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# ON THE ACTION OF WEIGHT-PRESERVING SETS

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# ON THE ACTION OF WEIGHT-PRESERVING SETS

MATTHEW BADGER

ABSTRACT. We introduce weight-preserving sets of binary words. Any transformation that respects the row and column weights of a 0-1 matrix can be decomposed as a composition of two types of actions on the matrix. We conjecture that weight-preserving sets perform only one type of action, permutations of rows and columns; i.e., weight-preserving sets are cwatsets.

## 1. INTRODUCTION

Let $\mathbb{Z}_2^d$ denote the space of *binary words* in *degree* $d$; i.e., the set of all expressions $a = a_1a_2\cdots a_d$ such that $a_i \in \{0,1\}$, for all $1 \leq i \leq d$. The *weight* of $a$, denoted $\mathrm{w}(a)$, is determined by $\mathrm{w}(a) = \#\{i : a_i = 1\}$; that is, the number of "1"s which appear in $a$. If $b = b_1b_2\cdots b_d$ is also a binary word, let the *sum* of $a$ and $b$, denoted $a+b$, be defined by

$$a + b = c_1c_2\cdots c_d \in \mathbb{Z}_2^d, \qquad c_i = a_i + b_i \pmod 2.$$

In degree 6, for example,

$$\begin{array}{lll} a = 011011 & b = 110100 & a+b = 101111 \\ \mathrm{w}(a) = 4 & \mathrm{w}(b) = 3 & \mathrm{w}(a+b) = 5 \end{array}$$

Given a set of binary words $A \subseteq \mathbb{Z}_2^d$, a *matrix representation* of $A$ is obtained by interpreting the elements of $A$ as rows in a 0-1 matrix; the *row* and *column weights* of $A$ are the row and column weights, respectively, of a matrix representation of $A$. Expanding on the previous example, for $A = \{011011, 110100, 101111\} \subseteq \mathbb{Z}_2^6$,

$$\begin{array}{c} \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \mathit{4} \\ \mathit{3} \\ \mathit{5} \end{matrix} \\ \begin{matrix} \mathit{2} & \mathit{2} & \mathit{2} & \mathit{2} & \mathit{2} & \mathit{2} \end{matrix} \end{array} \quad \text{and} \quad \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \mathit{3} \\ \mathit{4} \\ \mathit{5} \end{matrix} \\ \begin{matrix} \mathit{2} & \mathit{2} & \mathit{2} & \mathit{2} & \mathit{2} & \mathit{2} \end{matrix} \end{array}$$

are two equivalent matrix representations of $A$; the row weights of $A$ are (in ascending order) 3,4,5 and the column weights of $A$ are 2,2,2,2,2,2. In the sequel, we occasionally drop the distinction between a set of binary words and its matrix representations, when no confusion results.

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If $B \subseteq \mathbb{Z}_2^d$ is another set, with the same row weights (in any order) and column weights (in any order) as $A$, then we write $\mathrm{w}(A) = \mathrm{w}(B)$. Now we can define our object of interest.

**Definition 1.1.** Let $A \subseteq \mathbb{Z}_2^d$. Then, $A$ is said to be *weight-preserving* if, and only if, $\mathrm{w}(A + a) = \mathrm{w}(A)$, for all $a \in A$. ⊣

As an immediate consequence of the definition, observe that every weight-preserving set $A$ contains the word of all zeros, denoted by $\bar{0}$. Indeed, for any $a \in A$, $\bar{0} = a + a \in A + a$; hence, $A$ must contain a row of weight zero, or equivalently, $\bar{0} \in A$. This observation implies that the running example in degree 6 is not a weight-preserving set, since that example does not contain a row of weight zero. Here is the simplest non-trivial example of a weight-preserving set.

**Example 1.2** (Friedman, [10]). Let $F = \{000, 110, 101\} \subseteq \mathbb{Z}_2^3$. Then, simple inspection shows that the cosets of $F$ under addition,

$$F = F + 000 = \begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} & \begin{matrix} \mathit{0} \\ \mathit{2} \\ \mathit{2} \end{matrix} \\ \begin{matrix} \mathit{2} & \mathit{1} & \mathit{1} \end{matrix} & \end{array}, \qquad F + 110 = \begin{array}{cc} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} & \begin{matrix} \mathit{2} \\ \mathit{0} \\ \mathit{2} \end{matrix} \\ \begin{matrix} \mathit{1} & \mathit{2} & \mathit{1} \end{matrix} & \end{array},$$

$$F + 101 = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{matrix} \mathit{2} \\ \mathit{2} \\ \mathit{0} \end{matrix} \\ \begin{matrix} \mathit{1} & \mathit{1} & \mathit{2} \end{matrix} & \end{array},$$

have the same row weights (0,2,2) and column weights (1,1,2) as $F$. Thus, $F$ is a weight-preserving set, in degree 3. ⊣

In this paper, we wish to study the actions which send a matrix representation of a weight-preserving set to that of its additive cosets. For an arbitrary 0-1 matrix, the related question is, what are all of the transformations of the matrix (rearrangements of the 0s and 1s) that respect its row and column weights? In **Section 2**, we develop a class of weight-preserving sets, the cwatsets, whose actions permute the rows and the columns of the set. In **Section 3**, we then answer our question. Any transformation that respects the row and column weights of a 0-1 matrix can be decomposed as composition of two types of actions on the matrix: permutations of rows and columns, and matrix "four-flips". In **Section 4**, we conjecture that weight-preserving sets perform only one type of action, namely permutations of rows and columns—every weight-preserving set is a cwatset. To date, this conjecture has been verified through degree five. In **Section 5**, we provide ancillary results in number theory, which refine previously known divisor conditions on the order of cwatsets in a given degree. For arbitrary cwatsets, this is the best possible improvement of the result.

## 2. Cwatsets and Permutation

An obvious class of weight-preserving sets is the collection of all subgroups of $\mathbb{Z}_2^d$. Since a group $G \subseteq \mathbb{Z}_2^d$ is closed and transitive under addition, for any $g \in G$, the rows of the coset $G+g$ are a permutation of the rows of $G$. For instance, if $G = \{00, 10, 01, 11\} = \mathbb{Z}_2^2$ and $g = 10$,

$$G = \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} & \begin{matrix} \mathit{0} \\ \mathit{1} \\ \mathit{1} \\ \mathit{2} \end{matrix} \\ \begin{matrix} \mathit{2} & \mathit{2} \end{matrix} & \end{array}, \qquad G + g = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} & \begin{matrix} \mathit{1} \\ \mathit{0} \\ \mathit{2} \\ \mathit{1} \end{matrix} \\ \begin{matrix} \mathit{2} & \mathit{2} \end{matrix} & \end{array} = G^{(1,2)_R(3,4)_R},$$

the coset $G+g$ exchanges the first and second, and the third and fourth rows of $G$. Yet, permuting the rows of a 0-1 matrix changes at most the order of the row weights, not their values; the column weights are unaffected. Hence, for arbitrary $G$ and $g \in G$, $\mathrm{w}(G+g) = \mathrm{w}(G)$.

Similarly, permuting the columns of a 0-1 matrix changes at most the order of the column weights and leaves the row weights unaltered. In view of our goal to study actions that preserve row and column weights, this fact and the previous paragraph motivate the following definition. Let $A$ and $B$ be two 0-1 matrices, with equal dimensions; then $A$ and $B$ are said to be *equivalent*, and we write $A \sim B$, if there exists a permutation of rows $\pi_R$ and there exists a permutation of columns $\pi_C$ such that

$$A = B^{\pi_R \pi_C}.$$

For sets of binary words, we have the following companion definition.

**Definition 2.1.** Let $A \subseteq \mathbb{Z}_2^d$. Then, $A$ is said to be a *cwatset* if, and only if, $A + a \sim A$, for all $a \in A$. $\dashv$

**Remark 2.2.** Our definition of a cwatset is logically equivalent to the definition given in [10]. Sherman and Wattenberg define the sets of binary words that are closed with a twist, to study an extension of Hartigan's typical value theorem [1, 4, 9]. Since their introduction in 1993, cwatsets have been developed from an algebraic viewpoint, as a generalization of groups, and from a graph theoretic point of view, as a certain class of hypergraphs. Articles with results in these directions include [2, 3, 5, 6, 7]. $\dashv$

Following the reasoning of this section's first paragraph, every group $G \subseteq \mathbb{Z}_2^d$ is a cwatset with $\pi_C = \mathrm{id}$ (only rows of a coset are permuted). We have already seen a non-group cwatset, as well.

**Example 2.3** (Example 1.2 Revisited). Let $F = \{000, 110, 101\} \subseteq \mathbb{Z}_2^3$. We have previously shown that $F$ is a weight-preserving set. In fact, $F$ is a cwatset, since

$$F = F + 000 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = F^{\mathrm{id}} = F^{(2,3)_R(2,3)_C},$$

$$F + 110 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = F^{(1,2)_R(1,2)_C} = F^{(1,2,3)_R(1,2,3)_C},$$

$$F + 101 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = F^{(1,3)_R(1,3)_C} = F^{(1,3,2)_R(1,3,2)_C},$$

where $\pi_R$ and $\pi_C$ denote permutations of rows and columns, respectively. While it is enough for each coset to be written as just one permutation of the original set, the example shows that more than one permutation may exist for each coset. Although true in this example, in general the permutation of rows need not equal the permutation of columns. ⊣

The next lemma summarizes our earlier discussion, stating that every cwatset is a weight-preserving set. Finding a proof or a counterexample of the converse statement remains open and is the topic of §4.

**Lemma 2.4.** *If $A \subseteq \mathbb{Z}_2^d$ is a cwatset, then $A$ is weight-preserving.*

*Proof.* For fixed $a \in A$, there exists a permutation of rows $\pi_R$ and a permutation of columns $\pi_C$ such that $A + a = A^{\pi_R \pi_C}$. Yet, permutations of the rows and permutations of the columns do not alter the value of the row and column weights, respectively. Thus, $\mathrm{w}(A + a) = \mathrm{w}(A^{\pi_R \pi_C}) = \mathrm{w}(A)$. □

In group theory, Lagrange's Theorem provides a necessary divisor theoretic condition for the existence of a subgroup of $\mathbb{Z}_2^d$ with a given order (set cardinality). An algebraic study of cwatset structure yields the following parallel result for cwatsets, the proof of which we omit. We present a technical refinement of this proposition in section 5.

**Proposition 2.5** (Proposition 5, [10]). *If $A \subseteq \mathbb{Z}_2^d$ is a cwatset in degree $d$ and order $n$, then $n \big| 2^d d!$.* ⊣

While Proposition 2.5 gives a necessary condition for the existence of a cwatset with a given order, the condition is not sufficient. Consider, for instance, the space of binary words in degree 5. Although $15|2^5 5!$ and $30|2^5 5!$, there are no degree 5 cwatsets (or weight-preserving sets) with order 15 or 30. To check this, we refer the reader the appendix, which contains an exhaustive list of all inequivalent weight-preserving sets through degree 5.

## 3. Weight-Preserving Actions

In the previous section, it was shown that the action under addition of certain weight-preserving sets is to permute the rows and columns of their matrix representations. Unfortunately, not all actions, which preserve row and column weights of a 0-1 matrix, permute only rows and columns. For example, consider the action which sends $U$ to $V$,

$$U = \begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathbf{0} & \mathbf{1} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \begin{matrix} \mathit{0} \\ \mathit{2} \\ \mathit{2} \\ \mathit{2} \\ \mathit{4} \\ \mathit{4} \end{matrix} \\ \begin{matrix} \mathit{2} & \mathit{2} & \mathit{3} & \mathit{3} & \mathit{4} \end{matrix} & \end{array}, \qquad V = \begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathbf{1} & \mathbf{0} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \begin{matrix} \mathit{0} \\ \mathit{2} \\ \mathit{2} \\ \mathit{2} \\ \mathit{4} \\ \mathit{4} \end{matrix} \\ \begin{matrix} \mathit{2} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{3} \end{matrix} & \end{array}$$

by transposing the entries in the second row and right-most columns. To see that this action is not the permutation of rows or columns, examine the equality of the two weight 3 columns; while these columns are equal in $U$, the columns are different in $V$. But any permutation of rows or columns will send two equal columns to two equal columns. Thus, permutations of rows and columns do not suffice to classify all of the weight-preserving actions on 0-1 matrices.

In this section, we fill this gap and describe all weight-preserving actions on 0-1 matrices. Such actions can be written as the product of permutations of rows and columns, and the matrix "four-flips" defined below.

**Definition 3.1.** Let $A$ be an $m \times n$ 0-1 matrix, let $1 \leq r_1 < r_2 \leq m$, let $1 \leq c_1 < c_2 \leq n$, and let $A_s$ be a $2 \times 2$ submatrix of $A$ such that

$$A_s = \begin{bmatrix} a_{r_1 c_1} & a_{r_1 c_2} \\ a_{r_2 c_1} & a_{r_2 c_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The transformation of $A$ which transposes the rows of $A_s$ is said to be a *four-flip* in $A$ at rows $r_1, r_2$ and columns $c_1, c_2$. ⊣

**Example 3.2.** In the following example, $S$ is the image of a four-flip in $R$ at rows 3,4 and columns 2,4:

$$R = \begin{array}{cl} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & \mathbf{1} & 1 & \mathbf{0} & 1 \\ 1 & \mathbf{0} & 1 & \mathbf{1} & 1 \end{bmatrix} & \begin{matrix} \mathit{2} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \end{matrix} \\ \begin{matrix} \mathit{1} & \mathit{2} & \mathit{2} & \mathit{3} & \mathit{3} \end{matrix} & \end{array} \qquad S = \begin{array}{cl} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & \mathbf{0} & 1 & \mathbf{1} & 1 \\ 1 & \mathbf{1} & 1 & \mathbf{0} & 1 \end{bmatrix} & \begin{matrix} \mathit{2} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \end{matrix} \\ \begin{matrix} \mathit{1} & \mathit{2} & \mathit{2} & \mathit{3} & \mathit{3} \end{matrix} & \end{array}$$

The four boldface entries in $R$ now have "flipped" binary values in $S$. Observe that the four-flip preserves the row and column weights of $R$, with the weights in $S$ appearing in the same order as those in $R$. Yet, the four-flip sending $R$ to $S$ is not a permutation of rows or columns. To verify this, consider the "1" entry in the column of weight 1. In $R$, the "1" lies in the row with a "0" and "1" in the weight 2 columns. In $S$, however, the "1" lies in the row with a "1" and "1" in the weight 2 columns. ⊣

We require the following lemma, on certain products of four-flips.

**Lemma 3.3.** *Let $A$ be a 0-1 matrix. If $f$ is a four-flip in $A$, then the $i$-th row (column) of $A^f$ has the same weight as the $i$-th row (column) as $A$. Moreover, if $g_k$ is a transformation of a submatrix $A_S$ such that,*

$$A_S = \begin{array}{cl} \begin{bmatrix} 1 & 0 & \\ 0 & * & 1 \\ & 1 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array} \xrightarrow[g_3]{} \begin{array}{cl} \begin{bmatrix} 0 & 1 & \\ 1 & * & 0 \\ & 0 & 1 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array},$$

*or, such that,*

$$A_S = \begin{array}{cl} \begin{bmatrix} 1 & 0 & & \\ 0 & * & 1 & \\ & 1 & * & 0 \\ & & 0 & 1 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} & \end{array} \xrightarrow[g_4]{} \begin{array}{cl} \begin{bmatrix} 0 & 1 & & \\ 1 & * & 0 & \\ & 0 & * & 1 \\ & & 1 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} & \end{array},$$

*etc., where $r_1, \dots, r_k$ are distinct rows (in any order) and $c_1, \dots, c_k$ are distinct columns (in any order), then there exist four-flips $f_1, \dots, f_{k-1}$ such that $g_k = f_1 \circ \cdots \circ f_{k-1}$.*

*Remark.* In this context, compositions are applied from left to right, $f_1$ is a four-flip in $A_1 := A$, and $f_{i+1}$ is a four-flip in $A_{i+1} := A_i^{f_i}$.

*Proof.* First, we must show that four-flips do not change the order of the weights of a matrix. Let $f$ be a four-flip in $A$ at rows $r_1, r_2$ and columns $c_1, c_2$. Because the four-flip alters $A$ at only the intersections of these rows and columns, it suffices to check that the weights do not change under $f$, in the rows and columns of the submatrix $A_s$,

$$A_s = \begin{bmatrix} a_{r_1c_1} & a_{r_1c_2} \\ a_{r_2c_1} & a_{r_2c_2} \end{bmatrix}.$$

Since, without loss of generality,

$$A_s = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{matrix} \mathit{1} \\ \mathit{1} \end{matrix} \\ \begin{matrix} \mathit{1} & \mathit{1} \end{matrix} & \end{array} \qquad \text{and} \qquad A_s^f = \begin{array}{cc} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{matrix} \mathit{1} \\ \mathit{1} \end{matrix} \\ \begin{matrix} \mathit{1} & \mathit{1} \end{matrix} & \end{array},$$

the weights are preserved.

Next, we must show that transformations defined by $g_k$ can always be written as a product of $k-1$ four-flips. Proceed by induction on $k$, where $A_S$ is a $k \times k$ submatrix.

Base Case: If $k = 3$, then

$$A_S = \begin{array}{cc} \begin{bmatrix} 1 & 0 & \\ 0 & x & 1 \\ & 1 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array} \xrightarrow[g_3]{} \begin{array}{cc} \begin{bmatrix} 0 & 1 & \\ 1 & x & 0 \\ & 0 & 1 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array}.$$

There are two subcases. If $x = 1$, apply a four-flip $f_1$ in the upper-left corner first, followed by a four-flip $f_2$ in the lower-right corner second:

$$A_S = \begin{array}{cc} \begin{bmatrix} 1 & 0 & \\ 0 & 1 & 1 \\ & 1 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array} \xrightarrow[f_1]{} \begin{array}{cc} \begin{bmatrix} 0 & 1 & \\ 1 & 0 & 1 \\ & 1 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array} \xrightarrow[f_2]{} \begin{array}{cc} \begin{bmatrix} 0 & 1 & \\ 1 & 1 & 0 \\ & 0 & 1 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array}.$$

Otherwise, if $x = 0$, apply a four-flip $f_1$ in the lower-right corner first, followed by a four-flip $f_2$ in the upper-left corner second:

$$A_S = \begin{array}{cc} \begin{bmatrix} 1 & 0 & \\ 0 & 0 & 1 \\ & 1 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array} \xrightarrow[f_1]{} \begin{array}{cc} \begin{bmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ & 0 & 1 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array} \xrightarrow[f_2]{} \begin{array}{cc} \begin{bmatrix} 0 & 1 & \\ 1 & 0 & 0 \\ & 0 & 1 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array}.$$

In both subcases, there exist four-flips $f_1, f_2$ such that $g_3 = f_1 \circ f_2$. Thus, the base case holds.

Induction Step: Suppose there exists $\ell \geq 3$ so that for all $3 \leq k \leq \ell$, any map of the type $g_k$ can be written as the product of $k-1$ four-flips. We must show that a transformation of type $g_{\ell+1}$ can be decomposed as the product of $\ell$ four-flips.

Recall that

$$A_S = \begin{array}{cc} \begin{bmatrix} 1 & 0 & & & \\ 0 & x_2 & 1 & & \\ & 1 & \ddots & y & \\ & & y & x_\ell & z \\ & & & z & y \end{bmatrix} & \begin{array}{l} r_1 \\ r_2 \\ \vdots \\ r_\ell \\ r_{\ell+1} \end{array} \\ \begin{array}{ccccc} c_1 & c_2 & \cdots & c_\ell & c_{\ell+1} \end{array} & \end{array} \quad \xrightarrow[g_{\ell+1}]{} \quad \begin{array}{cc} \begin{bmatrix} 0 & 1 & & & \\ 1 & x_2 & 0 & & \\ & 0 & \ddots & z & \\ & & z & x_\ell & y \\ & & & y & z \end{bmatrix} & \begin{array}{l} r_1 \\ r_2 \\ \vdots \\ r_\ell \\ r_{\ell+1} \end{array} \\ \begin{array}{ccccc} c_1 & c_2 & \cdots & c_\ell & c_{\ell+1} \end{array} & \end{array}$$

where $y = 1 - z = \ell \pmod 2$ and where $x_2, \ldots, x_\ell \in \{0, 1\}$ are arbitrary. We claim that, for some $i$, $1 \leq i \leq \ell$,

$$(\Diamond_i) \qquad \begin{bmatrix} a_{r_i c_i} & a_{r_i c_{i+1}} \\ a_{r_{i+1} c_i} & a_{r_{i+1} c_{i+1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

To verify this, assume that $(\Diamond_i)$ fails for $1 \leq i \leq \ell - 1$. Then,

$$\begin{gathered} x_2 = 0, \text{ otherwise } (\Diamond_1) \text{ holds;} \\ x_3 = 1, \text{ otherwise } (\Diamond_2) \text{ holds;} \\ \ldots; \text{ and,} \\ x_\ell = y, \text{ otherwise } (\Diamond_{\ell-1}) \text{ holds.} \end{gathered}$$

But $x_\ell = y$ implies that $(\Diamond_\ell)$ holds. Therefore, $(\Diamond_i)$ is true for some $i$, $1 \leq i \leq \ell$, as claimed. Fix any such $i$, and define $f_1$ to be the four-flip in $A$ at rows $r_i, r_{i+1}$ and columns $c_i, c_{i+1}$. Then,

$$A_S^{f_1} = \begin{array}{cc} \left[\begin{array}{cccc|cccc} 1 & 0 & & & & & & \\ 0 & \ddots & u & & & & & \\ & u & x_{i-1} & v & & & & \\ & & v & \overline{x_i} & \overline{u} & & & \\ \hline & & & \overline{u} & \overline{x_{i+1}} & v & & \\ & & & & v & x_{i+2} & u & \\ & & & & & u & \ddots & z \\ & & & & & & z & y \end{array}\right] & \begin{array}{l} r_1 \\ \vdots \\ r_{i-1} \\ r_i \\ r_{i+1} \\ r_{i+2} \\ \vdots \\ r_{\ell+1} \end{array} \\ \begin{array}{cccccccc} c_1 & \cdots & c_{i-1} & c_i & c_{i+1} & c_{i+2} & \cdots & c_{\ell+1} \end{array} & \end{array}$$

where $v = 1 - u = i \pmod 2$ and where the $\overline{x} = 1 - x$ are four-flipped. Let $\alpha$ be an action on the upper-left submatrix of $A_S^{f_1}$ of type $g_i$ and, let $\beta$ be an action on the lower-right submatrix of $A_S^{f_1}$ of type $g_{\ell+1-i}$. Then, observe that

$$(\heartsuit) \qquad A_S \xrightarrow[g_{\ell+1}]{} \left(A_S^{f_1}\right)^{\alpha \circ \beta}.$$

By the induction hypothesis, there is a product of four-flips such that $\alpha = f_2 \circ \cdots \circ f_i$ and a product of four-flips such that $\beta = f_{i+1} \circ \cdots \circ f_\ell$. Therefore, by ($\heartsuit$), $g_{\ell+1} = f_1 \circ f_2 \circ \cdots f_i \circ f_{i+1} \circ \cdots f_\ell$, as required. $\square$

All weight-preserving actions, which respect the order of row and column weights, can be written as a product of four-flips:

**Theorem 3.4.** *Let $A$ and $B$ be 0-1 matrices, with equal dimensions. If $A \neq B$ and if the $i$-th row (column) of $A$ has the same weight as the $i$-th row (column) of $B$, then there exist four-flips $f_1, \ldots, f_k$ such that $A = B^g$ where $g = f_1 \circ \cdots \circ f_k$.*

*Proof.* Define the difference matrix $D_B = B - A$, interpreting $A$ and $B$ as matrices over $\mathbb{R}$. Then, $D$ records the positions, at which $B$ differs from $A$, as follows. If $A = (a_{ij})$, $B = (b_{ij})$ and $D_B = (d_{ij}^B)$, then

$$d_{ij}^B = \begin{cases} 0, & \text{if } a_{ij} = b_{ij} \\ +1, & \text{if } a_{ij} = 0 \text{ and } b_{ij} = 1 \\ -1, & \text{if } a_{ij} = 1 \text{ and } b_{ij} = 0. \end{cases}$$

(For the remainder of the proof, we write "+" and "−" as abbreviations for "+1" and "−1", respectively.) More generally, non-zero entries of $D_B$ represent exactly those entries of $B$, which if flipped, send $B$ to $A$. Our plan is to "remove" all the non-zero entries from $D_B$, by applying four-flips ($g = f_1 \circ \cdots f_k$) to $B$, until $D_{B^g} = 0$ and $A = B^g$.

Suppose that

$$D_B \supset \begin{array}{cc} \begin{bmatrix} + & - \\ - & + \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \end{array} \\ \begin{array}{cc} c_1 & c_2 \end{array} & \end{array} \qquad \left( B \supset \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right),$$

where the notation signifies that $D_B$ contains the specified submatrix, formed by the intersection of rows $r_1, r_2$ and columns $c_1, c_2$. If $f$ is the four-flip in $B$ at rows $r_1, r_2$ and columns $c_1, c_2$, then

$$D_{B^f} \supset \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \end{array} \\ \begin{array}{cc} c_1 & c_2 \end{array} & \end{array} \qquad \left( B^f \supset \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right).$$

Choose four-flips $f_1, \cdots, f_j$ in $B$ that correspond to non-zero entries in $D_B$ and such that, for all rows $r_1, r_2$ and columns $c_1, c_2$, this holds: writing $g_1 := f_1 \circ \cdots \circ f_j$ and $B_1 := B^{g_1}$,

($\clubsuit$) $$D_{B_1} \not\supset \begin{array}{cc} \begin{bmatrix} + & - \\ - & + \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \end{array} \\ \begin{array}{cc} c_1 & c_2 \end{array} & \end{array}.$$

In other words, remove any four-flips in $B$ that correspond to the non-zero entries in $D_B$ and label the resulting matrices $B_1$ and $D_{B_1}$.

If $D_{B_1} = 0$, we're done. Otherwise, differences between $A$ and $B_1$ remain. Without loss of generality, suppose that $D_{B_1}$ contains a "+" at row $r_1$ and column $c_1$. We must show how to remove the difference, by applying four-flips in $B_1$. Observe that the following property holds:

> **Fact A.** *In any row or column of $D_{B_1}$, the number of "+" entries is equal to the number of "−" entries.*
>
> *Proof.* The $i$-th row (column) of $B_1$ has the same weight as the $i$-th row (column) of $B$, by Lemma 3.3. But the $i$-th row (column) of $B$ has the same weight as the $i$-th row (column) of $A$, by the theorem hypothesis. Thus, the weight of any row or column of $D_{B_1}$ is zero. $\square$

Below is a procedure to remove the "+" at row $r_1$, column $c_1$ in $D_{B_1}$, by applying four-flips in $B_1$. The output is a product of four-flips $g_2$ such that $B_2 = B_1^{g_2}$ and such that $D_{B_2}$ contains fewer non-zero entries than $D_{B_1}$.

To start, we are given that

$$(\spadesuit_1) \qquad D_{B_1} \supset \begin{array}{cc} \begin{bmatrix} + \end{bmatrix} & r_1 \\ c_1 & \end{array}.$$

By Fact $A$, row $r_1$ and column $c_1$ must contain a "−" entry, as well. Hence, there exist a row $r_2$ and a column $c_2$ such that

$$(\spadesuit_2) \qquad D_{B_1} \supset \begin{array}{cc} \begin{bmatrix} + & - \\ - & * \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \end{matrix} \\ \begin{matrix} c_1 & c_2 \end{matrix} & \end{array}.$$

Again, by Fact A, row $r_2$ and column $c_2$ must contain a "+" entry. However, the corner entry $* \neq$ "+", by ($\clubsuit$). Hence, there exist a row $r_3$ and a column $c_3$ such that

$$(\spadesuit_3) \qquad D_{B_1} \supset \begin{array}{cc} \begin{bmatrix} + & - & * \\ - & -/0 & + \\ * & + & x_3 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array}.$$

To proceed, we examine the value of $x_3$. If $x_3 = $ "$-$", then we're done: by Lemma 3.3, there is a product of four-flips $g_2 = f_{j+1} \circ f_{j+2}$ such that

$$D_{B_1}^{g_2} \supset \begin{array}{cc} \begin{bmatrix} 0 & 0 & * \\ 0 & {\scriptstyle -/0} & 0 \\ * & 0 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 \end{matrix} & \end{array}.$$

Otherwise, $x_3 = 0$ or $+$, and we repeat the argument. By Fact A, row $r_3$ and column $c_3$ must contain a "$-$" entry. However, each of the entries $* \neq$ "$-$", by (♣). Hence, there exist a row $r_4$ and a column $c_4$ such that

$$(\spadesuit_4) \qquad D_{B_1} \supset \begin{array}{cc} \begin{bmatrix} + & - & {\scriptstyle 0/+} & * \\ - & {\scriptstyle -/0} & + & * \\ {\scriptstyle 0/+} & + & {\scriptstyle 0/+} & - \\ * & * & - & x_4 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} & \end{array}.$$

To continue, we now examine the value of $x_4$. If $x_4 = $ "$+$", then we're done: by Lemma 3.3, there is a product of four-flips $g_2 = f_{j+1} \circ f_{j+2} \circ f_{j+3}$ such that

$$D_{B_1}^{g_2} \supset \begin{array}{cc} \begin{bmatrix} 0 & 0 & {\scriptstyle 0/+} & * \\ 0 & {\scriptstyle -/0} & 0 & * \\ {\scriptstyle 0/+} & 0 & {\scriptstyle 0/+} & 0 \\ * & * & 0 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{matrix} \\ \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} & \end{array}.$$

Otherwise, $x_4 = -$ or $0$, and we repeat the argument.... By induction, use Fact A and (♣) to define ($\spadesuit_5$), ($\spadesuit_6$), ($\spadesuit_7$), ($\spadesuit_8$), ..., as necessary. Since the dimension of $D_{B_1}$ is finite, the sequence ($\spadesuit_i$) must terminate. (Otherwise, $D_{B_1}$ contains an infinite number of rows and columns.) That is, there exists some $i \geq 5$, for which there exists a product of four-flips $g_2 = f_{j+1} \circ f_{j+2} \circ f_{j+3} \circ \cdots \circ f_{j+i-1}$ such that

$$D_{B_1} \supset \begin{array}{cc} \begin{bmatrix} + & - & * & * & * \\ - & * & + & * & * \\ * & + & \ddots & \pm & * \\ * & * & \pm & * & \mp \\ * & * & * & \mp & \pm \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ \vdots \\ r_{i-1} \\ r_i \end{matrix} \\ \begin{matrix} c_1 & c_2 & \cdots & c_{i-1} & c_i \end{matrix} & \end{array}, \quad D_{B_1}^{g_2} \supset \begin{array}{cc} \begin{bmatrix} 0 & 0 & * & * & * \\ 0 & * & 0 & * & * \\ * & 0 & \ddots & 0 & * \\ * & * & 0 & * & 0 \\ * & * & * & 0 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ \vdots \\ r_{i-1} \\ r_i \end{matrix} \\ \begin{matrix} c_1 & c_2 & \cdots & c_{i-1} & c_i \end{matrix} & \end{array}.$$

This concludes the procedure.

If $D_{B_2} = 0$, we're done. Otherwise, differences between $A$ and $B_2$ remain. Use induction on the procedure above to produce a sequence of four-flip products $g_3, g_4, \ldots,$ as necessary such that for all $i \geq 3$, $B_i = B_{i-1}^{g_i}$ and $D_{B_i}$ contains fewer non-zero entries than does $D_{B_{i-1}}$. Since each $D_{B_i}$ has a fixed and finite dimension, it follows that $D_{B_\ell} = 0$, for some $\ell \geq 3$. Thus, $A = B^g$ where $g = g_1 \circ g_2 \circ \cdots \circ g_\ell$ is a composition of products of four-flip. Therefore, a product of four-flips $g = f_1 \circ \cdots \circ f_k$ sends $B$ to $A$, as claimed. $\square$

To recap, Theorem 3.4 states that if two 0-1 matrices $A$ and $B$ have the same row and column weights, with weights appearing in the same order, then there is a sequence of four-flips $f_1, \ldots f_k$ that send $B$ to $A$. Therefore, the class of weight-preserving actions on a 0-1 matrix that also preserve the order of row and column weights is exactly the set of all legal products of four-flips in the matrix.

For arbitrary weight-preserving actions on 0-1 matrices, the following generalization of the theorem easily follows. Any weight-preserving action on a 0-1 matrix is just a permutation of rows or columns, followed by a product of four-flips:

**Corollary 3.5.** *Let $A$ and $B$ be 0-1 matrices, with equal dimensions. If $\mathrm{w}(A) = \mathrm{w}(B)$ and if $A$ and $B$ are not equivalent, then there exist a permutation of rows $\pi_R$, a permutation of columns $\pi_C$, and four-flips $f_1, \ldots, f_k$ such that $A = B^g$ where $g = \pi_R \pi_C \circ f_1 \circ \cdots \circ f_k$.*

*Proof.* Since $\mathrm{w}(A) = \mathrm{w}(B)$, we can choose $\pi_R$ and $\pi_C$ such that the $i$-th row (column) of $A$ has the same weight as the $i$-th row (column) of $B_1 := B^{\pi_R \pi_C}$. But $A$ and $B$ are not equivalent, so $A \neq B_1$. Thus, by Theorem 3.4, there exists a product of four-flips $g_1 = f_1 \circ \cdots \circ f_k$ such that $A = B_1^{g_1}$. Therefore, $A = (B^{\pi_R \pi_C})^{g_1}$, as required. $\square$

Next, let's illustrate the procedure of Theorem 3.4 and Corollary 3.5, using the example of $U$ and $V$. Recall that $U$ is sent to $V$,

$$U = \begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathbf{0} & \mathbf{1} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \begin{matrix} \mathit{0} \\ \mathit{2} \\ \mathit{2} \\ \mathit{2}' \\ \mathit{4} \\ \mathit{4} \end{matrix} \\ \begin{matrix} \mathit{2} & \mathit{2} & \mathit{3} & \mathit{3} & \mathit{4} \end{matrix} & \end{array} \qquad V = \begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathbf{1} & \mathbf{0} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \begin{matrix} \mathit{0} \\ \mathit{2} \\ \mathit{2} \\ \mathit{2}' \\ \mathit{4} \\ \mathit{4} \end{matrix} \\ \begin{matrix} \mathit{2} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{3} \end{matrix} & \end{array}$$

by a transposition $\tau$ in the second row and right-most columns of $U$. Since $U$ and $V$ have the same weights, but are not equivalent (as shown above), apply the method of Corollary 3.5, with $A = V$ and $B = U$.

First, we must select a permutation of rows $\pi_R$ and a permutation of columns $\pi_C$ such that the weights of $U^{\pi_R\pi_C}$ appear in the same order as the weights of $V$. Because the row weights of $U$ are already in the same order as the row weights of $V$, it is simplest to choose $\pi_R = \mathrm{id}$. Also, since only two column weights are out of order, it is easiest to choose the permutation that transposes these two columns; namely, the permutation $\pi_C = (4,5)_C$ that exchanges the fourth and fifth columns. Under these choices, $U^{\pi_R\pi_C}$ equals

$$U^{(4,5)_C} = \begin{array}{cl} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \begin{array}{l} \mathit{0} \\ \mathit{2} \\ \mathit{2} \\ \mathit{2} \\ \mathit{4} \\ \mathit{4} \end{array} \\ \begin{array}{ccccc} \mathit{2} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{3} \end{array} & \end{array}.$$

At this point, it is easy to verify that the weights of $U^{(4,5)_C}$ do appear in the same order as the weights of $V$. The next step in the corollary is to apply Theorem 3.4, with $A = V$ and $B = U^{(4,5)_C}$. Following the theorem's procedure, we define the difference matrix $D = U^{(4,5)_C} - V$, by interpreting the entries of the matrices as real numbers. In this case,

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & - \\ 0 & 0 & 0 & - & + \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where "+" and "−" are abbreviations for "+1" and "−1", respectively. Our next goal is to "remove" all of the non-zero entries of $D$, by using four-flips in $B$ $(= U^{(4,5)_C})$. In this instance, the difference matrix clearly suggests applying a four-flip $f$ at rows $3,4$ and columns $4,5$:

$$D^f = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (U^{(4,5)_C})^f = \begin{array}{cl} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \begin{array}{l} \mathit{0} \\ \mathit{2} \\ \mathit{2} \\ \mathit{2} \\ \mathit{4} \\ \mathit{4} \end{array} \\ \begin{array}{ccccc} \mathit{2} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{3} \end{array} & \end{array} = V.$$

Hence, we have decomposed the weight-preserving action $\tau$ on $U$ as the product of a permutation of columns and a four-flip, $\tau = (4,5)_C f$.

**Remark 3.6.** Permutations of rows and columns, and four-flips are both necessary and sufficient to classify the weight-preserving actions on 0-1 matrices. On one hand, some actions are not permutations of rows and columns, such as the action of $\tau$ on $U$. On the other hand, some permutations of rows and columns are not a product of four-flips, since permutations may change the order of weights. Thus, both types of action are necessary for classification; the sufficiency of the two types of action follows from Corollary 3.5. $\dashv$

## 4. Two Conjectures

Let's return our discussion to the action of weight-preserving sets. Examples 1.2 and 2.3 looked at a weight-preserving set $F$, which is sent to its cosets under addition by permutations of rows and columns of its matrix representations; i.e., $F$ is a cwatset. Section 3 demonstrated that permutations do not cover all weight-preserving actions on 0-1 matrices. Do other actions occur in the weight-preserving sets, as well?

Recall that $U = \{00000, 10001, 01001, 00110, 10111, 01111\} \subseteq \mathbb{Z}_2^5$. We verify below that $U$ is a weight-preserving set. But while the matrix representations of $U$ are compatible with weight-preserving actions that are not permutations of rows and columns (such as $\tau$), the actions which send $U$ to its cosets under addition are as follows:

$$U + 00000 = U^{\mathrm{id}}$$

$$U + 10001 = U^{(1,2)_R(4,5)_R(1,5)_C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$U + 01001 = U^{(1,3)_R(4,6)_R(2,5)_C}$$

$$U + 00110 = U^{(1,4)_R(2,5)_R(3,6)_R}$$

$$U + 10111 = U^{(1,5)_R(2,4)_R(3,6)_R(1,5)_C}$$

$$U + 01111 = U^{(1,6)_R(2,5)_R(3,4)_R(2,5)_C}$$

(Note the permutations are relative to the matrix representation in §3.) Hence, $U$ is a cwatset. In other words, a weight-preserving action on the matrix representations of $U$, which is not just a permutation of rows and columns, does not take $U$ to any of its cosets under addition. This observation is circumspect. Is every weight-preserving set a cwatset?

We suspect an answer, in the affirmative:

**Conjecture 4.1.** *Let $A \subseteq \mathbb{Z}_2^d$. Then, $A$ is a weight-preserving set if, and only if, $A$ is a cwatset.* $\dashv$

Yet, a stronger conjecture is that the collection of equivalence classes of weight-preserving sets is partitioned by row and column weights:

**Conjecture 4.2.** *Let $A, B \subseteq \mathbb{Z}_2^d$ be weight-preserving sets. Then, $A$ and $B$ are equivalent if, and only if, $\mathrm{w}(A) = \mathrm{w}(B)$.* $\dashv$

**Remark 4.3.** Observe that Conjecture 4.1 follows from Conjecture 4.2 and the following lemma. $\dashv$

**Lemma 4.4.** *Let $A \subseteq \mathbb{Z}_2^d$ be a weight-preserving set. The coset $A + a$ is a weight-preserving set if, and only if, $a \in A$.*

*Proof.* Let $A$ be a weight-preserving set.

($\Leftarrow$) Suppose that $A + a$ is a weight-preserving set. Since $\bar{0} \in A + a$, there exists $a_1 \in A$ such that $a_1 + a = \bar{0}$. Thus, $a = a_1 \in A$, as required.

($\Rightarrow$) Suppose that $a \in A$. For any $a_2 + a \in A + a$, it follows that $\mathrm{w}((A + a) + (a_2 + a)) = \mathrm{w}(A + a_2) = \mathrm{w}(A) = \mathrm{w}(A + a)$, because $A$ is a weight-preserving set and $a, a_2 \in A$. $\square$

**Remark 4.5.** To date, we have verified that Conjectures 4.1 and 4.2 hold in degree $d$, for all $1 \leq d \leq 5$. For degrees 1 and 2, the conjectures are trivial. For degrees 3 through 5, the reader can compare the lists of all inequivalent weight-preserving sets in the appendix with the lists of all inequivalent cwatsets in [8]. $\dashv$

Motivated by the classification of weight-preserving actions on 0-1 matrices given in section 3, we suggest the following line of attack on the two conjectures. Let $A \subseteq \mathbb{Z}_2^d$ be a weight-preserving set. Show that if $g = f_1 \circ \cdots \circ f_k$ is a product of four-flips in $A$ such that $A^g$ is a weight-preserving set, then $g$ is a permutation of the rows and columns of $A$. Then, Conjecture 4.2 follows from Theorem 3.5 and Lemma 4.4. Of course, there is a possibility that Conjecture 4.2 fails, yet 4.1 holds. One might prove Conjecture 4.1 directly using an induction argument on the degree of weight-preserving sets.

The key application of Conjecture 4.1 is to speed up the detection of cwatsets in computer code. Given a set of binary words $A \subseteq \mathbb{Z}_2^d$, suppose that we want to determine if $A$ is a cwatset. To check that Definition 2.1 is satisfied, we must be able to enumerate the symmetric group on $d$ symbols. But as the degree $d \to \infty$, the order of $S_d$ grows superexponentially ($O(d!)$). This prevents the detection of cwatsets, even in degrees as low as 9. On Conjecture 4.1, however, it suffices to sort row and column weights of $A + a$ for comparison with the weights of $A$, which is almost linear ($O(n \log n)$) in the dimensions of $A$.

## 5. Miscellaneous Results

To close, we provide a technical refinement of the Lagrange type theorem for cwatsets, cited in section 2. Recall that Proposition 2.5 states that, for an arbitrary cwatset $A$ in degree $d$ of order $n$,

$$(\flat) \qquad n \mid 2^d d!.$$

But $n = |A| \leq 2^d$, since $A \subseteq \mathbb{Z}_2^d$. Hence, the power of 2 in the divisor condition $(\flat)$ is not sharp, in the following sense. If $\alpha(d)$ denotes the highest power of 2 dividing $d!$, then

$$(\natural) \qquad n \mid 2^{d-\alpha(d)} d!.$$

Yet, $\alpha(d) \to \infty$ as $d \to \infty$. Hence, the condition $(\natural)$ improves over $(\flat)$. To make $(\natural)$ sharp, we must lower bound $\alpha(d)$, for arbitrary $d$.

**Definition 5.1** (Greatest Dividing Exponent)**.** Let $b \geq 2$ be an integer. We define $\psi_b : \mathbb{N}^+ \to \mathbb{N}$ such that $\psi_b(n) := \max\{k : b^k | n\}$. ⊣

**Lemma 5.2.** *Let $m \geq 1$, $n \geq 1$ be integers, $p \geq 2$ prime. Then,*

1. $\psi_p(n) > 0$ *if, and only if,* $p|n$
2. $\psi_p(mn) = \psi_p(m) + \psi_p(n)$
3. $\psi_p(p^n) = n$
4. $\psi_p(p^n!) = (p^n - 1)/(p-1)$

*Proof.* Claims (1), (2), and (3) follow easily from Definition 5.1 and the Fundamental Theorem of Arithmetic. For Claim (4), we observe that

$$\begin{aligned}
\psi_p(p^{n+1}!) &= \underbrace{\psi_p(p) + \cdots + \psi_p(p^n)}_{1} \\
&\quad + \underbrace{\psi_p(p^n + p) + \cdots + \psi_p(2p^n)}_{2} \\
&\quad + \cdots + \underbrace{\psi_p((p-1)p^n + p) + \cdots + \psi_p(p^{n+1})}_{p} \\
&= \underbrace{\psi_p(p) + \cdots + \psi_p(p^n)}_{1} \\
&\quad + \underbrace{\psi_p(p) + \cdots + \psi_p(p^n)}_{2} \\
&\quad + \cdots + \underbrace{\psi_p(p) + \cdots + \psi_p(p^n)}_{p} + 1
\end{aligned}$$

to find the recurrence relation

$$(\star) \qquad \psi_p(p^{n+1}!) = p\psi_p(p^n!) + 1, \qquad \psi_p(p^0!) = \psi_p(1) = 0.$$

Therefore, solving $(\star)$ yields $\psi_p(p^n!) = (p^n - 1)/(p-1)$. □

**Theorem 5.3.** *Let $n \geq 2$ be an integer. Then, $1 \leq n - \psi_2(n!) \leq \lceil \lg n \rceil$. Both inequalities are obtained for an infinite class of $n$.*

*Proof.* Fix an integer $k \geq 1$. By Lemma 5.2, $\psi_2(2^k!) = 2^k - 1$; i.e.,

$$(\diamond) \qquad 2^k - \psi_2(2^k!) = 1.$$

Moreover, $\psi_2((2^{k+1}-1)!) = \psi_2(2^{k+1}!) - \psi_2(2^{k+1}) = 2^{k+1} - 1 - (k+1)$. Yet, since $k + 1 = \lceil \lg(2^{k+1} - 1) \rceil$,

$$2^{k+1} - 1 - \psi_2((2^{k+1}-1)!) = \lceil \lg(2^{k+1} - 1) \rceil.$$

Thus, we know that the claimed lower and upper bounds for $n - \psi_2(n!)$ are achieved infinitely often. It remains to show that for each integer $n \in (2^k, 2^{k+1} - 1)$, $1 \leq n - \psi_2(n!) \leq \lceil \lg n \rceil = k + 1$.

In fact, because $\psi_2((n+1)!) = \psi_2(n!)$ for all integers $n \geq 2$, $n$ even, it suffices to show that for all $n \in \mathcal{A}_k := \{2^k + 2, 2^k + 4, \ldots, 2^{k+1} - 2\}$ that $1 < n - \psi_2(n!) < k + 1$. Proceed by induction on $k$.

Base Case: For $k = 2$, $\mathcal{A}_k = \{6\}$ and $6 - \psi_2(6!) = 2 \in (1, 3)$.

Induction Step: Suppose there exists $k \geq 2$ such that $n - \psi_2(n!) \in (1, k+1)$, for all $n \in \mathcal{A}_k$. It is useful notation to define two sequences $(\sigma_i)_{i=1}^{2^{k-1}}$ and $(\tau_i)_{i=1}^{2^k}$, by $\sigma_i := \psi_2(2^k + 2i)$ and $\tau_i := \psi_2(2^{k+1} + 2i)$, respectively. Then, by $(\diamond)$, the induction hypothesis states that

$$(\circ) \qquad 0 < 2i - \sigma_1 - \cdots - \sigma_i < 2^k - \sigma_1 - \cdots - \sigma_{2^{k-1}}$$

for all $1 \leq i < 2^{k-1}$. Now, we must verify that $n - \psi_2(n!) \in (1, k+2)$, for all $n \in \mathcal{A}_{k+1}$. Again, by $(\diamond)$, this means we must show that

$$(\bullet) \qquad 0 < 2i - \tau_1 - \cdots - \tau_i < 2^{k+1} - \tau_1 - \cdots - \tau_{2^k}$$

for all $1 \leq i < 2^k$. But, for all $1 \leq i < 2^{k-1}$,

$$\tau_i = \tau_{i+2^{k-1}} = \sigma_i, \qquad \text{and} \qquad \tau_{2^{k-1}} = \sigma_{2^{k-1}} - 1.$$

Therefore, $(\bullet)$ follows from $(\circ)$, as required. $\square$

**Corollary 5.4.** *If $A \subseteq \mathbb{Z}_2^d$ is a cwatset in degree $d$ and order $n$, then*

$$(\natural) \qquad n \big| 2^{\lceil \lg d \rceil} d!$$

*Proof.* By Proposition 2.5, we know that $n \big| 2^d d!$. However, $n \leq 2^d$, since $A \subseteq \mathbb{Z}_2^d$, and $2^{d - \lceil \lg d \rceil} \big| d!$, by Theorem 5.3. Therefore, $n \big| 2^{\lceil \lg d \rceil} d!$. $\square$

**Remark 5.5.** The power of 2 in $(\natural)$ cannot be reduced for arbitrary $d$, because the lower bound $d - \lceil \lg d \rceil \leq \psi_2(d!) = \alpha(d)$ in Theorem 5.3 is obtained for an infinite class of $d$, namely $d = 2^{k+1} - 1$, $k \geq 1$. Hence, Corollary 5.4 is the best refinement of Proposition 2.5 in this direction for arbitrary $d$. At the same time, the upper bound in Theorem 5.3 shows that if $A$ is a cwatset in degree $d = 2^k$, $k \geq 1$, then $|A| \big| 2d!$. $\dashv$

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## Appendix

### Small Degree Weight-Preserving Sets

The following lists contain representatives of all inequivalent classes of weight-preserving sets through degree five. These were generated using software developed by the author, which is available online at:

<http://www.pitt.edu/~mab39/awp>.

Degree 1: (2 Classes)

1a. $\{0\}$
2a. $\{0,1\} = \mathbb{Z}_2^1$

Degree 2: (4 Classes)

1a. $\{00\}$
2a. $\{00,10\}$

2b. $\{00, 11\}$
4a. $\{00, 10, 01, 11\} = \mathbb{Z}_2^2$

Degree 3: (10 Classes)

1a. $\{000\}$
2a. $\{000, 100\}$
2b. $\{000, 110\}$
2c. $\{000, 111\}$
3a. $\{000, 110, 101\}$
4a. $\{000, 100, 010, 110\}$
4b. $\{000, 100, 011, 111\}$
4c. $\{000, 110, 101, 011\}$
6a. $\{000, 100, 010, 101, 011, 111\}$
8a. $\{000, 100, 010, 110, 001, 101, 011, 111\} = \mathbb{Z}_2^3$

Degree 4: (22 Classes)

1a. $\{0000\}$
2a. $\{0000, 1000\}$
2b. $\{0000, 1100\}$
2c. $\{0000, 1110\}$
2d. $\{0000, 1111\}$
3a. $\{0000, 1100, 1010\}$
4a. $\{0000, 1000, 0100, 1100\}$
4b. $\{0000, 1000, 0110, 1110\}$
4c. $\{0000, 1000, 0111, 1111\}$
4d. $\{0000, 1100, 1010, 0110\}$
4e. $\{0000, 1100, 1010, 1001\}$
4f. $\{0000, 1100, 0011, 1111\}$
4g. $\{0000, 1100, 1011, 0111\}$
6a. $\{0000, 1000, 0100, 1010, 0110, 1110\}$
6b. $\{0000, 1000, 0110, 1110, 0101, 1101\}$
6c. $\{0000, 1100, 1010, 0101, 0011, 1111\}$
8a. $\{0000, 1000, 0100, 1100, 0010, 1010, 0110, 1110\}$
8b. $\{0000, 1000, 0100, 1100, 0011, 1011, 0111, 1111\}$
8c. $\{0000, 1000, 0110, 1110, 0101, 1101, 0011, 1011\}$
8d. $\{0000, 1100, 1010, 0110, 1001, 0101, 0011, 1111\}$
12a. $\{0000, 1000, 0100, 1100, 0010, 1010, 0101, 1101, 0011, 1011, 0111, 1111\}$
16a. $\{0000, 1000, 0100, 1100, 0010, 1010, 0110, 1110, 0001, 1001, 0101, 1101, 0011, 1011, 0111, 1111\} = \mathbb{Z}_2^4$

Degree 5: (53 Classes)

1a. $\{00000\}$
2a. $\{00000, 10000\}$
2b. $\{00000, 11000\}$
2c. $\{00000, 11100\}$
2d. $\{00000, 11110\}$
2e. $\{00000, 11111\}$
3a. $\{00000, 11000, 10100\}$
4a. $\{00000, 10000, 01000, 11000\}$
4b. $\{00000, 10000, 01100, 11100\}$
4c. $\{00000, 10000, 01110, 11110\}$
4d. $\{00000, 10000, 01111, 11111\}$
4e. $\{00000, 11000, 10100, 01100\}$
4f. $\{00000, 11000, 10100, 10010\}$
4g. $\{00000, 11000, 00110, 11110\}$
4h. $\{00000, 11000, 10110, 01110\}$
4i. $\{00000, 11000, 10110, 10101\}$
4j. $\{00000, 11000, 00111, 11111\}$
4k. $\{00000, 11000, 10111, 01111\}$
4l. $\{00000, 11100, 10011, 01111\}$
5a. $\{00000, 11000, 10100, 10010, 10001\}$
5b. $\{00000, 11000, 00110, 11101, 10111\}$
6a. $\{00000, 10000, 01000, 10100, 01100, 11100\}$
6b. $\{00000, 10000, 01100, 11100, 01010, 11010\}$
6c. $\{00000, 11000, 10100, 01010, 00110, 11110\}$
6d. $\{00000, 11000, 10100, 00011, 11011, 10111\}$
6e. $\{00000, 11000, 10100, 01011, 00111, 11111\}$
8a. $\{00000, 10000, 01000, 11000, 00100, 10100, 01100, 11100\}$
8b. $\{00000, 10000, 01000, 11000, 00110, 10110, 01110, 11110\}$
8c. $\{00000, 10000, 01000, 11000, 00111, 10111, 01111, 11111\}$
8d. $\{00000, 10000, 01100, 11100, 01010, 11010, 00110, 10110\}$
8e. $\{00000, 10000, 01100, 11100, 01010, 11010, 01001, 11001\}$
8f. $\{00000, 10000, 01100, 11100, 00011, 10011, 01111, 11111\}$
8g. $\{00000, 10000, 01100, 11100, 01011, 11011, 00111, 10111\}$
8h. $\{00000, 11000, 10100, 01100, 10010, 01010, 00110, 11110\}$
8i. $\{00000, 11000, 10100, 01100, 00011, 11011, 10111, 01111\}$
8j. $\{00000, 11000, 10100, 01100, 10011, 01011, 00111, 11111\}$
8k. $\{00000, 11000, 00110, 11110, 10101, 01101, 10011, 01011\}$
10a. $\{00000, 10000, 01000, 10100, 01010, 10101, 01011, 10111, 01111, 11111\}$

10b. $\{00000, 10000, 01100, 11010, 01110, 10101, 11101, 00011, 11011, 00111\}$

10c. $\{00000, 11000, 10100, 10010, 01110, 10001, 01101, 01011, 00111, 11111\}$

10d. $\{00000, 11000, 10100, 10010, 01001, 00101, 11101, 00011, 11011, 10111\}$

12a. $\{00000, 10000, 01000, 11000, 00100, 10100, 01010, 11010, 00110, 10110, 01110, 11110\}$

12b. $\{00000, 10000, 01000, 11000, 00110, 10110, 01110, 11110, 00101, 10101, 01101, 11101\}$

12c. $\{00000, 10000, 01000, 10100, 01100, 11100, 00011, 10011, 01011, 10111, 01111, 11111\}$

12d. $\{00000, 10000, 01100, 11100, 01010, 11010, 00101, 10101, 00011, 10011, 01111, 11111\}$

12e. $\{00000, 11000, 10100, 01100, 10010, 01010, 00101, 11101, 00011, 11011, 10111, 01111\}$

16a. $\{00000, 10000, 01000, 11000, 00100, 10100, 01100, 11100, 00010, 10010, 01010, 11010, 00110, 10110, 01110, 11110\}$

16b. $\{00000, 10000, 01000, 11000, 00100, 10100, 01100, 11100, 00011, 10011, 01011, 11011, 00111, 10111, 01111, 11111\}$

16c. $\{00000, 10000, 01000, 11000, 00110, 10110, 01110, 11110, 00101, 10101, 01101, 11101, 00011, 10011, 01011, 11011\}$

16d. $\{00000, 10000, 01100, 11100, 01010, 11010, 00110, 10110, 01001, 11001, 00101, 10101, 00011, 10011, 01111, 11111\}$

16e. $\{00000, 11000, 10100, 01100, 10010, 01010, 00110, 11110, 10001, 01001, 00101, 11101, 00011, 11011, 10111, 01111\}$

20a. $\{00000, 10000, 01000, 00100, 10010, 01010, 11010, 00110, 10110, 01110, 10001, 01001, 11001, 00101, 10101, 01101, 11011, 10111, 01111, 11111\}$

24a. $\{00000, 10000, 01000, 11000, 00100, 10100, 01100, 11100, 00010, 10010, 01010, 11010, 00101, 10101, 01101, 11101, 00011, 10011, 01011, 11011, 00111, 10111, 01111, 11111\}$

32a. $\{00000, 10000, 01000, 11000, 00100, 10100, 01100, 11100, 00010, 10010, 01010, 11010, 00110, 10110, 01110, 11110, 00001, 10001, 01001, 11001, 00101, 10101, 01101, 11101, 00011, 10011, 01011, 11011, 00111, 10111, 01111, 11111\} = \mathbb{Z}_2^5$

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