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Abstract

Cwatsets were originally defined as subsets of $\mathbb{Z}_2^d$ that are "closed with a twist." Attempts have been made to generalize them, but the generalizations have failed to produce notions of subcwatset and quotient cwatset that behave naturally.

We present a new, abstract definition that appears to avoid these problems. The relationship between this new definition and its predecessor is similar to that between the abstract definition of "group" and its original meaning as a set of permutations. To justify the broader definition, we use small cancellation theory to prove a result analogous to the statement that every group is isomorphic to some permutation group. After developing the notion of a quotient cwatset, we prove an analogue of the First Homomorphism Theorem.

1 Introduction

1.1 History and Motivation

The term "cwatset" was coined in [5] to describe a certain type of subset of $\mathbb{Z}_2^d$ that has a "closure with a twist" property. The set

$$F = \{000, 110, 101\},$$

for example, is not a subgroup of $\mathbb{Z}_2^3$ since it is not closed. However,

$$F + 110 = \{110, 000, 011\} = F^{(1,2)}$$

and

$$F + 101 = \{101, 011, 000\} = F^{(1,3)},$$

so F is closed "with a twist." Cwatsets were initially motivated by statistics, but have since been studied purely for algebraic properties. Further background is discussed by Biss [1].

The notions of cwatset morphism and subcwatset proposed in [1] and [2] have the disadvantage that the homomorphic image of a subcwatset may not be a subcwatset of the codomain. We will present an alternative definition that solves this problem.

We also propose the first definition of a quotient cwatset. This requires us first to abstract the definition of cwatset, as one needs to abstract the definition of group to define quotient groups.

In the case of groups, the important structural information is the group's multiplication table. With regards to a cwatset, it seems to be a particular transitive group action on the cwatset. This action, represented by a group of permutations known as the "L-group" of the cwatset, contains all of the structure of the cwatset. This allows us to define a cwatset abstractly, and not specifically as a subset of $\mathbb{Z}_2^d$. The precise definition will be given in Section 5.

A cwatset, as classically defined, is a subset of $\mathbb{Z}_2^d$; however, the procedure that produces the L-group relies only on the fact that $\mathbb{Z}_2^d$ is a group. It is natural, then, to ask what happens if we extend the definition of a cwatset to an arbitrary group. We will explore such generalized cwatsets in Sections 3 and 4.

Returning to our analogy, we see that since every abstract group is isomorphic to some permutation group, the abstraction does not produce essentially new groups. Similarly, we will show in Section 4 that the same is true of abstracting generalized cwatsets.

In Section 5, we abstract the definition of a cwatset and develop the definition of cwatset homomorphism in this abstract context. In Section 6, we develop the definition of subcwatset, pointing out certain problems along the way. Section 7 offers a resolution of these problems, and we arrive in Section 8 at the definition of a quotient cwatset, where we prove an analogue to the First Homomorphism Theorem. Section 9 demonstrates that we can naturally reformulate the definition of normality in terms of inner automorphisms.

1.2 Notation and Preliminaries

$\mathrm{Sym}(A)$ denotes the symmetric group on a set A, defined to be the set of all bijections from A to itself. If n is a positive integer, then $\mathrm{Sym}(n)$ denotes the symmetric group on some set of cardinality n.

We will adopt the convention of many group theorists to compose functions from left to right: if $f : A \to B$ and $g : B \to C$, then $fg : A \to C$ denotes the composition map of f followed by g. Because of this, we generally write $(a)f$ instead of $f(a)$.

If G is a group, $g \in G$, and $\alpha \in \mathrm{Aut}(G)$, then we may write g^α to mean $\alpha(g)$.

2 Prior Work

Definition 2.1. *Let $C \subseteq \mathbb{Z}_2^d$ be nonempty, and let Sym(d) act on $\mathbb{Z}_2^d$ by permuting the d components. We say C is a* **cwatset** *if for every $x \in C$, there is some $\sigma \in Sym(d)$ such that $C^\sigma + x = C$.*

We first observe that $0 \in C$ for every cwatset C, since if $x \in C$ and $C^\sigma + x = C$, then there is some $c \in C$ such that

$$c^\sigma + x = x,$$

implying that $c = 0$. Now, since $\mathrm{Sym}(d)$ acts on $\mathbb{Z}_2^d$ by automorphisms, we have a semidirect product $\Gamma = \mathbb{Z}_2^d \rtimes \mathrm{Sym}(d)$. Associating $\mathbb{Z}_2^d$ and $\mathrm{Sym}(d)$ with their isomorphic copies in Γ, we can write $\Gamma = \mathrm{Sym}(d)\mathbb{Z}_2^d$, and then can define an action of Γ on the set $\mathbb{Z}_2^d$ by

$$x \cdot (\sigma g) = x^\sigma + g$$

for $x, g \in \mathbb{Z}_2^d$ and $\sigma \in \mathrm{Sym}(d)$. We next define the **M-group** of a cwatset C, denoted M_C, to be the (set) stabilizer of C under the action of Γ.

Lemma 2.2. *If $C \subseteq \mathbb{Z}_2^d$ is a cwatset, then M_C acts transitively on C.*

Proof. It is sufficient to show that the element $0 \in C$ can be mapped to an arbitrary $x \in C$ by some element of M_C. For each $x \in C$, there is by definition some $\sigma \in \mathrm{Sym}(d)$ such that $C \cdot (\sigma x) = C$, so $\sigma x \in M_C$. Now

$$0 \cdot (\sigma x) = 0^\sigma + x = x,$$

as required. $\square$

We now define the **L-group** of a cwatset C, denoted L_C, to be the image of the permutation representation of the action of M_C on C; hence, L_C is a transitive subgroup of $\mathrm{Sym}(C)$.

In [6], Smith attempted to generalize this definition to subsets of an arbitrary group G through what he called a "gc-set." His definition is equivalent to the following:

Definition 2.3. *Let G be an arbitrary group, and let $C \subseteq G$. We say that C is a* **gc-set** *if for every $x \in C$, there is some $\sigma \in Aut(G)$ such that $C^\sigma x = C$.*

It is clear that every cwatset is a gc-set, since $\mathrm{Sym}(d)$ acts on $\mathbb{Z}_2^d$ by automorphisms, so we can think of $\mathrm{Sym}(d) \subseteq \mathrm{Aut}(\mathbb{Z}_2^d)$. However, we quickly run into a problem that Smith does not discuss. We can follow the same line of reasoning as with cwatsets to construct an L-group of a gc-set. However, if C is a cwatset, then the L-group of C as a cwatset can be different from the L-group of C as a gc-set. The problem arises from the fact that a gc-set allows for *any* automorphism of the group, while a cwatset only deals with automorphisms generated by $\mathrm{Sym}(d)$.

For example, the set $C = \mathbb{Z}_2^2$ is a cwatset whose L-group is a Sylow 2-subgroup of $\mathrm{Sym}(C)$. Viewed as a gc-set, however, the L-group of C is the entire group $\mathrm{Sym}(C)$ since there are more automorphisms of $\mathbb{Z}_2^2$ to work with. If the underlying structure of a cwatset is to be its L-group, then a cwatset is a *different object* viewed as a gc-set. Accordingly, we will modify the definition of a gc-set to accommodate this problem.

Nevertheless, this seems like the right track: In $\mathbb{Z}_7$, $C = \{0, 1, 3\}$ is a gc-set with L-group isomorphic to the alternating group in Sym(3). As shown by Goodwin and Lin [3], there is no cwatset with such an L-group. Thus, the gc-set indeed provides a tool for constructing *structurally new* cwatsets.

3 Basic Theory

Definition 3.1. *Let G be a group and $A \subseteq Aut(G)$ be a subgroup. We say that a nonempty subset $C \subseteq G$ is a* (G, A)**-cwatset** *if for each $x \in C$, there is some $\sigma \in A$ such that $C^\sigma x = C$.*

Note that a classical cwatset C, that is, a cwatset according to Definition 2.1, is a $(\mathbb{Z}_2^d, \mathrm{Sym}(d))$-cwatset. If the groups G and A are understood, then we simply call C a cwatset.

Now, the condition that C be nonempty implies that $1 \in C$, since if x, σ are as above, then $x \in C = C^\sigma x$, so $x = y^\sigma x$ for some $y \in C$, and hence $y = 1$.

It may happen that for a given $x \in C$, there are many automorphisms σ such that $C^\sigma x = C$, and in such a case, it is unclear whether there may be a preferred automorphism. It is useful, then, to study the set

$$M_C = \{(x, \sigma) \mid C^\sigma x = C\} \subseteq G \times A$$

consisting of all such pairs that "work." Note that C is a subgroup of G precisely when $(x, \mathrm{id}) \in M_C$ for all $x \in C$. In particular, any subgroup of G is a (G, A)-cwatset.

We now give a more useful description of M_C. Since $A \subseteq \mathrm{Aut}(G)$, we have a natural semidirect product $\Gamma = G \rtimes A$, and we will identify G and A with their isomorphic copies in Γ, writing $\Gamma = AG$. We then can define an action of Γ on G by

$$x \cdot (\sigma g) = x^\sigma g$$

for all $x, g \in G$ and $\sigma \in A$.

Extending this action to the subsets of G, we note that for any subset $C \subseteq G$, the (set) stabilizer Γ_C is set M_C defined above. Since M_C stabilizes C, the action of Γ on G induces an action of M_C on C. The group M_C is referred to as the **M-group** of C.

Lemma 3.2. *If $C \subseteq G$ is a (G, A)-cwatset, then M_C acts transitively on C.*

Proof. The proof is identical to that in the classical case (Lemma 2.2). □

Definition 3.3. *Let C be a cwatset. The* **L-group** *of C, denoted L_C, is defined to be the image of the permutation representation of the action of M_C on C.*

Note that if $\sigma g \in M_C$ induces the permutation $\mu \in L_C$, then $g = 1 \cdot \mu$. We define the **projection map** $\pi : L_C \to C$ by

$$\mu \mapsto 1 \cdot \mu.$$

We now get the following correspondence.

Lemma 3.4. *Let G be any group. Considering G to be a (G, A)-cwatset for some $A \subseteq Aut(G)$, suppose $H \subseteq L_G$ is subgroup. Then the image of H under π is a (G, A)-cwatset.*

Proof. Fix $x \in H\pi$, and let $\nu \in H$ with $x = \nu\pi$. By definition of the L-group, there is some $\sigma \in A$ such that $\sigma x \in G \rtimes A$ induces the permutation ν. Therefore

$$(H\pi) \cdot (\sigma x) = (H\pi) \cdot \nu = (1 \cdot H) \cdot \nu = 1 \cdot (H\nu) = 1 \cdot H = H\pi,$$

so $H\pi$ is a G-cwatset. $\square$

The group H is called a **covering group** for $H\pi$ and we say that $H\pi$ is **covered** by H. We summarize the previous results in the following theorem.

Theorem 3.5. *Let G be any group, viewed as a (G, A)-cwatset for some $A \subseteq Aut(G)$. Then $C \subseteq G$ is a cwatset iff it is covered by some subgroup $H \subseteq L_G$.*

Proof. Lemma 3.4 shows that all subgroups cover some cwatset. Conversely, a cwatset C is always covered by its L-group L_C. $\square$

4 How Many Cwatsets Are There?

Since every cwatset C determines some transitive subgroup $L_C \subseteq \mathrm{Sym}(C)$, we pose the following natural converse: Suppose we started with some set C and a transitive subgroup $L \subseteq \mathrm{Sym}(C)$. Does there exist some group $G \supseteq C$ and some subgroup $A \subseteq \mathrm{Aut}(G)$ such that C is a (G, A)-cwatset with L-group equal to L? The answer, at least for finite sets C, is yes, and we present an even stronger result.

Theorem 4.1. *Let C be a finite set and let $L \subseteq Sym(C)$ be a transitive subgroup. There exists some group $G \supseteq C$ such that C is a $(G, Aut(G))$-cwatset with L-group equal to L.*

Note that here we demand that the subgroup $A \subseteq \mathrm{Aut}(G)$ be the full group of automorphisms $\mathrm{Aut}(G)$.

Proof. We will actually construct the desired group. Writing $n = |C| - 1$, label the elements of C as $\{1, a_1, \ldots, a_n\}$, and let $X = \{a_1, \ldots, a_n\}$, with $F = F(X)$ the free group on the set X of generators. To each element $\sigma \in \mathrm{Sym}(C)$, we define $f_\sigma : F \to F$ by

$$f_\sigma(a) = (a \cdot \sigma)(1 \cdot \sigma)^{-1},$$

for all generators $a \in C$, and extend this to a homomorphism of F (note that $f_\sigma(1) = 1$ by the above formula). It is easy to check that each f_σ is a bijection.

Note that if $\sigma \in \mathrm{Sym}(C)$ and $y \in C$, then

$$y \cdot \sigma = (y \cdot \sigma)(1 \cdot \sigma)^{-1}(1 \cdot \sigma) = y^{f_\sigma}(1 \cdot \sigma), \tag{1}$$

so $(f_\sigma)(1 \cdot \sigma) \in F \rtimes \mathrm{Aut}(F)$ acts like σ on C.

Suppose we have a set of relations R such that the group $G = \langle X \mid R \rangle$ has the following properties:

1. The quotient mapping $C \to G$ is an injection.
2. For each $\sigma \in L$, f_σ induces an automorphism of G.
3. For each $\tau \in \mathrm{Sym}(C)$ such that $\tau \notin L$, f_τ does not induce an automorphism of G.

Let $\Gamma = G \rtimes \mathrm{Aut}(G)$. We claim that C is a $(G, \mathrm{Aut}(G))$-cwatset with $L_C = L$. First of all, property (1) allows us to embed C in G, and property (2) implies that $(f_\sigma)(1 \cdot \sigma) \in \Gamma$ for all $\sigma \in \mathrm{Sym}(C)$.

Since L is transitive, for each $x \in C$, there is some $\sigma \in L$ such that $1 \cdot \sigma = x$. By equation (1),

$$C \cdot (f_\sigma x) = C \cdot \sigma = C,$$

implying that C is indeed a $(G, \mathrm{Aut}(G))$-cwatset.

Similarly, for each $\sigma \in L$, the element $f_\sigma(1 \cdot \sigma) \in \Gamma$ induces the action of σ on C, so $L \subseteq L_C$.

Conversely, suppose $\sigma \in L_C$. Then there is some element $\alpha g \in \Gamma$ that induces the action of σ on C; i.e., for all $y \in C$,

$$y \cdot \sigma = y \cdot (\alpha g) = y^\alpha g,$$

so

$$y^\alpha = (y \cdot \sigma) g^{-1}$$

for all $y \in C$. Since α is an automorphism,

$$1 = 1^\alpha = (1 \cdot \sigma) g^{-1},$$

so $g = 1 \cdot \sigma$. We then see that α has the same action as f_σ on C, so $\alpha = f_\sigma$ since $G = \langle C \rangle$. Thus $f_\sigma \in \mathrm{Aut}(G)$, so property (3) implies that $\sigma \in L$, and hence $L = L_C$.

The proof is then reduced to showing the existence of the set R, which is asserted by the Lemma below. □

Lemma 4.2. *Under the assumptions of Theorem 4.1, there exists a set of relations R such that the group $G = \langle X \mid R \rangle$ has the following properties:*

1. *The quotient mapping $C \to G$ is an injection.*
2. *For each $\sigma \in L$, f_σ induces an automorphism of G.*

3. *For each $\tau \in Sym(C)$ such that $\tau \notin L$, f_τ does not induce an automorphism of G.*

To find the proper set of relations R, we need some terminology from small cancellation theory.

Let F be a free group on a set X of generators, and let R be a set of words of F. We say that R is **symmetrized** if every element of R is cyclically reduced, and for each $r \in R$, all cyclically reduced conjugates of both r and r^{-1} are in R. If $r, s \in R$ are distinct elements, with $r = ab$ and $s = ac$ (with equality holding letter for letter without cancellation), then we say a is a **piece relative to the set** R. If the set R is understood, then we simply say that a is a piece. For a fixed $\lambda > 0$, the set R **satisfies** $C'(\lambda)$ if the following holds:

If $r \in R$ and $r = ab$ where a is a piece, then $|a| < \lambda|r|$.

We will use the following theorem (see, e.g., [4, Chapter V, Theorem 4.4]).

Theorem 4.3. *Let F be a free group. Let R be a symmetrized subset of F and N the normal closure of R. If R satisfies $C'(\frac{1}{6})$, then every nontrivial element $w \in N$ contains a subword s of some $r \in R$ with $|s| > \frac{1}{2}|r|$.*

Proof of Lemma 4.2. If $n = 1$, then the lemma is trivial, so we assume that $n > 1$.

Let
$$w = \prod_{k=1}^{100} \prod_{i=1}^{n} a_i^{k \cdot 10^n + 10i},$$
where the product is taken writing from left to right, and let S be the symmetrized set generated by $\{f_\sigma(w) \,|\, \sigma \in \mathrm{Sym}(C)\}$. We claim that S satisfies $C'(\frac{1}{6})$.

We need to show that for any $\sigma, \tau \in \mathrm{Sym}(C)$, no cyclic permutations of $f_\sigma(w)$ and $f_\tau(w)$ (or their inverses) contain a common initial segment of length greater than $\frac{1}{6}$ of either's size. We will check one such case.

Let $\sigma, \tau \in \mathrm{Sym}(C)$ with $\sigma \neq \tau$ and $1 \cdot \sigma = 1 \cdot \tau \neq 1$, and let $y = 1 \cdot \sigma$. We will show that no cyclic permutation of $f_\sigma(w)$ and $f_\tau(w)$ contain a common initial segment of length greater than $\frac{1}{6}$ of either's size. Suppose instead that this is false. Then there is some subword v of both $f_\sigma(w)$ and $f_\tau(w)$ with length greater than $\frac{1}{12}$ of either word. We will now analyze the structure of $f_\sigma(w)$:

Let $x_i = a_i \cdot \sigma$. Since σ is a bijection, we have $x_i \neq y$ for all i, and so there is no cancellation between each power of $f_\sigma(a_i) = x_i y^{-1}$. Thus the only cancellation occurs for the i for which $x_i = 1$, for which we get the string
$$x_{i-1} y^{-(k \cdot 10^n + 10i + 1)} x_{i+1}$$

as a subword in $f_\sigma(w)$. Note that the above holds true for τ as well, and so $|f_\sigma(w)| = |f_\tau(w)|$. Since $|v| > \frac{1}{12}|f_\sigma(w)|$, v contains some subword of the form

$$(x_1y^{-1})^{k\cdot 10^n+10} \cdots x_{i-1}y^{-(k\cdot 10^n+10i+1)}x_{i+1}\cdots(x_ny^{-1})^{k\cdot 10^n+10n}.$$

Because the power $-(k\cdot 10^n + 10i+1)$ of y is found uniquely in $f_\tau(w)$, this subword must match exactly with the corresponding subword in $f_\tau(w)$, implying that $x_i = a_i \cdot \tau$ for all i. But this contradicts the fact that $\sigma \neq \tau$, and hence no cyclic permutation of $f_\sigma(w)$ and $f_\tau(w)$ contain a common initial segment of length greater than $\frac{1}{6}$ of either's size, as desired.

Now let R be the symmetrized set generated by $\{f_\sigma(w) \mid \sigma \in L\}$. Since $R \subseteq S$ and S satisfies $C'(\frac{1}{6})$, certainly R does as well. Let N be the normal closure of R. We claim that the group $G = F/N$ satisfies properties (1) - (3) above.

Let $x, y \in C$, and suppose that $x = y$ in G, so $xy^{-1} \in N$. Since both x and y are either letters or the identity, $|xy^{-1}| \leq 2$. If $xy^{-1} \neq 1$ in F, then Theorem 4.3 implies that xy^{-1} contains some subword s of some $r \in R$ with $|s| > \frac{1}{2}|r|$. But all elements of R certainly have lengths greater than 5, so the subword s would have length greater than 3, which is a contradiction since $|xy^{-1}| \leq 2$. Thus $xy^{-1} = 1$ in F, so $x = y$, and hence property (1) holds.

Property (2) follows trivially from the definition of G. Now suppose that for $\tau \in \mathrm{Sym}(C)$ such that $\tau \notin L$, f_τ induces to an automorphism of G, i.e., $f_\sigma(w) = 1$ in G. By Theorem 4.3, $f_\sigma(w)$ contains some subword s of some $r \in R$ with $|s| > \frac{1}{2}|r|$. Thus some cyclic permutation w' of $f_\sigma(w)$ and r' of r begin with the same subword s. But $w', r' \in S$, and S satisfies $C'(\frac{1}{6})$, implying that $|s| < \frac{1}{6}|r|$, a contradiction, so property (3) follows. ☐

Question 4.4. *Does Theorem 4.1 generalize to infinite sets?*

5 Cwatsets and Homomorphism

Definition 5.1. *Let X be a set with a special element $1 \in X$, and let M be a group that acts transitively on X. We say that $C \subseteq X$ is an (X, M)-***cwatset** *if there is some subgroup $H \subseteq M$ that covers C, i.e., such that $C = \{1 \cdot \mu \mid \mu \in H\}$.*

Note that if G is a group, then every (G, A)-cwatset is a (G, L_G)-cwatset by Theorem 3.5. Henceforth, unless otherwise specified, the term *cwatset* will be used with reference to the above definition. The element 1 is referred to as the **identity** element of X. As before, we define the map $\pi : M \to X$ by $\mu \mapsto 1 \cdot \mu$.

Definition 5.2. *Let $C \subseteq X$ be an (X, M)-cwatset. The* **M-group** *of C, denoted M_C, is defined to be the set stabilizer of C under the action of M. We then define the* **L-group** *of C, denoted L_C, to be the permutation representation of the action of M_C on C.*

Note that since X is an (X, M)-cwatset, $M_X = M$. Therefore, we will write (X, M_X) instead of (X, M). Also, these definitions coincide with the previous definitions in the case of a (G, A)-cwatset.

Definition 5.3. *Suppose C is an (X, M_X)-cwatset and D is a (Y, M_Y)-cwatset, and let $\phi : C \to D$. We say that ϕ is a* **homomorphism** *if there exists a group homomorphism $\Phi : L_C \to L_D$ such that the following diagram commutes:*

$$\begin{array}{ccc} L_C & \xrightarrow{\Phi} & L_D \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow{\phi} & D \end{array}$$

In this case, Φ is called a **lift** *of ϕ. An injective homomorphism is called a* **monomorphism**, *a surjective homomorphism is called an* **epimorphism**, *and a bijective homomorphism is called an* **isomorphism**.

If $\phi : C \to D$ is a homomorphism, the set

$$\ker(\phi) = \{x \in C \mid \phi(x) = 1\}$$

is the **kernel** *of ϕ.*

The analogous definition suggested in [1] replaces the L-groups L_C and L_D with the corresponding M-groups M_C and M_D. In the case of classical cwatsets, these definitions are equivalent, since L_C is found isomorphically in M_C for any classical cwatset C (its image in M_C is known as the **action group**) [2]. However, this is not true in general, so the two definitions are not equivalent, and we must therefore make a choice. A close reading of [1] and [2] shows that the (equivalent) L-group statement is used. In light of this, we feel justified in choosing the L-group for Definition 5.3.

Example 5.4. Let $C = \mathbb{Z}_2$ be a $(\mathbb{Z}_2, \mathrm{Aut}(\mathbb{Z}_2))$-cwatset and let $D = \{0, 2\} \subseteq \mathbb{Z}$ be a $(\mathbb{Z}, \mathrm{Aut}(\mathbb{Z}))$-cwatset. Note that $L_C \cong \mathrm{Sym}(2) \cong L_D$, and so we obviously have $C \cong D$. The cwatset D is Example 1.5 of [6], and we see now that it is isomorphic to a classical cwatset.

Lemma 5.5. *If $\phi : C \to D$ and $\psi : D \to E$ are homomorphisms, then:*

1. *$\phi\psi$ is a homomorphism.*

2. *$\ker(\phi) = 1$ iff ϕ is injective.*

Proof. Immediate from the definition. □

Lemma 5.6. *Let $\phi : C \to D$ be a homomorphism of cwatsets with lift $\Phi : L_C \to L_D$. If ϕ is injective, then Φ is injective.*

Proof. Let $\mu \in \ker(\Phi)$. Then

$$1 = \mu\Phi\pi = \mu\pi\phi,$$

so $\mu\pi \in \ker(\phi)$, and hence $\mu\pi = 1$. Fix $x \in C$, and set $y = x \cdot \mu$. Since L_C acts transitively on C, there is some $\nu \in L_C$ with $x \cdot \nu = 1$. Since $\mu^\nu \in \ker(\Phi)$, we have $(\mu^\nu)\pi \in \ker(\phi) = 1$, so

$$1 = (\mu^\nu)\pi = 1 \cdot (\nu^{-1}\mu\nu) = x \cdot (\mu\nu) = y \cdot \nu.$$

We now have $x = 1 \cdot \nu^{-1} = y = x \cdot \mu$, so $\mu = 1$. Thus $\ker(\Phi) = 1$, as desired. $\square$

We would like the analogous statement to be true of surjectivity; that is, we ask if ϕ is surjective, is Φ surjective as well? Unfortunately, that does not hold. For example, let D be any cwatset with $L_D < \mathrm{Sym}(D)$, and $\phi : C \to D$ a surjective homomorphism with $\Phi : L_C \to L_D$ surjective. We may now view D as a $(D, \mathrm{Sym}(D))$-cwatset, thus expanding its L-group. The map Φ is still a homomorphism, but now it only maps onto the old L_D, and not the entire group $\mathrm{Sym}(D)$.

Nevertheless, this concept is important, so we capture it in the following definition.

Definition 5.7. *Let $\phi : C \to D$ be a homomorphism of cwatsets. We say that ϕ is* **totally surjective** *if ϕ is surjective and its lift $\Phi : L_C \to L_D$ is surjective.*

6 Subcwatset

The definition of a homomorphism paves the way for a provisional definition of a subcwatset.

Definition 6.1. *Let C be an (X, M_X)-cwatset. A subset $B \subseteq C$ is a* **subcwatset** *of C if it is an (X, M_X)-cwatset and the inclusion map $i : B \to C$ is a homomorphism.*

Lemma 6.2. *Let C be an (X, M_X)-cwatset. If $B \subseteq C$ is a subcwatset of C and $A \subseteq B$ is a subcwatset of B, then A is a subcwatset of C.*

Proof. This follows from Lemma 5.5. $\square$

Suppose that groups G and H act on a set Ω and that $\theta : H \to G$ is a monomorphism. Then $\theta(H)$ acts on Ω in two ways: first by inheriting the action of G, and second by the induced action of H. When these two actions are the same, we say that $\theta(H)$ **respects the action of** H **on** Ω.

Theorem 6.3. *Let C be an (X, M_X)-cwatset, and suppose that $B \subseteq C$. Then the following are equivalent:*

1. *B is a subcwatset of C.*

2. *B is an (X, M_X)-cwatset, and L_C contains an isomorphic image of L_B that respects the action of L_B on B.*

Proof. First suppose (1), and let $i : B \to C$ be the inclusion with pre-lift $I : L_B \to L_C$. By Lemma 5.6, I is injective, so L_C contains an isomorphic image of L_B. Now let $b \in B$ with $\beta \in L_B$ such that $\beta\pi = b$, and let $\mu \in L_B$. We have

$$b \cdot (\mu I) = (\beta\pi i) \cdot (\mu I) = (\beta I)(\mu I)\pi = (\beta\mu)I\pi = (\beta\mu)\pi i = \beta\pi \cdot \mu = b \cdot \mu,$$

as desired.

Now suppose (2), and let $I : L_B \to L_C$ be an injective homomorphism such that $L_B I$ respects the action on B. We claim that I respects the injection $i : B \to C$. Let $\mu \in L_B$. Then

$$\mu I\pi = 1 \cdot (\mu I) = 1 \cdot \mu = \mu\pi = \mu\pi i,$$

so $I\pi = \pi i$. Thus i is a homomorphism with lift I, and hence B is a subcwatset of C. $\square$

Unfortunately, we now run into two significant problems.

Fact 6.4. *The homomorphic image of a subcwatset need not be a subcwatset.*

In fact, the isomorphic image of a subcwatset need not be a subcwatset. Furthermore, this problem is not a consequence of our generalization, as the following classical counterexample shows.

Writing a classical cwatset as a matrix whose rows are its elements, let

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

and let $\phi : C \to D$ be defined by mapping row i of C to row i of D; this is an isomorphism [3]. Now, it is straightforward though tedious to show that

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

is a subcwatset of C, but

$$B\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

is not a subcwatset of D.

Fact 6.5. *The homomorphic preimage of a subcwatset may not be a subcwatset.*

In particular, the kernel of a homomorphism need not be a subcwatset. Again, this is already a problem for classical cwatsets, as the following counterexample [2] shows.

Let

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and let $\phi : C \to D$ be defined by mapping the first three elements to $0 \in D$, and the second three to $1 \in D$. Here, the kernel

$$\ker(\phi) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \subseteq C$$

is not a subcwatset of C.

7 More Subcwatsets and Normality

Facts 6.4 and 6.5 make it clear that the definitions presented thus far are flawed. What remains unclear is whether the problem lies in the definition of a subcwatset or in that of a homomorphism. We propose a possible solution to this problem through a modification of the definition of subcwatset.

Definition 7.1. *Let C be an (X, M_X)-cwatset, and let $B \subseteq C$. We say that B is a* **subcwatset** *of C if it is a (C, L_C)-cwatset.*

We will see at the end of this section that this new definition fixes most of the problems of the old one. For the remainder of this paper, we will use Definition 7.1 when referring to a subcwatset. Note that this definition would be *impossible to formulate* outside of our abstract setting.

If B is a subcwatset of C, then some subgroup of L_C covers B. This is similar to the "alternative definition" of a subcwatset presented in [2] with one crucial difference. In that setting, the subcwatset B is still treated (using our notation) as an (X, M_X)-cwatset, so M_B might not be contained in L_C. Here, however, B is treated as a (C, L_C)-cwatset, meaning that $M_B \subseteq L_C$ *by definition.*

If $B \subseteq C$ is a subset, then we define

$$L_B^C = \{\mu \in L_C \mid \mu\pi \in B\}.$$

The following observation will be used in Section 9.

Lemma 7.2. *Let C be a cwatset and $B \subseteq C$ be a subcwatset. Then $L_1^C M_B = L_B^C$.*

Proof. The inequality $L_1^C M_B \subseteq L_B^C$ is trivial. For the reverse inequality, we let $\mu \in L_B^C$. Since M_B covers B, we know there is some $\nu \in M_B$ with $\nu\pi = \mu\pi$. Then $\mu\nu^{-1}\pi = 1$, implying that $\mu\nu^{-1} \in L_1^C$, so we can write $\mu = (\mu\nu^{-1})(\nu) \in L_1^C M_B$. Thus $L_1^C M_B = L_B^C$, as desired. □

We are now ready to define normality.

Definition 7.3. *Let $N \subseteq C$ be a subset. We say that N is a* **normal subcwatset** *of C if L_N^C is a subgroup of L_C. In this case, we write $N \triangleleft C$.*

Note that although we don't require N to be a subcwatset of C, it is immediate from the above definition that all normal subcwatsets are indeed subcwatsets.

We also note that L_1^C is a subgroup of L_C for all cwatsets C, so $1 \triangleleft C$. Furthermore, $L_C^C = L_C$, so $C \triangleleft C$.

Lemma 7.4. *Let $\phi : C \to D$ be a cwatset homomorphism with $\Phi : L_C \to L_D$ a lift. Then*

1. *If $B \subseteq C$ is any subset, then $L_B^C \Phi \subseteq L_{B\phi}^D$. Suppose in addition that ϕ is totally surjective. If $B \triangleleft C$ and $B \supseteq \ker(\phi)$, then $L_B^C\Phi = L_{B\phi}^D$.*

2. *If $E \subseteq D$ is any subset, then $L_E^D \Phi^{-1} = L_{E\phi^{-1}}^C$.*

Proof. Let $B \subseteq C$, and choose some $\mu \in L_B^C$. Then

$$\mu\Phi\pi = \mu\pi\phi \in B\phi,$$

so $\mu\Phi \in L_{B\phi}^D$, implying that $L_B^C\Phi \subseteq L_{B\phi}^D$.

For the second part of (1), we need to show that $L_{B\phi}^D \subseteq L_B^C\Phi$, so we let $\nu \in L_{B\phi}^D$. We have $\nu\pi \in B\phi$, so choose $b \in B$ such that $\nu\pi = b\phi$, and choose $\beta \in L_B^C$ such that $b = \beta\pi$. Since ϕ is totally surjective, there is some $\mu \in L_C$ such that $\nu = \mu\Phi$. Now

$$\mu\Phi\pi = \nu\pi = b\phi = \beta\pi\phi = \beta\Phi\pi,$$

so

$$1 = (\mu\Phi)(\beta\Phi)^{-1}\pi = (\mu\beta^{-1})\Phi\pi = (\mu\beta^{-1})\pi\phi.$$

Thus

$$(\mu\beta^{-1})\pi \in \ker(\phi) \subseteq B,$$

so $\mu\beta^{-1} \in L_B^C$. Therefore

$$\mu = (\mu\beta^{-1})(\beta) \in L_B^C$$

since L_B^C is a group. Since $\nu = \mu\Phi$, we have $\nu \in L_B^C\Phi$, as desired.

Now let $E \subseteq D$, and choose some $\mu \in L_C$. Then

$$\mu\Phi \in L_E^D \Leftrightarrow \mu\Phi\pi \in E \Leftrightarrow \mu\pi\phi \in E \Leftrightarrow \mu\pi \in E\phi^{-1} \Leftrightarrow \mu \in L_{E\phi^{-1}}^C,$$

which proves (2). □

We now see the purpose our new definition of subcwatset:

Theorem 7.5. *Let $\phi : C \to D$ be a cwatset homomorphism. Then*

1. *If B is a subcwatset of C, then $B\phi$ is a subcwatset of D. If, in addition, ϕ is totally surjective and B is a normal subcwatset that contains* $\ker(\phi)$, *then $B\phi \lhd D$.*

2. *If E is a normal subcwatset of D, then $E\phi^{-1}$ is a normal subcwatset of C.*

Proof. Let $\Phi : L_C \to L_D$ be a lift of ϕ. If B be a subcwatset of C, then $M_B\Phi$ clearly covers $B\phi$, which proves the first part of (1). Assuming that ϕ is totally surjective and B is a normal subcwatset that contains $\ker(\phi)$, Lemma 7.4 (1) implies that $L^C_B\Phi = L^D_{B\phi}$, so $B\phi \lhd D$, as desired.

For (2), Lemma 7.4 (2) implies that $L^C_{E\phi^{-1}} = L^D_E\Phi^{-1}$. If $E \lhd D$, then $L^D_E\Phi^{-1}$ is a group and equals $L^C_{E\phi^{-1}}$, so $E\phi^{-1} \lhd C$. □

Theorem 7.5 mirrors the corresponding result for groups, with two exceptions:

1. In the second half of part (1), we have the extra hypothesis that $B \supseteq \ker(\phi)$.

2. If E is a subcwatset of D, we do not know whether $E\phi^{-1}$ is a subcwatset of C.

The resolution of these issues is currently open.

8 Quotients

The idea of normality allows us to capture the structure of kernels of morphisms.

Lemma 8.1. *If $\phi : C \to D$ be a cwatset homomorphism, then* $\ker(\phi) \lhd C$.

Proof. Since $1 \lhd D$, Theorem 7.5 (2) implies that $\ker(\phi) = 1\phi^{-1} \lhd C$. □

We now work towards constructing a quotient cwatset.

Lemma 8.2. *Let $N \lhd C$ be (X, M_X)-cwatsets, and let $\mu, \nu \in M_X$. If $\mu\pi = \nu\pi$, then $L^C_N\mu = L^C_N\nu$.*

Proof. Since $\mu\pi = \nu\pi$, $\mu\nu^{-1}\pi = 1$, so $\mu\nu^{-1} \in L^C_1 \subseteq L^C_N$. Thus $L^C_N\mu = L^C_N\nu$, as desired. □

Definition 8.3. *Let $N \lhd C$ be (X, M_X)-cwatsets, and choose $x \in X$. The* **right coset of N associated to x**, *denoted N^x, is defined to be the set $(L^C_N\mu)\pi$ for some $\mu \in M_X$ with $\mu\pi = x$.*

This set is well-defined due to Lemma 8.2.

Lemma 8.4. *Let $N \lhd C$. Then the set of right cosets of N in C partitions C.*

Proof. Lemma 8.2 implies that distinct right cosets of N are disjoint, because the corresponding cosets of L_N^C are disjoint. Since every $c \in C$ is obviously in some right coset, the result follows. $\square$

Also note that if $N \lhd C$ are (X, M_X)-cwatsets, then the right cosets of N in X partitions X as well. We introduce the notation C/N for the set of right cosets of N in C (and similarly X/N for the set of right cosets of N in X).

Lemma 8.5. *Let $N \lhd C$. For every $x \in C$, we have*

$$|N| = |N^x|.$$

Proof. Choose $\mu \in L_C$ with $\mu\pi = x$. The map $\theta : N \to N^x$ defined by $\theta(n) = n \cdot \mu$ is a bijection, which implies that $|N| = |N^x|$. $\square$

If $N \lhd C$, then the **index** of N in C, denoted $|C : N|$, is the number of distinct right cosets of N in C. We now have an analog to Lagrange's theorem.

Theorem 8.6. *Let $N \lhd C$. Then $|C| = |N||C : N|$. In particular, if C is finite, then $|N|$ divides $|C|$ and $|C|/|N| = |C : N|$.*

Proof. The cwatset C is the disjoint union of $|C : N|$ right cosets, each of cardinality equal to $|N|$. $\square$

Theorem 8.7. *Let $N \lhd C$ be (X, M_X)-cwatsets. Then C/N is an $(X/N, M_X)$-cwatset (with identity of X/N equal to N) whose M-group is M_C.*

Proof. We must first provide a transitive action of M_X on X/N. Since M_X acts naturally on the cosets of L_N^C, we simply take the projection of this action, i.e., for $(L_N^C\mu)\pi \in X/N$ and $\nu \in M_X$, we define

$$(L_N^C\mu)\pi \cdot \nu = (L_N^C\mu\nu)\pi.$$

Transitivity of this action follows from that of the action of M_X on X.

We now have the appropriate objects for a cwatset, and we claim that the group M_C covers C/N. Let $S = \{N \cdot \mu \mid \mu \in M_C\}$, and we claim that $S = C/N$. If $\mu \in M_C$, then

$$N \cdot \mu = (L_N^C)\pi \cdot \mu = (L_N^C\mu)\pi \in C/N,$$

so $S \subseteq C/N$. Similarly, if $N^c \in C/N$ for some $c \in C$, then $N^c = (L_N^C\mu)\pi$ for some $\mu \in M_C$ with $\mu\pi = c$. Thus $N^c = N \cdot \mu$, and hence $C/N \subseteq S$. It follows that $S = C/N$, and so C/N indeed is an $(X/N, M_X)$-cwatset.

The above argument shows that $M_C \subseteq M_{C/N}$. To show equality, we let $\mu \in M_{C/N}$. If $c \in C$, then $c \cdot \mu \in N^c \cdot \mu \subseteq C$, so $\mu \in M_C$, as desired. $\square$

Lemma 8.8. *If $N \lhd C$, then the canonical map $p : C \to C/N$ is a totally surjective cwatset homomorphism.*

Proof. By Theorem 8.7, the M-group of C/N is precisely the M-group of C, so we have a homomorphism $P' : M_C \to L_{C/N}$. Furthermore, this factors through the quotient to a homomorphism $P : L_C \to L_{C/N}$, since if two elements of M_C induce the same permutation of C, then they certainly induce the same permutation of C/N. By construction, P respects the map p, so p is a homomorphism. $\square$

Theorem 8.9. *Let C be an arbitrary cwatset. Then the normal subcwatsets of C are precisely the kernels of all homomorphisms defined on C.*

Proof. This follows from Lemma 8.1, and the fact that $N \triangleleft C$ implies that N is the kernel of the canonical homomorphism $p : C \to C/N$. $\square$

Theorem 8.10 (First Homomorphism Theorem for Cwatsets). *If $\phi : C \to D$ is a totally surjective cwatset homomorphism with kernel K, then there is an isomorphism $f : C/K \to D$ with a lift $F : L_{C/K} \to L_D$ that is a group isomorphism.*

Proof. Suppose that $\Phi : L_C \to L_D$ is a lift of ϕ, and define $f : C/K \to D$ by

$$f(K^c) = (c)\phi$$

for $c \in C$. We first show that this map is well-defined.

Suppose that $K^b = K^c$ for $b, c \in C$. We claim that $b\phi = c\phi$. Choose some $\mu, \nu \in L_C$ such that $\mu\pi = b$ and $\nu\pi = c$. Since $K^b = K^c$, we have $L_K^C \mu = L_K^C \nu$, so $\mu\nu^{-1} \in L_K^C$. This means that $(\mu\nu^{-1})\pi \in K$, so

$$\begin{aligned} 1 &= (\mu\nu^{-1})\pi\phi = (\mu\nu^{-1})\Phi\pi = (\mu\Phi)(\nu^{-1}\Phi)\pi \\ &= (\mu\Phi\pi) \cdot (\nu\Phi)^{-1} = (b\phi) \cdot (\nu\Phi)^{-1}. \end{aligned}$$

Thus

$$b\phi = 1 \cdot (\nu\Phi) = \nu\Phi\pi = \nu\pi\phi = c\phi.$$

We now claim that f is a bijection. First choose some $d \in D$. Since ϕ is surjective, there is some $c \in C$ such that $c\phi = d$. Thus $f(K^c) = c\phi = d$, so f is surjective. Now suppose that $f(K^b) = f(K^c)$ for some $b, c \in C$, so $b\phi = c\phi$, and choose some $\mu, \nu \in L_C$ such that $\mu\pi = b$ and $\nu\pi = c$. Then

$$\begin{aligned} 1 &= b\phi \cdot (\nu\Phi)^{-1} = (\mu\Phi\pi) \cdot (\nu\Phi)^{-1} \\ &= (\mu\Phi)(\nu^{-1}\Phi)\pi = (\mu\nu^{-1})\Phi\pi = (\mu\nu^{-1})\pi\phi, \end{aligned}$$

so $(\mu\nu^{-1})\pi \in K$. Thus $\mu\nu^{-1} \in L_K^C$, so $K^b = K^c$.

We next define a lift $F : L_{C/K} \to L_D$ as follows: If $\mu \in L_{C/K}$, there is some $\nu \in L_C$ that induces the action of μ on C/K. We define

$$F(\mu) = \nu\Phi.$$

We now show that this map is well-defined by showing that the action of $\nu\Phi$ on D is independent of our choice of ν. It will follow immediately that F is a homomorphism since Φ is a homomorphism. For $d \in D$, we choose $\delta \in L_D$ such that $\delta\pi = d$. Since Φ is surjective, there is a $\gamma \in L_C$ such that $\gamma\Phi = \delta$. Note that

$$(K^{\gamma\pi})f = (\gamma\pi)\phi = d.$$

We now have

$$\begin{aligned} d \cdot (\nu\Phi) &= (\delta(\nu\Phi))\pi = (\gamma\nu)\Phi\pi = (\gamma\nu)\pi\phi \\ &= (\gamma\pi \cdot \nu)\phi = (K^{\gamma\pi\cdot\nu})f = (K^{\gamma\pi} \cdot \mu)f \\ &= ((d)f^{-1} \cdot \mu)f, \end{aligned}$$

which clearly does not depend on the choice of μ.

We now claim that F respects f. Let $\mu \in L_{C/K}$. From the above calculation,

$$\mu F\pi = ((1)f^{-1} \cdot \mu)f = (K \cdot \mu)f = \mu\pi f,$$

as desired.

To show that F is a bijection, first let $\delta \in L_D$. Since Φ is surjective, there is some $\gamma \in L_C$ with $\gamma\Phi = \delta$. If $\mu \in L_{C/K}$ denotes the action of γ on cosets, then $F(\mu) = \gamma\Phi = \delta$, so F is surjective. Injectivity follows from Lemma 5.6. □

9 Automorphisms and Normality

An isomorphism $\phi : C \to C$ from a cwatset to itself is called an **automorphism**. Note that if $\mu \in L_1^C$, then the map $\phi_\mu : C \to C$ defined by

$$\phi_\mu(x) = x \cdot \mu$$

is an automorphism: If $\Phi_\mu : L_C \to L_C$ denotes conjugation by μ, then Φ_μ is a lift of ϕ_μ, so ϕ_μ is a homomorphism, and it is easy to check that it is a bijection as well. An automorphism induced in this manner is an **inner automorphism** of C.

Theorem 9.1. *Let C be a cwatset and $B \subseteq C$ be a subcwatset. Then $B \lhd C$ iff B is fixed by all inner automorphisms of C.*

Proof. First suppose that B is fixed by all inner automorphisms of C. Therefore $L_1^C \in \mathbf{N}_{L_C}(M_B)$, so $L_1^C M_B$ is a group. By Lemma 7.2, that group equals L_B^C, so indeed $B \lhd C$.

Now let $B \lhd C$. By Lemma 7.2, $L_1^C \subseteq L_B^C$, so $L_B^C L_1^C = L_B^C$ since L_B^C is a group. Thus $B \cdot L_1^C = B$; that is, B is fixed by all inner automorphisms. □

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