $\mathcal{D}_{2n}$, $\mathcal{T}$, $\mathcal{O}$, $\mathcal{I}$.

Surface	Singularity Type
$\mathbb{C}^2/\mathcal{C}_m$	A_s where $s = m - 1$
$\mathbb{C}^2/\mathcal{D}_{2n}$	D_t where $t = n + 2$
$\mathbb{C}^2/\mathcal{T}$	E_6
$\mathbb{C}^2/\mathcal{O}$	E_7
$\mathbb{C}^2/\mathcal{I}$	E_8

For instance, the singularity that we resolved earlier, which arose from $\mathcal{D}_{2(2)}$ is referred to as D_{2+2} or D_4. The number of vertices (4) in its Dynkin diagram gives us an insight into why the subscripts have been renumbered in this manner!

We begin, (for good reason!) with the surface arising from $\mathcal{I}$, which has the singularity type E_8 . Its matrix representation is given by:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This E_8 singularity blows up into a projective line which has a singularity in the second chart at $[0 : 1 : 0]$, i.e., at the origin of the local affine coordinate system. In this chart, the surface is described by:

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which is the singular surface that arises from the group $\mathcal{O}$ that contains a singularity of type E_7. Blowing this singularity up we get another projective line, with a singularity in the first chart at $[1:0:0]$–or locally speaking, at the origin. In this chart the blown up surface is given by:

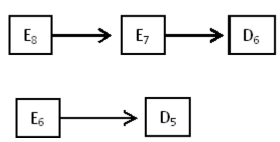
$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

I.e., it is the surface $X^2Y + XY^3 + Z^2 = 0$, which, after the invertible change of coordinates:

$$\begin{aligned} X &\to \frac{Y - Z^2}{\sqrt[5]{2}} \\ Y &\to Z\sqrt[5]{4} \\ Z &\to X \end{aligned}$$

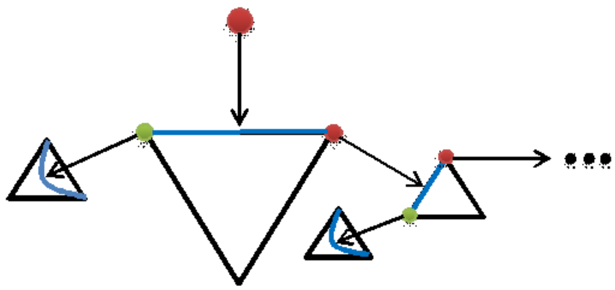
becomes $X^2 + Y^2Z + Z^5$.

One might recognize this as the surface arising from the group $\mathcal{D}_{2n}$ for $n = 4$ (up to a sign change, which happens not to affect the singularity type). Therefore it contains a singularity of type D_6. As we will see shortly the chain of blow-ups for a D_t singularity for any $t \geq 1$ only contains singularities of types $D_{t'<t}$ and A_2, so, in fact, the singularity E_6 does not appear in the chain of blow-ups of E_8, but has its own chain which proceeds similarly. In particular the E_6 singularity is blown up into a projective line with a singularity of type D_5. So far we have:

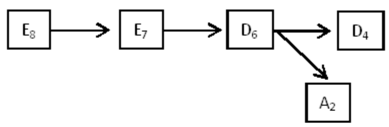


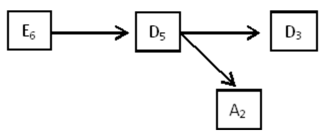
Hence our next task is to resolve the D_t singularity. For $t \geq 5$ the D_t singularity is blown up to a projective line containing both a D_{t-2} singularity and an A_2 singularity. For $s \geq 2$ an A_s singularity blows up into an A_{s-2} singularity. In particular, the blow-up of an A_2 singularity is smooth.

In the image below the large red dot represents a D_t singularity for some large t. It is blown up into a projective line (represented by the blue line on the edge of the large triangle) that is described by $a = 0$ in the projective coordinates of the large triangle representing $\mathbb{C}P^2$. This line has two singularities on the corners of $\mathbb{C}P^2$. The medium size red dot represents a D_{t-2} singularity and the medium size green dot represents an A_2 singularity. The green dot is blown up into a smooth projective line given by $a^2 = bc$ in the projective coordinates of the leftmost $\mathbb{C}P^2$ triangle. The red dot is blown up in the same way as before.

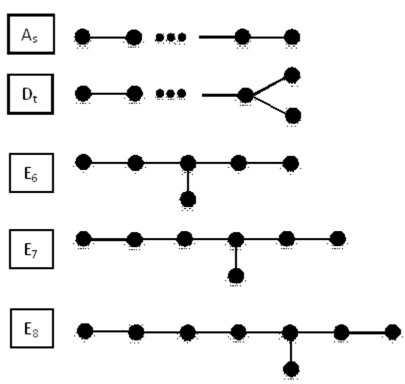


The process continues until we reach, depending on whether t was even or odd respectively, the D_4 singularity, which we have already resolved, or the D_3 singularity, which can also be easily resolved. This lets us extend our blow-up chains for E_8 and E_6 as follows:





Which allows us to compute the Dynkin diagrams for each singularity type based on that of D_4, D_3, and A_2. We have:



7 The Invariant Lattices

Our next goal is to find a rank four lattice in $\mathbb{C}^2$ for each subgroup $\mathcal{C}_m$, $\mathcal{D}_{2n}$, $\mathcal{T}$, $\mathcal{O}$, $\mathcal{I}$ that is invariant under the action of that subgroup, or, if we cannot, to show that there does not exist an invariant lattice for that subgroup. For instance, the subgroup $\mathcal{C}_4$, generated by: $\left\langle \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right\rangle$ preserves the integral lattice: $\{(a+bi, c+di) : a, b, c, d \in \mathbb{Z}\}$, which we can write in terms of its generators as $\langle (1,0),(i,0),(0,1),(0,i)\rangle$ (where the span is that of an additive group).

Theorem 6. *Suppose $\mathcal{C}_m$ is generated by the element $\begin{bmatrix}\beta & 0\\ 0 & \beta^{-1}\end{bmatrix}$, where β is an m^{th} root of unity, then there exists a rank four lattice in $\mathbb{C}^2$ that is invariant under the action of $\mathcal{C}_m$ if and only if $\beta+\beta^{-1}\in\mathbb{Z}$.*

Proof. Let $\mathcal{C}_m$ be generated by the element $B=\begin{bmatrix}\beta & 0\\ 0 & \beta^{-1}\end{bmatrix}$, where β is an m^{th} root of unity, and suppose $\beta+\beta^{-1}\notin\mathbb{Z}$. Suppose Λ is a lattice invariant under $\mathcal{C}_m$. Let $(x,y)\in\Lambda$. Then:

$$B(x,y)+B^{-1}(x,y)=(\beta x,\beta^{-1}y)+(\beta^{-1}x,\beta y)=([\beta+\beta^{-1}]x,[\beta+\beta^{-1}]y)\in\Lambda$$

Since $[\beta+\beta^{-1}]\in\mathbb{R}$ was assumed not to be an integer, we can find $n\in\mathbb{Z}$ s.t. $0<[\beta+\beta^{-1}-n]<1$. Since $n\in\mathbb{Z}$, $n(x,y)=(nx,ny)\in\Lambda$. Hence:

$$([\beta+\beta^{-1}-n]x,[\beta+\beta^{-1}-n]y)=[\beta+\beta^{-1}-n](x,y)\in\Lambda$$

By induction, $[\beta+\beta^{-1}-n]^m(x,y)\in\Lambda$, $\forall m\in\mathbb{N}$. Since $0<[\beta+\beta^{-1}-n]<1$, Λ contains points arbitrarily close to the origin, so is not a lattice.

Conversely, note that since $|\beta|\le 1$, we have $|\beta+\beta^{-1}|\le 2$. Therefore if $[\beta+\beta^{-1}]$ is an integer, it is -2, -1, 0, 1, or 2.
If $[\beta+\beta^{-1}]=-2$ then $\beta=-1$ and $\langle B\rangle=\mathcal{C}_2$.
If $[\beta+\beta^{-1}]=-1$ then $\beta=-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}$ and $\langle B\rangle=\mathcal{C}_3$.
If $[\beta+\beta^{-1}]=0$ then $\beta=\pm i$ and $\langle B\rangle=\mathcal{C}_4$.
If $[\beta+\beta^{-1}]=1$ then $\beta=\frac{1}{2}\pm\frac{i\sqrt{3}}{2}$ and $\langle B\rangle=\mathcal{C}_6$.
If $[\beta+\beta^{-1}]=2$ then $\beta=1$ and $\langle B\rangle=\mathcal{C}_1$.

We will construct invariant lattices for each of these groups shortly, thereby establishing the existence of a lattice invariant under said group. □

Corollary 2. *There is no rank four lattice in $\mathbb{C}^2$ invariant under the action of $\mathcal{O}$.*

Proof. $\mathcal{C}_8\subset\mathcal{O}$. Since no such lattice is invariant under $\mathcal{C}_8$, no lattice is invariant under $\mathcal{O}$. □

Corollary 3. *There is no rank four lattice in $\mathbb{C}^2$ invariant under the action of $\mathcal{I}$*

Proof. $\mathcal{C}_{12} \subset \mathcal{I}$. Since no such lattice is invariant under $\mathcal{C}_{12}$, no lattice is invariant under $\mathcal{I}$. □

Corollary 4. *If $n > 3$ there is no rank four lattice in $\mathbb{C}^2$ invariant under the action of $\mathcal{D}_{2n}$*

Proof. $\mathcal{C}_{2n} \subset \mathcal{D}_{2n}$. Since no such lattice is invariant under $\mathcal{C}_{2n}$ for $n > 3$, no lattice is invariant under $\mathcal{D}_{2n}$ for $n > 3$. □

8 Construction of Invariant Lattices

We now complete Theorem 6 by explicitly constructing invariant lattices for $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, and $\mathcal{C}_6$ and noting that any choice of lattice in $\mathbb{C}^2$ is invariant under $\mathcal{C}_1$. We also construct invariant lattices for $\mathcal{D}_{2(2)}$ and $\mathcal{D}_{2(3)}$.

In the following tables we have constructed a lattice invariant under the each of these groups and, for the four simplest groups, have listed all the points of its invariant lattice that are fixed by the action of that group. For the groups that contain nontrivial subgroups, we have also listed all the points of said lattice that are fixed by each nontrivial subgroup. For each such subgroup, we have grouped together the fixed points of that subgroup that lie in the same orbit under the action of the group.

$\mathcal{C}_2$: Invariant Lattice: $\Lambda_{\mathcal{C}_2} = \mathbb{C}^2/(1,0),(i,0),(0,1),(0,i)$

Subgroup	Fixed Points
$\mathcal{C}_2$	$((a+b),(c+d))$

Where $a, b \in \{0, \frac{1}{2}\}$ and $c, d \in \{0, \frac{i}{2}\}$

$\mathcal{C}_3$: Invariant Lattice: $\Lambda_{\mathcal{C}_3} = \mathbb{C}^2/(1,0),(\frac{1}{2}+i\frac{\sqrt{3}}{2},0),(0,1),(0,\frac{1}{2}+i\frac{\sqrt{3}}{2})$

Subgroup	Fixed Points
$\mathcal{C}_2$	(a,b)

Where $a, b \in \{0, i\sqrt{3}, -i\sqrt{3}$

$\mathcal{C}_4$: Invariant Lattice: $\Lambda_{\mathcal{C}_4} = \mathbb{C}^2/(1,0),(i,0),(0,1),(0,i)$

Variable Assignments: $a = \frac{1}{2}, b = \frac{i+1}{2}$

Subgroup	Fixed Points
$\mathcal{C}_4$	$(0,0)$
	$(b,0)$
	$(0,b)$
	(b,b)
$\mathbb{C}^2$	$(a,0) \equiv (ai,0)$
	$(a,b) \equiv (ai,bi)$
	$(a,a) \equiv (ai,ai)$
	$(a,ai) \equiv (ai,a)$
	$(b,a) \equiv (bi,ai)$
	$(0,a) \equiv (0,ai)$

In order to find the fixed points of all subgroups of $\mathcal{D}_{2(2)}$ we need to know a bit more about its structure, namely, we need to be able to identify its subgroups. Recall that $\mathcal{D}_{2(2)}$ is a group of eight elements generated by $\langle I, J\rangle$ where:

$$I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad J = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Further, it is easy to check that these generators satisfy the relations:

$$I^2 = J^2, \quad IJI = J$$

Which are the sufficient conditions for us to conclude that $\mathcal{D}_{2(2)}$ is Q_8, the quaternion group of eight elements. In accordance with convention we write $IJ = K$ and $I^2 = -1$. Hence we know that the proper subgroups of $\mathcal{D}_{2(2)}$ are the three cyclic groups of order four generated by, respectively, I, J, and K, and the one cyclic group of order two generated by -1.

$\mathcal{D}_{2(2)}$: Invariant Lattice: $\Lambda_{\mathcal{D}_{2(2)}} = \mathbb{C}^2/(1,0),(i,0),(0,1),(0,i)$
Variable Assignments: $a = \frac{1}{2}, b = \frac{i+1}{2}$

Subgroup	Fixed Points
$\mathcal{D}_{2(2)}$	$(0,0)$
	(b,b)
$\langle I\rangle$	$(b,0) \equiv (0,bi)$
$\langle J\rangle$	$(a,ai) \equiv (ai,a)$
$\langle K\rangle$	$(a,a) \equiv (ai,ai)$
$\langle -1\rangle$	$(a,0) \equiv (ai,0) \equiv (0,a) \equiv (0,ai)$

For the last two cases, we will give a bit less information as the number of fixed points is quite large. When we write the number of fixed points in the form $a \times b$, we mean there are a orbits of points fixed by the subgroup (where the orbit is the orbit of the action of the whole group), each of which contains b points.

$\mathcal{C}_6$: Invariant Lattice: $\Lambda_{\mathcal{C}_6} = \mathbb{C}^2/(1,0),(\frac{1}{2}+i\frac{\sqrt{3}}{2},0),(0,1),(0,\frac{1}{2}+i\frac{\sqrt{3}}{2})$

Subgroup	Number of Fixed Points
$\mathcal{C}_6$	1
$\mathcal{C}_3$	4×2
$\mathcal{C}_2$	5×3

In our standard presentation of $\mathcal{D}_{2(3)}$ we chose two generators, $\tilde{R}$ of order six, and $\tilde{s}$ of order four. However, recall that we could equivalently have chosen generators, r, $\tilde{s}$ of orders three and four respectively. Then, writing

$$\mathcal{D}_{2(3)} = \langle \tilde{r}, \tilde{s} \rangle = \langle \begin{bmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \rangle$$

where δ is a $3^r d$ root of unity, we can easily make some observations:
1) $\langle \tilde{r} \rangle \cap \langle \tilde{s} \rangle = 1$
2) $\langle \tilde{r} \rangle \langle \tilde{s} \rangle = \mathcal{D}_{2(3)}$
3) $\langle \tilde{r} \rangle \trianglelefteq \mathcal{D}_{2(3)}$

Now, let φ be the homomorphism from $\langle \tilde{s} \rangle$ into $\mathrm{Aut}(\langle \tilde{r} \rangle)$ defined by:

$$\varphi(\tilde{s}^n) \to (\phi)^n \text{where} \phi : \tilde{r} \to \tilde{r}^2$$

Then we have that:

$$\langle \tilde{r} \rangle \ltimes_{\varphi} \langle \tilde{s} \rangle = \mathcal{D}_{2(3)}$$

In particular we have the relation $\tilde{s}\tilde{r} = \varphi_{(\tilde{s})}(\tilde{r})\tilde{s} = \tilde{r}^2\tilde{s} = \tilde{r}^{-1}\tilde{s}$. This group is sometimes called the dicyclic group of order 12. One can check that its only subgroups are cyclic of orders 2, 3, 4, and 6. The group of order 6 fixes only the origin, which is already a fixed point of the whole group. The groups of order 2 and 3 are normal subgroups and we may proceed as usual:

$\mathcal{D}_{2(3)}$: Invariant Lattice: $\Lambda_{\mathcal{D}_{2(3)}} = \mathbb{C}^2/(1,0), (\frac{1}{2} + i\frac{\sqrt{3}}{2}, 0), (0,1), (0, \frac{1}{2} + i\frac{\sqrt{3}}{2})$

Subgroup	Order	Number of Fixed Points
$\mathcal{D}_{2(3)}$	12	1
$\langle \tilde{r} \rangle$	3	1×4
$\langle \tilde{s}^2 \rangle$	2	2×6

However the 3 subgroups of order 4 are not normal. Any one of them fixes 3 points. The 3 points fixed by any one subgroup of order 4 lie in distinct orbits under the action of $\mathcal{D}_3(2)$, that is, they are all DISTINCT points in $T_{\mathcal{D}_{3(2)}}/\mathcal{D}_{3(2)}$. At first, this might lead one to expect that we will have a total of 9 distinct fixed points. But we DO still have equivalent fixed points showing up here, only now the points that are equivalent are each fixed by a different subgroup of order 4. If we denote each orbit of equivalent fixed points by FP1, FP2, and FP3, then, in particular, FP1 is formed by 3 equivalent fixed points, one of which is fixed by $\langle \tilde{s} \rangle$, one fixed by $\langle \tilde{r}\tilde{s} \rangle$, and one fixed by $\langle \tilde{s}\tilde{r} \rangle$. Similarly for FP2 and FP3.

Fixed Points of the Order 4 Subgroups
Letting $a = \frac{1}{2}$

Subgroup	FP1	FP2	FP3
$\langle \tilde{s} \rangle$	(a, ai)	$(\delta a, \delta ai)$	$(\delta^2 a, \delta^2 ai)$
$\langle \tilde{r}\tilde{s} \rangle$	$(\delta a, \delta^{-1} ai)$	$(\delta^2 a, ai)$	$(a, \delta ai)$
$\langle \tilde{s}\tilde{r} \rangle$	$(\delta^2 a, \delta^{-2} ai)$	$(a, \delta^{-1} ai)$	$(\delta a, ai)$

We will not do so explicitly here, but it is also possible to construct a rank four lattice invariant under the action of $\mathcal{T}$. To do this we will introduce the Hurwitz integers:

Definition 3. *The Hurwitz integers, $\mathcal{H}$, are the subring of the quaternion numbers whose elements take form: $a+bi+ci+dk$, where either all of $a, b, c, d \in \mathbb{Z}$ or all of $a, b, c, d \in \mathbb{Z}+\frac{1}{2}$.*

Next, we note that $\mathcal{T}$ is isomorphic to the group of units of $\mathcal{H}$, denote $\mathcal{H}^\times$. To establish this isomorphism, we replace the generator, $\tilde{d}$ of degree 4, of $\mathcal{T}$ with a new generator, $\tilde{f}$ of order 6, given by $\tilde{f} = \tilde{d}^3\tilde{c}^2$. Then, we may write $\mathcal{T} = \langle \tilde{c}, \tilde{f} \rangle$. This new set of generators satisfies the relations $\tilde{c}^3 = \tilde{f}^3 = (\tilde{c}\tilde{f})^2$—a set of relations that uniquely specifies the abstract binary tetrahedral group. Moreover, the elements $\frac{1+i+j+k}{2}$ and $\frac{1+i+j-k}{2}$ generate the group $\mathcal{H}^\times$ and also satisfy these relations. Thus $\mathcal{H}^\times$ is also isomorphic to the abstract binary tetrahedral group, and hence, isomorphic to $\mathcal{T}$. Explicitly, ϕ defined below is an isomorphism from $\mathcal{T}$ to $\mathcal{H}^\times$:

$$\phi(\tilde{c}) = \frac{1+i+j+k}{2}$$
$$\phi(\tilde{f}) = \frac{1+i+j-k}{2}$$

Moreover, there is a natural lattice that is preserved by a group action of $\mathcal{H}^\times$. Indeed, if we take $\{1, i, j, k\}$ as a basis of $\mathbb{R}^4$, the elements of $\mathcal{H}$ itself give a lattice that is invariant under left multiplication by elements of $\mathcal{H}^\times$. From this, one can construct a corresponding lattice in $\mathbb{C}^2$ invariant under the action of $\mathcal{T}$. For more on this construction see [2, pp. 4-5].

9 Construction of the Corresponding K3 Surfaces

Definition 4. *A K3 surface (over $\mathbb{C}$) is a smooth complex surface which is compact, simply connected, and has trivial canonical bundle.*

Perhaps the most famous examples of K3 surfaces come from the minimal resolution of Kummer surfaces:

Definition 5. *Suppose A is an abelian surface. Then the surface $A/\mathcal{C}_2$, i.e., the surface obtained from A by identifying $a \to -a, \forall a \in A$, is called a Kummer surface.*

(Sometimes the smooth resolution of a Kummer surface is also referred to as a Kummer surface. However, here, when we speak of Kummer surfaces, we will be referring to the singular quartic surface described above.)

Below we construct a Kummer surface, and use a similar construction to create 5 additional surfaces whose minimal resolutions are K3 surfaces. In the latter cases we quotient an abelian variety by 5 of the 6 nontrivial subgroups of $SU(2)$ for which there exist invariant lattices (we omit the case of $\mathcal{T}$). One might call these additional surfaces "generalized" Kummer surfaces.

Let Λ_Γ be a rank 4 lattice in $\mathbb{C}^2$ invariant under the action of a finite subgroup, $\Gamma \subset SU(2)$. Consider the action of Γ on the quotient space $T_\Gamma = \mathbb{C}^2/\Lambda_\Gamma$. We have seen in the preceding section that the action of Γ and the action of Γ's subgroups fix certain points in T_Γ.

Theorem 7. *(Extension of Theorem 4). Let $\gamma \subseteq \Gamma \subset SU(2)$ and Λ_Γ be a lattice invariant under Γ. Let $T_\Gamma = \mathbb{C}^2/\Lambda_\Gamma$. Then each Γ–orbit of γ–fixed points in T_Γ corresponds to one singularity of the space T_Γ/Γ. Further:*
1) If $\gamma \trianglelefteq \Gamma$, this singularity is locally isomorphic to the singularity of $\mathbb{C}^2/\Gamma$ at the origin.
2) Otherwise, let the smallest subgroup of Γ that acts transitively on the points of each orbit of γ–fixed points be denoted ν. Then if $\nu \trianglelefteq \Gamma$, let $\mu = \Gamma/\nu$. Then the singularity described above is locally isomorphic to the singularity of $\mathbb{C}^2/\mu$ at the origin.

Remark 1. *Note that if both γ and ν are normal in Γ then $\Gamma/\nu = \gamma$. Thus $\mu = \gamma$ and the statements (1) and (2) are equivalent.*

Remark 2. *Note that each γ–fixed point of T_Γ does not necessarily correspond to a unique singularity of the space T_Γ/Γ. Often, multiple γ–fixed points will lie in the same orbit under action of Γ and therefore be the same point in the space T_Γ/Γ. Other times, (namely when γ is not normal in Γ) a fixed point of γ will be equivalent to a fixed point of a conjugate subgroup of γ.*

Our results from last section may be summarized as:

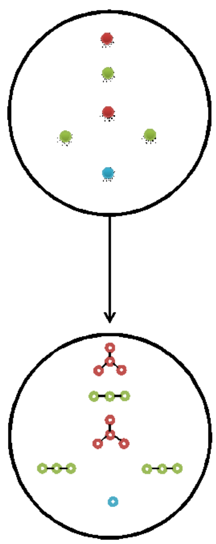
.	# of Pts Fixed By a Group $\cong$ to:					
Quotient Space	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_6$	$\mathcal{D}_{2(2)}$	$\mathcal{D}_{2(3)}$
$T_{\mathcal{C}_2}$	16					
$T_{\mathcal{C}_3}$		9				
$T_{\mathcal{C}_4}$	6		4			
$T_{\mathcal{C}_6}$	5	4		1		
$T_{\mathcal{D}_{2(2)}}$	1		3		2	
$T_{\mathcal{D}_{2(3)}}$	2	1	3			1

So, according to Theorem 7 we have:

.	# of Singularities of Type:					
Singular Surface	A_1	A_2	A_3	A_5	D_4	D_5
$T_{\mathcal{C}_2}/\mathcal{C}_2$	16					
$T_{\mathcal{C}_3}/\mathcal{C}_3$		9				
$T_{\mathcal{C}_4}/\mathcal{C}_4$	6		4			
$T_{\mathcal{C}_6}/\mathcal{C}_6$	5	4		1		
$T_{\mathcal{D}_{2(2)}}/\mathcal{D}_{2(2)}$	1		3		2	
$T_{\mathcal{D}_{2(3)}}/\mathcal{D}_{2(3)}$	2	1	3			1

In particular statement (1) of Theorem 7 has been used for every case except $T_{\mathcal{D}_{2(3)}}/\mathcal{D}_{2(3)}$, where statement (2) has been applied with $\nu = \langle \tilde{r} \rangle$.

Since we have already worked out the resolutions of these singularity types we know what the resolutions of these singular surfaces are! For instance, the surface $T_{\mathcal{D}_{2(2)}}/\mathcal{D}_{2(2)}$ has 6 singularities, 2 of type D_4, 3 of type A_3 and 1 of type A_1. We resolve each of these singularities by replacing them with the copie(s) of $\mathbb{C}P^1$ that we get from blowing up that singularity. In the diagram below the solid dots of colors red, green, and blue, respectively represent the singularities of type D_4, A_3, and A_1 of the singular surface $T_{\mathcal{D}_{2(2)}}/\mathcal{D}_{2(2)}$. Each of these points is replaced by the blow-up of that point, represented by the configuration of hollow dots of the same color (where the black lines are used to connect intersecting copies of $\mathbb{C}P^1$). The resulting surface is a smooth K3 surface.



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Appendix A.
Some Sample Matlab Code for Singularity Resolutions

Given the matrix representation of a surface with a singularity at the origin, the following code blows up said singularity.

```
function [ ] = Blow( A )
%   UNTITLED2 Summary of this function goes here
%   Given a variety in matrix form this function outputs the surface
%   blow-up in all three charts as well as giving the image of the point
%   (0,0,0) in each chart, as a two element column.  For instance,
%   ba_ca=[x;y] represents the projective lines(s) (b/a)^x+(c/a)^y=0.
S=sum(A);
M=min(S);
R=[M M M];
A_a=[S-R;A(2,:);A(3,:)]
A_b=[A(1,:);S-R;A(3,:)]
A_c=[A(1,:);A(2,:);S-R]
anonz=Zero_a(A_a);
bnonz=Zero_b(A_b);
cnonz=Zero_c(A_c);
ba_ca=[anonz(2,:);anonz(3,:)]
ab_cb=[bnonz(1,:);bnonz(3,:)]
ac_bc=[cnonz(1,:);cnonz(2,:)]
end
```

Where Zero-a is defined as:

```
function [ az ] = Zero_a( B )
%
%   Given a blown up surface, this gives the image of the blow-up
%   of the point (0,0,0) in the first chart, as
%   a three term column, where [u;v;w] represents the equation
%   X^u +(b/a)^v +(c/a)^w=0. Zero_b and Zero_c are defined similarly.
if B(1,1)>0
    C1=[];
else C1=B(1:3,1);
end
if B(1,2)>0
    C2=[];
else C2=B(1:3,2);
end
if B(1,3)>0
    C3=[];
else C3=B(1:3,3);
end
az=[C1';C2';C3']';
end
```

Zero-b and Zero-c are defined similarly.