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### Recommended Citation

Davis, Noah (2014) "Squaring The Circle In The Hyperbolic Disk," *Rose-Hulman Undergraduate Mathematics Journal*: Vol. 15 : Iss. 1 , Article 2.  
Available at: <https://scholar.rose-hulman.edu/rhumj/vol15/iss1/2>

**ROSE-  
HULMAN  
UNDERGRADUATE  
MATHEMATICS  
JOURNAL**

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VOLUME 15, No. 1, SPRING 2014

Sponsored by

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## SQUARING THE CIRCLE IN THE HYPERBOLIC DISK

Noah Davis

**Abstract.** Bolyai ended his 1832 introduction to non-Euclidean geometry with a strategy for constructing regular quadrilaterals (squares) and circles of the same area. In this article, we provide the steps for these constructions in the Poincaré disk. We come to the surprising conclusion that Bolyai's strategy of building the circle and square separately is the only way to perform the constructions. That is, we cannot in general construct the square from the circle, nor vice versa.

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**Acknowledgements:** The author recognizes the late Sr. Mary Martens, O.P. She knew enough to leave her first edition of [5] on the shelves at Aquinas College for me to find, just when I needed it. Unfortunately, she passed on a few months before I used her book and I did not get the chance to thank her in person. My advisor, Dr. Michael McDaniel, and I, also thank the Mohler-Thompson Program for supporting our work in summer 2012.

## 1 Introduction

In 1840, János Bolyai shook the foundations of mathematics and almost nobody noticed. He investigated the consequences of some axioms of Euclidean geometry combined with the assumption that intersecting lines could all miss a given line, and then published his study of this new geometry as an appendix to his father's book. History of mathematics and college geometry courses usually include a summary of Bolyai's insights and their discouraging reception. He closed his presentation with an enticement: one of the problems from antiquity, the squaring of the circle, could be constructed in hyperbolic geometry. Without a model or even a distance formula, he reasoned a way to construct circles and regular quadrilaterals with the same hyperbolic areas. At that time, constructing a Euclidean square with the same area as a given circle was an open problem, waiting for Lindemann's 1882 proof of the impossibility of squaring the circle in the Euclidean plane. Bolyai knew squaring the circle was an intractable, ancient, and famous problem in Euclidean geometry and he was the first person to have something interesting to say about it in centuries: a good way to draw attention to his ideas.

In this paper, we will see how to accomplish his strategy for squaring the circle in Poincaré's disk model of hyperbolic geometry and we will see a surprising conclusion: that Bolyai was more correct than he may have known. That is, we will see that Bolyai's strategy is the only viable construction for squaring the circle in hyperbolic geometry. Another mathematician, William Jagy, also found this same conclusion, but in Bolyai's context [6]. The work in this article shares the strategy of using known unconstructible entities with Jagy's work; but in this article we actually construct the objects which we need. Jagy worked in the purely formal realm which Bolyai had laid out.

This paper provides a step-by-step construction for squaring the circle in hyperbolic geometry. In the next few pages, we will see how to make a regular quadrilateral and a circle which have the same hyperbolic area. Figure 1 shows an example of our goal.

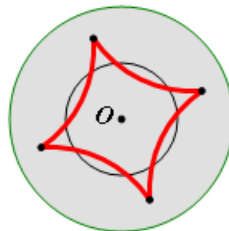


Figure 1. A squared circle.

## 2 The Poincaré Disk

In 2004, Jeremy Grey presented Bolyai's original Latin, a translation and then Grey's own explanation of Bolyai's ideas in his book *János Bolyai, Non-Euclidean Geometry and the Nature*

of *Space*. Remaining faithful to the source material, a step-by-step construction in a hyperbolic model could not be mentioned because Bolyai had no such model. But we do.

The Poincaré disk model of hyperbolic geometry consists of the interior of a Euclidean disk with center  $O$  and we will use Euclidean radius 1. The Euclidean points inside the disk are the points of the hyperbolic space. Arcs of circles orthogonal to the boundary disk and diameters of the disk are the hyperbolic lines. The distance formula computes the hyperbolic length of a segment  $\overline{AB}$  using the complex coordinates of  $A$  and  $B$  along with the coordinates of the endpoints of the hyperbolic line. This leads to an unusual situation where two segments of equal Euclidean length can have wildly different hyperbolic lengths, depending on their proximity to the boundary. Imprecisely, the closer a segment gets to the boundary, the larger its hyperbolic length.

Constructing these hyperbolic lines and computing distances requires the use of points outside the hyperbolic space. Thinking like this means we are using a Euclidean compass and straightedge for our work in this paper: our hyperbolic objects have Euclidean pedigrees. (Some geometers pursue constructions using hyperbolic compass and straightedge, objectives for which we have no physical tools.) A consequence of this hyperbolic model inside a Euclidean setting is we must use the adjectives hyperbolic and Euclidean much more often than if we were concerned with only one geometry. (See [3] for an introduction to hyperbolic constructions.)

Hyperbolic angles have sides on hyperbolic lines, which may be Euclidean arcs. We measure these angles using tangents. Building circles whose tangents meet at a certain angle size and which are orthogonal to the boundary of the disk is one of our basic constructions because this is how we create regular quadrilaterals with certain areas.

### 3 Construction of a circle and square

Bolyai reasoned that the hyperbolic area of triangle  $ABC$  is  $\pi - (m\angle A + m\angle B + m\angle C)$ , times a constant which we will take as 1,  $m\angle A$  means the measure of angle  $A$  in radians). The formula illustrates a key property of hyperbolic triangles: the sum of their angles is less than  $\pi$ , not a fixed amount as in Euclidean geometry because the angle sum is adjustable. He realized the implications for squaring the circle, connecting his new geometry to an ancient conundrum (taking a square as a regular quadrilateral, since the hyperbolic area formula will not allow a hyperbolic quadrilateral with four right angles). He found the first two of the following three formulas, with radius  $r$  in hyperbolic distance.

$$|\square ABCD| = 2\pi - 4m\angle A \quad (1)$$

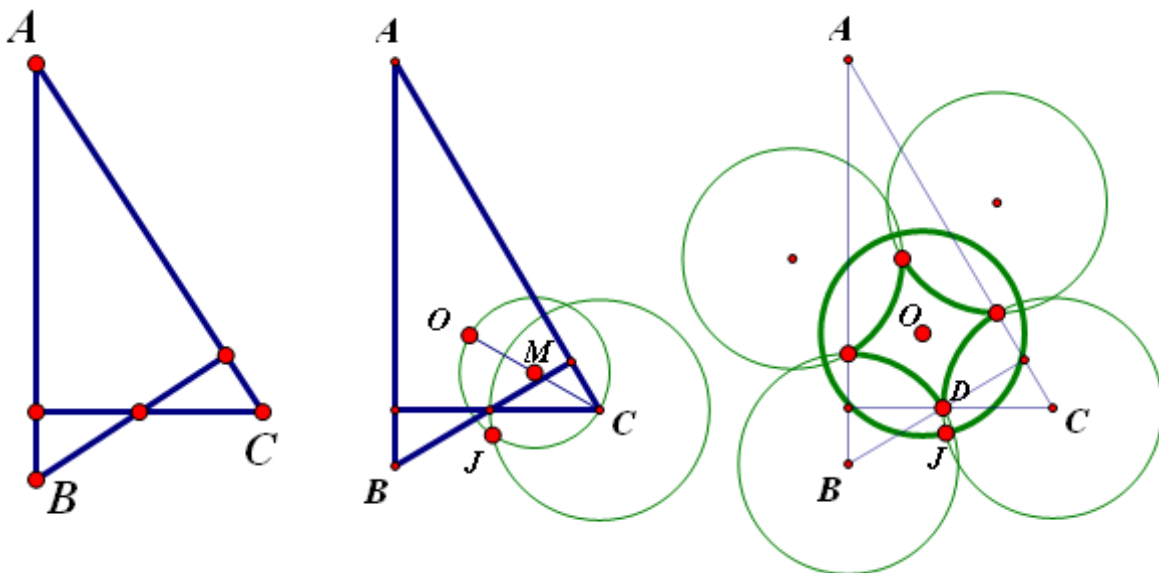
$$|\odot O| = 4\pi \sinh^2 \frac{r}{2} \quad (2)$$

$$OX = \ln \frac{1+x}{1-x} \quad (3)$$

Let's be clear about these three formulas. In (1), the area of the quadrilateral is 2 minus 4 times the measure of any one of angles  $A$ ,  $B$ ,  $C$  or  $D$  because all 4 angles are the same size. Equation (2) gives us the hyperbolic area of a hyperbolic circle centered at  $O$  with hyperbolic radius length  $r$ . The third equation is a handy modification of the hyperbolic distance formula due to Poincaré, taking advantage that our  $X$  is always a point on the positive real axis inside the disk, with Euclidean coordinate  $x$ . When  $x = .5$ ;  $OX = \ln 3$ . (This simplified form of the distance formula also illustrates that the hyperbolic length of  $\overline{OX}$  approaches infinity as  $x$  approaches 1 from below.)

So, a circle with hyperbolic radius  $\ln 3$  has hyperbolic area  $\frac{4\pi}{3}$ . Following Bolyai's advice to square this circle, which may be summed up as *build the two objects separately*, we solve  $\frac{4\pi}{3} = 2\pi - 4m\angle A$ . The angle  $A$  has to be  $\frac{\pi}{6}$ , a constructible angle: we can square this circle. Figures 2 - 4 below illustrate some of the steps of our construction of a regular hyperbolic quadrilateral with all angles  $\frac{\pi}{6}$ . All our steps are Euclidean constructions which will have hyperbolic interpretations. We start with a  $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$  triangle reflected and copied so that the  $\frac{\pi}{6}$  angles coincide. We construct circles centered at  $B$  and  $C$  with radii  $\overline{BD} \equiv \overline{CD}$ . We construct  $O$ , the incenter of both right triangles. We will prove that these two circles  $B$  and  $C$  contain hyperbolic sides of our desired square.

This construction works for any given size angle, a crucial property for our results later in the paper. Since we were constructing a pair of objects with the same fixed hyperbolic area, we had to construct our first  $\angle A$  to be the required size (these marks are not in Figure 2). Our construction has the nice property that the first Euclidean angle we use ends up being the size angle of the four vertices of the square. We have done a funny thing: we started without the disk being visible. In fact, we always construct our angle first and then build a disk of the appropriate size later.



Figures 2, 3, 4. Construction of the hyperbolic square.

That this construction produces a regular hyperbolic quadrilateral with angle size  $m\angle A$  requires a little support. The point  $M$  in Figure 3 is the midpoint of segment  $\overline{OC}$ : the circle centered at  $M$  through  $O$  intersects circle  $C$  at the point  $J$ , which we will use for the radius of the hyperbolic disk. The undrawn radii  $\overline{OJ}$  and  $\overline{CJ}$  are perpendicular since they are inscribed in a semicircle, giving us hyperbolic lines. The undrawn tangents at point  $D$  are perpendicular to their radii; writing the angles we know with the tangents drawn in gives us the hyperbolic  $\angle D$  congruent to our first  $\angle A$ . We finish the square with a reflection of our first two hyperbolic lines across the diagonal of the square opposite  $\angle D$ . (Symmetry requirements forced us to use the common incenter for  $O$ .)

At this point, we have our regular hyperbolic quadrilateral with the desired area. Now we have to construct the circle of the same area in this same disk. As we calculated above, to finish squaring the circle, we need a circle of Euclidean radius  $.5$  centered at  $O$ , using the disk radius as  $1$ . So we construct the midpoint of the disk radius and we have squared the circle. We took the square and disk from Figure 4, constructed the circle in the disk and hid all the construction marks for Figure 1.

This construction not only allows us to square the circle in the hyperbolic disk; but it also forms part of the proof of a theorem which adds a footnote to Bolyai's introduction to hyperbolic geometry.

#### 4 We must construct circle and square separately.

**Theorem** *Construction of a regular, hyperbolic quadrilateral and a hyperbolic circle of the same area must be done separately. That is, neither object may be constructed directly from the other.*

Bolyai's strategy to build the square and circle separately rather than one from another turns out to be the only way to square the circle in hyperbolic geometry. The careful reader probably noticed our first example seemed overly convenient, when we chose Euclidean length  $.5$  out of thin air and everything went just fine. The phrase "constructible angle" was crucial because building the two objects separately means we must be able to build the two objects! Now, Bolyai did not write that his way would be the only way; but seeing the only way to square the circle, without benefit of a model, remains a remarkable insight.

*Proof.* To see that we cannot build the circle from the square nor the square from the circle, we will use contradiction, assuming we have such a construction, each way. Since we require the area of the square to equal the area of the circle, substitution with equations (1) and (2) results in

$$2\pi - (4m\angle A) = 4\pi \sinh^2 \frac{r}{2}$$

Solving for  $m\angle A$ , we get  $m\angle A = \frac{\pi}{2} (1 - 2 \sinh^2 \frac{r}{2})$ . Substituting our simplified distance formula for the hyperbolic radius  $r$  and solving for  $x$ , we arrive at

$$x = \sqrt{\frac{\frac{\pi}{2} - m\angle A}{\frac{3\pi}{2} - m\angle A}}. \quad (4)$$

For the contradiction, let's find a constructible length which gives us a non-constructible angle. The angle  $\frac{\pi}{7}$  is a famously non-constructible angle thanks to Gauss. Constructing such an angle would allow the construction of a regular Euclidean heptagon, which he proved was impossible. Solving equation (4) for  $m\angle A$  gives us

$$m\angle A = \frac{\pi}{2} - \frac{\pi x^2}{(1-x^2)}. \quad (5)$$

Substituting  $\frac{\sqrt{95}}{19}$  for  $x$  in (5) results in  $\frac{\pi}{2} - \frac{\pi \frac{95}{361}}{(1-\frac{95}{361})} = \frac{\pi}{7}$ .

The given circle could have a constructible Euclidean radius of  $\frac{\sqrt{95}}{19}$ . When we used this value for  $x$  in (4), we got  $m\angle A = \frac{\pi}{7}$ , which we now can apparently construct. This is a contradiction; so there is no general construction for the square from the circle.

If we have the square, we cannot construct the circle directly, either. This time, we use the angles of triangles whose sides are Pythagorean triples because their acute angles are irrational numbers times  $\pi$  [4]. Thanks to equation (4), we can use another indirect proof.

So, we assume a construction exists for the circle of area equal to that of a given regular hyperbolic quadrilateral, built directly from the quadrilateral. We may construct a 3, 4, 5 Euclidean triangle and use our construction from Figures 2 - 4 to make a regular hyperbolic quadrilateral with angles  $\tan^{-1} \frac{3}{4}$ . Our assumption then constructs the circle of equal hyperbolic area. Using equation (4), we compute the Euclidean radius  $x$  as

$$x = \sqrt{\frac{\frac{\pi}{2} - \tan^{-1}(\frac{3}{4})}{\frac{3\pi}{2} - \tan^{-1}(\frac{3}{4})}} = \sqrt{\frac{\tan^{-1}(\frac{4}{3})}{\pi - \tan^{-1}(\frac{4}{3})}}.$$

To see this is our desired contradiction, we square both sides and take reciprocals (these two steps are constructible moves so we can use our tools on the Euclidean segment with length  $x$ .)



Then we have constructed the length  $\frac{1}{x^2} = \frac{\pi - \tan^{-1}(\frac{4}{3})}{\tan^{-1}(\frac{4}{3})} = \frac{\pi}{\tan^{-1}(\frac{4}{3})} - 1$ . Adding 1 to this length gives a member of a family of transcendental numbers [5]. Since the constructible numbers are all algebraic and the transcendental numbers are mutually exclusive with the algebraic numbers, we see that we have indeed accomplished the impossible. ■

## 5 Conclusion

We can meet the entire family of constructible circles and squares at a glance. Figure 5 starts with the limiting case as the Euclidean radius approaches zero. In that case, circle and square both approach area zero, with the angle approaching  $\frac{\pi}{2}$ ; they are the same point. In the middle, we show a pair similar to Figure 1 to stand for all the cases Bolyai had considered: they all look pretty much the same. The third member of the family shows the limiting case as the angle approaches zero and the Euclidean radius approaches  $\frac{\sqrt{3}}{3}$ , from equation (4). The vertices of this last square are not in hyperbolic space and its sides are not segments but hyperbolic lines. What a family! Two degenerates, a typical pair and an outcast with its circle.



Figure 5. A family portrait.

Most geometric constructions begin with putting the straightedge across two of the given points or placing the point of the compass at one given point and the pencil at another given point; that is, we build on some given figure. Squaring the circle in hyperbolic geometry does not work like that and Bolyai advised us to build the objects separately. Moreover, he wrote that, in hyperbolic geometry there are many ways to square the circle, so long as the constructible angle is a rational number times  $\pi$ , avoiding a mention of other constructible angles, like  $\tan^{-1}\frac{3}{4}$ . Since there are countably infinite constructible angles that are rational numbers times  $\pi$ , there are infinitely many ways to square the circle in hyperbolic geometry, something that is not the case in Euclidean geometry. As Grey noted, Bolyai stuck to the facts, even though he wrote at a time when intuition was allowed in formal math papers [2]. It is interesting that he nailed so many

details of squaring the circle, even though he could not measure nor model the segments.

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