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1 Introduction

The ability to study the interior of an object without destroying it is an important industrial tool. One method of recent interest is steady state thermal or impedance imaging. In this paper we will use the steady state heat equation to locate one or more circular inclusions within a two-dimensional region D, where the boundaries of the inclusions have partially disbonded from the surrounding material; this disbond is modelled as between the heat flux and jump discontinuity at the disbonded interface. We will first consider the case of one inclusion. We will explain an algorithm for mapping one inclusion inside the region D and give a numerical example. We will then give an algorithm for locating n inclusions and give a numerical example.

A similar inverse problem for a single circular inclusion was studied in [4], for a rather general case of the problem. Our approach, which adapts to multiple inclusions, is based on a variation of the "reciprocity gap" technique; see [1] or [2] for more information on this approach.

2 The Forward Problem

Let D be a bounded region in $\mathbb{R}^2$ with boundary ∂D. We assume, after appropriate scaling, that the thermal conductivity and diffusivity of D are both equal to one. Within D there is a circular inclusion B with boundary ∂B, centered at a point C^*, with radius R, where ∂B may have corroded and begun to disbond from the rest of the region. We apply a time-independent heat flux $g(x, y)$ to ∂D, for a sufficient amount of time so that the temperature within D stabilizes at some function $u(x, y)$. Then the function $u(x, y)$ satisfies the two-dimensional steady-state heat equation

$$\triangle u = 0$$

within $D \backslash B$ and B, where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. We also have the boundary data

$$\frac{\partial u}{\partial \mathbf{n}} = g(x, y) \text{ on } \partial D$$
$$\frac{\partial u^+}{\partial \mathbf{n}} = \frac{\partial u^-}{\partial \mathbf{n}} = k[u] \text{ on } \partial B$$

where $\mathbf{n}$ is a unit normal outward vector field on ∂D or a unit normal outward vector on ∂B. Here, we state the temperature difference across ∂B as $[u] = u^+ - u^-$, where we will use the superscript "+" to denote the limiting value of a quantity as approached from the outside of B and the superscript "-" to denote the limiting value of a quantity as approached from the inside of B. Note that the normal derivative of u is continuous over the interface ∂B (a consequence of the conservation of energy) while u itself has a jump. We call k the "transmission constant," where $k \geq 0$ and describes the extent of the corrosion of ∂B. If $k = 0$, then ∂B is completely disbonded from D. If k is large, then heat flows more easily over ∂B. We will also add the conditions that $\int_{\partial B} g\, ds = 0$ to ensure the existence of a solution and $\int_{\partial B} u\, ds = 0$ to ensure a unique solution.

Now let us consider the following inverse problem: suppose that we are unable to access the interior of D and are only able to measure $u(x, y)$ on ∂D for a given $g(x, y)$. Can we locate the center, radius, and transmission constant of B?

3 The Reciprocity Gap

We begin by recalling Green's Second Identity ([5]).

Theorem 1 (Green's Second Identity) *For any pair of functions u and w that are $C^2(\overline{D})$,*

$$\int\int_D (u\triangle w - w\triangle u)\, dA = \int_{\partial D}\left(u\frac{\partial w}{\partial \mathbf{n}} - w\frac{\partial u}{\partial \mathbf{n}}\right)\, ds.$$

Let us choose a function $w(x,y)$ that is harmonic throughout D, so that $\triangle w = 0$ throughout D. We also recall that our temperature function $u(x,y)$ is harmonic within B and $D\backslash B$, so we also have $\triangle u = 0$ within B and $D\backslash B$. By using Green's Second Identity on B we find

$$\int_{\partial B}(u^-\frac{\partial w^-}{\partial \mathbf{n}} - w\frac{\partial u^-}{\partial \mathbf{n}})\, ds \;=\; 0 \qquad (1)$$

and using Green's Second Identity on $D\backslash B$ yields

$$\int_{\partial D}(u^+\frac{\partial w^+}{\partial \mathbf{n}} - w\frac{\partial u^+}{\partial \mathbf{n}})\, ds - \int_{\partial B}(u^+\frac{\partial w^+}{\partial \mathbf{n}} - w\frac{\partial u^+}{\partial \mathbf{n}})\, ds \;=\; 0 \qquad (2)$$

If we add equations (1) and (2) we find

$$\begin{aligned}\int_{\partial D}\left(u^+\frac{\partial w^+}{\partial \mathbf{n}} - w\frac{\partial u^+}{\partial \mathbf{n}}\right) ds \;+\; &\int_{\partial B}\left(u^-\frac{\partial w^-}{\partial \mathbf{n}} - w\frac{\partial u^-}{\partial \mathbf{n}}\right) ds \\ -\; &\int_{\partial B}\left(u^+\frac{\partial w^+}{\partial \mathbf{n}} - w\frac{\partial u^+}{\partial \mathbf{n}}\right) ds = 0\end{aligned}$$

We note that on ∂D, $\frac{\partial w^+}{\partial \mathbf{n}} = \frac{\partial w^-}{\partial \mathbf{n}}$, $\frac{\partial u^+}{\partial \mathbf{n}} = \frac{\partial u^-}{\partial \mathbf{n}}$, and $w^+ = w^-$. Thus we have from the above, after cancellations,

$$\int_{\partial D}\left(u\frac{\partial w}{\partial \mathbf{n}} - wg\right)\, ds = \int_{\partial B}[u]\frac{\partial w}{\partial \mathbf{n}}\, ds. \qquad (3)$$

We call the left side of equation (3) the Reciprocity Gap functional, and we write it as $RG(w)$. Note that $RG(w) = \int_{\partial D}(u\frac{\partial w}{\partial \mathbf{n}} - wg)\, ds$ is a computable expression given g and u on ∂D, and therefore $RG(w)$ allows us to use information about ∂D to derive information about ∂B.

4 Locating the Center of B

We are able to use the Reciprocity Gap function to locate the center of the inclusion. In what follows we will identify $\mathbb{R}^2$ with the complex plane, and write $C^* = x^* + iy^*$ when convenient. We will first consider the following Lemma.

Lemma 2 *Suppose B is an inclusion with center $C^* = x^* + iy^*$ and radius R. Let w be the harmonic function $w(x,y) = \frac{1}{\eta}e^{\eta(x+iy)}$ where $\eta \neq 0$ is any complex number. The Reciprocity Gap of $w(x,y)$ can be approximated as*

$$RG(w) \approx Re^{\eta C^*}\int_0^{2\pi} e^{i\theta}[u](\theta)\,d\theta + O(R^2)\int_0^{2\pi} e^{i\theta}[u](\theta)\,d\theta$$

where $[u](\theta) = u(x^* + R\cos(\theta), y^* + R\sin\theta)$ and $O(R^2)$ denotes a quantity bounded by AR^2 for some positive constant A (which does not depend on R).

Proof For $w(x,y) = \frac{1}{\eta}e^{\eta(x+iy)}$, we have $\nabla w = \eta w < 1, i >$. If we parameterize ∂B in polar coordinates, we find that

$$x = x^* + R\cos(\theta), \quad y = y^* + R\sin(\theta).$$

We then have

$$\begin{aligned} RG(w) &= \int_{\partial B} \nabla w \cdot \mathbf{n}[u](\theta)\,ds \\ &= \int_0^{2\pi} e^{\eta C^*}e^{R(\cos(\theta)+i\sin(\theta))} < 1, i > \cdot < \cos(\theta), \sin(\theta) > [u](\theta)\,ds \\ &= Re^{\eta C^*}\int_0^{2\pi} e^{R(\cos(\theta)+i\sin(\theta))}e^{i\theta}[u](\theta)\,d\theta \end{aligned} \tag{4}$$

where we've used $ds = r\,d\theta$. We can approximate

$$e^{R(\cos(\theta)+i\sin(\theta))} = 1 + O(R)$$

and so from equation (4) we have

$$RG(w) = Re^{\eta C^*}\int_0^{2\pi} e^{i\theta}[u](\theta)\,d\theta + O(R^2)\int_0^{2\pi} e^{i\theta}[u](\theta)\,d\theta.$$

which proves the Lemma. ∎

The Reciprocity Gap functional has been defined as a function of $w(x,y)$. Let us now consider $RG(w)$ with $w(x,y) = \frac{1}{\eta}e^{\eta(x+iy)}$ as a function of η, and define $\phi(\eta) = RG(w)$. We also drop the higher order error term involving $O(R^2)$ (in doing so we assume that R is small). We consider the following corollary.

Corollary 3 *If the $O(R^2)$ term in Lemma 2 is dropped then the center C^* of a circular inclusion (in the form of a complex number), can be found as*

$$C^* = \frac{\phi'(\eta)}{\phi(\eta)}.$$

Proof If we drop the $O(R^2)$ term in Lemma 2, we have $\phi(\eta) = RG(w) = Re^{\eta C^*}\int_0^{2\pi} e^{i\theta}[u](\theta)\,d\theta$. Let us differentiate $\phi(\eta)$ with respect to η to find $\phi'(\eta) = C^*\phi(\eta)$ which immediately yields the Corollary. ∎

It is important to note here that we can compute $\phi'(\eta)$ from the boundary data, as

$$\phi'(\eta) = RG\left(\frac{\partial w}{\partial \eta}\right)$$

where $\frac{\partial w}{\partial \eta}$ can be computed explicitly.

We now have a method for locating the center of an inclusion B given only the boundary conditions of ∂D.

5 Finding R and k

We will now find the radius R of B as well as the transmission constant k, given C^* and the boundary data of ∂D. In what follows let $u_0(x,y)$ denote the temperature in the region D with no inclusion B, with $\frac{\partial u_0}{\partial \mathbf{n}} = g$ on ∂D; note u_0 is harmonic in D, and uniquely determined up to an additive constant.

Lemma 4 *Let $w(x,y)$ be some harmonic function on all of D. The Reciprocity Gap Function can be represented in terms of the radius, the center,* *and the transmission constant (R, C^*, and k, respectively) as*

$$\begin{aligned} RG(w) &= \frac{2\pi R^2}{1+2kR}\nabla u_0(C^*)\cdot\nabla w(C^*) \\ &+ \frac{\pi R^4}{1+kR}\left(\frac{\partial^2 w}{\partial x^2}(C^*)\frac{\partial^2 u_0}{\partial x^2}(C^*) + \frac{\partial^2 w}{\partial x \partial y}(C^*)\frac{\partial^2 u_0}{\partial x \partial y}(C^*)\right) + O(R^6). \end{aligned}$$

Proof Let $v = u - u_0$. That is, v is a small correction in the "no-inclusion" temperature when there is an inclusion in the region D. The function v satisfies

$$\begin{aligned} \triangle v &= 0 \text{ in } B,\ D\backslash B & (5)\\ \frac{\partial v}{\partial \mathbf{n}} &= 0 \text{ on } \partial D & (6)\\ \frac{\partial v}{\partial \mathbf{n}} &= k[u] - \frac{\partial u_0}{\partial \mathbf{n}} \text{ on } \partial B & (7)\\ & & (8) \end{aligned}$$

Because $[u_0] = 0$ we see that $[u] = [u_0] + [v] = [v]$ so that equation (7) can be written as

$$\frac{\partial v}{\partial \mathbf{n}} = k[v] - \frac{\partial u_0}{\partial \mathbf{n}}. \tag{9}$$

Note that the above conditions determine v up to an arbitrary additive constant.

We use (r, θ) as polar coordinates about the center of the inclusion C^*. We will explicitly write out v in terms of u_0 to good approximation.

Let $v_{D\backslash B}$ and v_B denote the restriction of v to $D\backslash B$ and B, respectively. We attempt an expansion of each as follows:

$$v_B(r,\theta) = \sum_{m=0}^{\infty}(c_m\cos(m\theta) + d_m\sin(m\theta))r^m \tag{10}$$

$$v_{D\backslash B}(r,\theta) = \sum_{m=0}^{\infty}(a_m\cos(m\theta) + b_m\sin(m\theta))r^{-m} \tag{11}$$

for constants a_m, b_m, c_m, d_m. Note that the expansion of (10) is definitely possible for suitable constants c_m, d_m. However, the expansion of equation

(11) ignores terms of the form $r^m \cos(m\theta)$ and $r^m \sin(m\theta)$. If R is sufficiently small, however, these terms will be negligible.

Our goal is to work out the coefficients a_m, b_m, c_m, d_m as explicitly as possible in terms of u_0, then related this to $RG(w)$.

It is required that $\frac{\partial v_{D\setminus B}}{\partial r} = \frac{\partial v_B}{\partial r}$ on $r = R$. Note that $\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial r}$ because the outward normal vector for the circle is in the radial direction. Thus we need

$$\sum_{m=1}^{\infty} m(c_m \cos(m\theta) \quad + \quad d_m \sin(m\theta))R^{m-1}$$
$$= \sum_{m=1}^{\infty} -m(a_m \cos(m\theta) + b_m \sin(m\theta))R^{-m-1}.$$

By matching terms we find that

$$a_m = -R^{2m} c_m \tag{12}$$

and

$$b_m = -R^{2m} d_m. \tag{13}$$

Let u_0 have Fourier expansion

$$u_0 = \sum_{m=0}^{\infty} (e_m \cos(m\theta) + f_m \sin(m\theta))r^m \tag{14}$$

(note that e_0 can be chosen arbitrarily, so we'll take $e_0 = 0$). From equation (9) we have for $r = R$ that

$$\sum_{m=1}^{\infty} m(c_m \cos(m\theta) + d_m \sin(m\theta))R^{m-1} - k[\cos(m\theta)(a_m R^{-m} - c_m R^m)$$
$$+ \sin(m\theta)(b_m R^{-m} - d_m R^m)] = -\sum_{m=1}^{\infty} m(e_m \cos(m\theta) + f_m \sin(m\theta))R^{m-1}.$$

Matching the cosine terms above yields

$$m c_m R^{m-1} - k(a_m R^{-m} - c_m R^m) \quad = \quad -m e_m R^{m-1}$$

so that with equation (12) we have

$$c_m = \frac{-me_m}{m+2kR}. \tag{15}$$

The same reasoning with equation (13) shows that $d_m = \frac{-mf_m}{m+2kR}$. Thus we have

$$\begin{aligned}
[u] &= [v] \\
&= \sum_{m=1}^{\infty} \cos(m\theta)(a_m R^{-m} - c_m R^m) + \sin(m\theta)(b_m R^{-m} - d_m R^m) \\
&= 2\sum_{m=1}^{\infty} \left(\frac{e_m \cos(m\theta) + f_m \sin(m\theta)}{m+2kR} \right) mR^m. \qquad (16)
\end{aligned}$$

This gives us $[u]$ in terms of the Fourier coefficients of u_0.

Let us perform a similar computation for any harmonic test function w. We expand

$$w = \sum_{m=0}^{\infty} (g_m \cos(m\theta) + h_m \sin(m\theta)) r^m \tag{17}$$

for coefficients g_m, h_m, so that when we evaluate the derivative $\frac{\partial w}{\partial \mathbf{n}}$ for $r = R$ we find

$$\frac{\partial w}{\partial r} = \sum_{m=1}^{\infty} m(g_m \cos(m\theta) + h_m \sin(m\theta)) R^{m-1}. \tag{18}$$

Now recall the Reciprocity gap functional $RG(w) = \int_{\partial B} [u] \frac{\partial w}{\partial \mathbf{n}}\, ds$. From the expansions (16) and (18) we obtain (after integrating term by term and using orthogonality)

$$RG(w) = \frac{2\pi R^2}{1+2kR}(e_1 g_1 + f_1 h_1) + \frac{4\pi R^4}{1+kR}(e_2 g_2 + f_2 h_2) + O(R^6). \tag{19}$$

We now expand our harmonic function given in equation 17.

$$w = g_0 + g_1 r\cos(\theta) + h_1 r \sin(\theta) + g_2 r^2 \cos(2\theta) + h_2 r^2 \sin(2\theta) + \dots \tag{20}$$

By using $\frac{\partial}{\partial r} = \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \theta} = -\frac{\sin(\theta)}{r}\frac{\partial}{\partial x} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial y}$ it's not hard to see that

$$\begin{aligned}
\nabla w(C^*) =&< g_1, h_1 > \qquad & \nabla u_0(C^*) =&< e_1, f_1 > \\
e_2 =& \frac{1}{2}\frac{\partial^2 u_0}{\partial x^2}(C^*) & f_2 =& \frac{1}{2}\frac{\partial^2 u_0}{\partial x \partial y}(C^*) \\
g_2 =& \frac{1}{2}\frac{\partial^2 w}{\partial x^2}(C^*) & h_2 =& \frac{1}{2}\frac{\partial^2 w}{\partial x \partial y}(C^*).
\end{aligned}$$

When we substitute the previous results into equation 19 we find the Reciprocity Gap function to be

$$\begin{aligned}
RG(w) &= \frac{2\pi R^2}{1+2kR}\nabla u_0(C^*) \cdot \nabla w(C^*) \\
&+ \frac{\pi R^4}{1+kR}\left(\frac{\partial^2 w}{\partial x^2}(C^*)\frac{\partial^2 u_0}{\partial x^2}(C^*) + \frac{\partial^2 w}{\partial x \partial y}(C^*)\frac{\partial^2 u_0}{\partial x \partial y}(C^*)\right) + O(R^6).
\end{aligned}$$

This proves the Lemma. ■

If we choose the harmonic function $w(x, y) = \frac{1}{\eta}e^{\eta(x+iy)}$ and choose two distinct values for η, we then have a system of two equations, $RG(w(\eta_1))$ and $RG(w(\eta_2))$, with two unknowns, R and k. Therefore we can solve for R and k. We have the found the center, radius, and transmission constant of the inclusion B.

6 A Numerical Example

Using several computer programs, we will now give a numerical example of our algorithm. We assume that D is the unit disk in two dimensions. Using a C program developed by Dr. Kurt Bryan, we are able to input the center, radius, and transmission constant of an inclusion, as well as an input heat flux, and the program outputs the values of $u(x, y)$ at n points along ∂D. For the following example we use an inclusion centered at $(0.3, 0.4)$ with a radius of 0.15 and a transmission constant of 0.9. We will set the heat flux to be $g = \sin(2\theta)$ and use the program to compute 100 values of $u(x, y)$ evenly spaced around ∂D.

This data is loaded into a Maple notebook. We can then choose a harmonic function $w(x, y)$ and numerically calculate $RG(w)$. In this example we use $w(x, y) = \frac{1}{\eta}e^{\eta(x+iy)}$ for various choices of η. As mentioned above, we define $\phi(\eta) = RG(w)$, and we can calculate $\phi'(\eta)$. In this example with $\eta = 1$ we find the center at $(0.315, 0.420)$. The choice of η has little effect on the estimate of the center.

Once we've located the center we can estimate the radius and transmission coefficient. In what follows we will take the center to be $(0.3, 0.4)$ (rather than the slightly erroneous estimate above) and calculate the radius and transmission constant of the inclusion. We choose two values of η and calculate $\phi(\eta)$ for each η. For our example, we choose $\eta_1 = 1 + i$ and $\eta_2 = 1 - i$. We obtain $\phi(\eta_1) = 0.009 + 0.050i$ and $\phi(\eta_2) = 0.086 + 0.080i$. We thus obtain the following system of two equations with two unknowns, R and k:

$$\begin{aligned}
0.009 + 0.050i &= \frac{2\pi R^2}{1+2kR}(0.4 + 0.3i)\ e^{(1+i)(0.3+0.4i)} \\
&+ \frac{\pi R^4}{1+kR}(i(1+i)e^{(1+i)(0.3+0.4i)}) \\
0.086 + 0.080i &= \frac{2\pi R^2}{1+2kR}(0.4 + 0.3i) \cdot e^{(1-i)(0.3+0.4i)} \\
&+ \frac{\pi R^4}{1+kR}(i(1-i)e^{(1-i)(0.3+0.4i)}).
\end{aligned}$$

We then solve with Newton's method to find $R = 0.1505$ and $k = 0.8527$.

However, if we use the slightly inaccurate value for C^* we obtain very poor results—large negative values for k, highly erroneous values for R. The computation is quite unstable with respect to the simultaneous estimates of k and R. If, however, we regard k as known (even approximately) we can recover R with good stability. In the present case using $k = 1$ in the equation $\phi(1 + i) = c$ (with $C^* = 0.315 + 0.42i$ as recovered, and where c is computed from the boundary data) yields $R \approx 0.1504$, while $k = 0.1$ yields $R \approx 0.133$. Alternatively, we can recover k stably if R is considered known. The precise stability of the problem of recovering both k and R simultaneously is a topic for further study.

7 Multiple Inclusions

We will now consider the case of multiple circular inclusions within a two-dimensional region. As with the one inclusion case, we will consider a two-dimensional region as D with boundary ∂D. Within D we assume that there exist N circular inclusions which we will denote by B_n; here N may be considered unknown. Each B_n has a radius of R_n and a transmission constant of k_n. We will use a slightly different approach to locating the inclusions within D. Most importantly (and unfortunately), we do not have a method for finding R_n and k_n simultaneously; we need R_n in order to find k_n, or we need k_n in order to find R_n.

We first recall our Reciprocity Gap test function $w(x, y) = \frac{1}{\eta}e^{\eta(x+iy)}$, and set $\phi(\eta) = RG(w)$. Also recall that in the case of a single inclusion B, from Lemma 1 we can write, to good approximation,

$$\phi(\eta) = Je^{\eta C^*}$$

where

$$J = \int_{\partial B} e^{i\theta}[u](\theta)\, ds \tag{21}$$

where $ds = R\, d\theta$. For multiple inclusions a similar argument shows that

$$\phi(\eta) = \sum_{j=1}^{N} J_j e^{\eta C_j^*}. \tag{22}$$

where the subscript "j" indexes the individual inclusions and

$$J_j = \int_{\partial B_j} e^{i\theta}[u](\theta)\, ds. \tag{23}$$

We may not know the exact number N of inclusions in our region since the interior cannot be accessed, but below we outline a procedure for finding N.

7.1 Locating N Centers

Because $\phi(\eta)$ is of the form (22), it must satisfy a constant-coefficient linear ODE of the form

$$c_M\phi^{(M)}(\eta) + c_{M-1}\phi^{(M-1)}(\eta) + \ldots + c_1\phi'(\eta) + c_0\phi(\eta) = 0 \tag{24}$$

for certain constants c_j any $M \geq N$. We can use our boundary data to compute $\phi(\eta)$ and its derivatives (recall we compute $\phi^{(k)}$ by computing $RG(\partial^k w/\partial\eta^k)$). If we choose $M \geq N$ distinct values of η, we obtain M equations with M unknowns. We can solve this system of equations for the coefficients c_j. As shown in [3] the rank of the resulting matrix gives the number of exponential terms in ϕ (which is the number of inclusions), provided we use a value of M which exceeds the true number of inclusions. We can then solve for the c_j. Given the coefficients c_j, we can solve for the roots of $p(x)$ (the characteristic equation for the ODE) where

$$p(x) = x^N + \sum_{j=1}^{N} c_j x^j.$$

The roots of $p(x)$ are the centers of the inclusions.

Note that once we have recovered the centers C_j^*, we can use equation (22) to evaluate $\phi(\eta_k)$ for various values of η_k and thereby recover the J_j by solving a system of linear equations.

7.2 The Jump Integral

As we stated earlier, we will present a method for finding either R_n or k_n, given that the other is known. The central result is the following lemma.

Lemma 5 *Let J be defined by equation (21), for the case in which D contains a single inclusion. Then J can be rewritten in terms of the center, radius, and transmission constant of the inclusion as*

$$|J| = \frac{2\pi R^2 \, |\nabla u_0(C^*)|}{1+2kR} + O(R^4).$$

Proof The proof of Lemma 5 closely mimics the proof of Lemma 4. We will define identical conditions for the functions $u(x,y)$, $u_0(x,y)$, and $v(x,y)$. We are given the same boundary data as well. Using $w = \frac{1}{\eta}e^{\eta(x+iy)}$, we get $\nabla w = < 1, i > e^{\eta(x+iy)}$. Lemma 4 states that

$$RG(w) = \frac{2\pi R^2}{1+2kR}\nabla u_0(C^*) \cdot \nabla w(C^*) + O(R^4)$$

Taking the magnitude of the Reciprocity Gap functional and using the above function $w(x, y)$ yields

$$\begin{aligned} |RG(w)| &= \frac{2\pi R^2}{1+2kR} |\nabla u_0(C^*)| \, e^{\eta C^*} \\ &= J e^{\eta C^*} \end{aligned}$$

We then find that

$$|J| = \frac{2\pi R^2}{1+2kR} |\nabla u_0(C^*)| + O(R^4)$$

■

For multiple inclusions, we expect the integral J_j as defined by equation (23) to be (to leading order) of the form

$$|J_j| = \frac{2\pi R_j^2 \left|\nabla u_0(C_j^*)\right|}{1+2k_j R_j}. \tag{25}$$

We now have the integrals J_j in terms of C^*, R_j, and k_j. We recall our multiple inclusion Reciprocity Gap function, equation (22), and consider it as a function of η. We can calculate $\phi(\eta)$ and we now have C_j^*. Therefore if we choose N distinct values of η, we have N equations of N unknowns (those unknowns being J_j). Thus we can solve for each value of J_j.

Once we have the J_j and C_j^* we can use equation (25) to solve for R_j if we are given k_j or vice-versa.

8 A Numerical Example of Multiple Inclusions

We will now present a numerical example that illustrates how to locate N inclusions within a region. To solve the forward problem we use a similar program to the one we used to demonstrate our single inclusion case. We will give the program the center, the radius, and transmission constant for each inclusion B_j within D, where D is again the unit disk. For our example we will use two inclusions within D. Our first inclusion B_1 will be centered at $(0.4, 0.6)$ with a radius of $R_1 = 0.15$ and a transmission constant of $k_1 = 0.9$. Our second inclusion B_2 will be centered at $(-0.3, 0.7)$ with a radius of $R_2 = 0.1$ and a transmission constant of $k_2 = 0.75$. We again use input flux $g(\theta) = \sin(2\theta)$ (so the harmonic function with the boundary data is $u_0(x, y) = xy$).

Let us first consider the case in which we know the actual number of inclusions. We use a C program to find the value of $u(x, y)$ at a given number of points (50 to 100) along ∂D. We feed these values into a Matlab notebook, we are able to calculate $\phi(\eta)$ for any given η as well as the derivatives of $\phi(\eta)$. We will choose two values for η, $\eta_1 = 1$ and $\eta_2 = -1$, and have Matlab calculate the first two derivatives of $\phi(\eta_1)$ and $\phi(\eta_2)$. We use these values to solve the following system of equations, where without loss of generality we let $c = 1$:

$$a\phi''(\eta_1) + b\phi'(\eta_1) + c\phi(\eta_1) = 0$$
$$a\phi''(\eta_2) + b\phi'(\eta_2) + c\phi(\eta_2) = 0$$

Once we have values for a and b we can solve for the roots of the quadratic equation

$$am^2 + bm + c = 0. \tag{26}$$

In the present case we find $a = -0.5596 + 0.1048i, b = -0.1087 - 1.3224i$ (recall $c = 1$). The roots of the characteristic equation (26) are $-0.3027 + 0.7072i$ and $0.4114 + 0.6152i$, quite close to the correct center values.

Once we have located the centers of the inclusions, we are easily able to calculate J_1 and J_2 by evaluating $\phi(\eta)$ for two values of η. In this case we take $\eta = 1$ and $\eta = -1$ and find $J_1 = 0.0706+0.0484i$, $J_2 = 0.0401-0.0167i$. From this we can calculate R or k, if we are given the other. Let us assume that we know $k_1 = 0.9$ and $k_2 = 0.75$. We calculate $R_1 = 0.1533$ and $R_2 = 0.1018$. Figure 1 below shows the accuracy of the reconstructions. Alternatively, we can consider R_1 and R_2 known and so estimate that $k_1 = 0.74$ and $k_2 = 0.565$, not quite as accurate as the radius estimation.

If we use equation (24) with a guess of $M = 5$ (and five distinct values for η; we use the fifth roots of unity) we find that the resulting linear system for the c_j has rank two, with two non-zero singular values 0.4665 and 0.0663; the remaining values are less than 10^{-5}. This indicates that only two inclusions are present, and we can proceed as above.

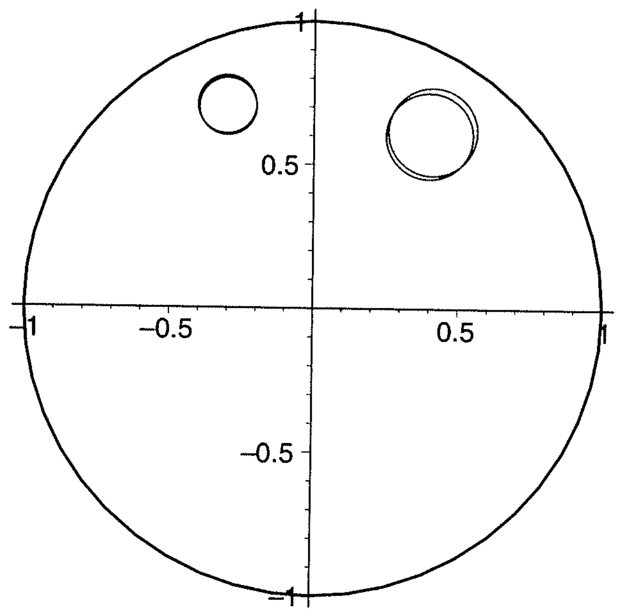

Figure 1: Reconstruction of two inclusions.

9 Conclusion and Future Work

We used the reciprocity gap approach with carefully chosen test functions to reduce the problem of identifying the centers of one or more inclusions to that of identifying the multipliers in a function ϕ which is a sum of exponentials. This is easily done by noting that such functions satisfy very simple ODE's. The radii of the inclusions are obtained by deriving a relation between the "jump" coefficients J_n which appear in ϕ and the radii. This relation involves the transmission coefficient for each inclusions, which we must consider known. However, in the case of a single inclusion a more careful analysis shows that we can recover both the radius and transmission coefficient. The k value was found to be more sensitive to noise than was R. Future work should be done to stabilize k.

It would be desirable to find a way to get both R and k simultaneously for the multiple inclusion case since there are more real world applications for that scenario. We would also like to find an algorithm for single and multiple inclusions in $\mathbb{R}^3$. Another case to consider would be a case where the material on the outside of the inclusion boundary is different from the material on the inside of the inclusion boundary. This would mean that $\frac{\partial u^+}{\partial r} = \alpha \frac{\partial u^-}{\partial r}$, or that the material on the outside of the inclusion has a different thermal conductivity than the material on the inside of the inclusion.

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