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Abstract

Let S be a hyperbolic surface tiled by kaleidoscopic triangles. Let R_e denote the set of fixed points by the reflection in an edge, e, of a triangle. We say that R_e is *separating* if $S - R_e$ has two components. Once we have a tiling, we can define a group of orientation preserving transformations, G. We develop a method for determining when a reflection is separating using the group algebra of G. Using this method we give necessary and sufficient conditions for a mirror to be separating when G is abelian. We also conjecture, that when G is simple there are no separating mirrors.

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1 Introduction

M. C. Escher and many other artists have used the mathematics of tilings to produce beautiful artwork. We have all seen the beautiful patterns obtained by tiling the hyperbolic plane by repeated reflections in the sides of a "kaleidoscopic" polygon. Though there are similar tilings of the sphere and Euclidean plane the positive and flat curvature of those surfaces do not provide as interesting pictures as the negative curvature of the hyperbolic plane. The following picture is one of the many examples of a tiling being turned into artwork.

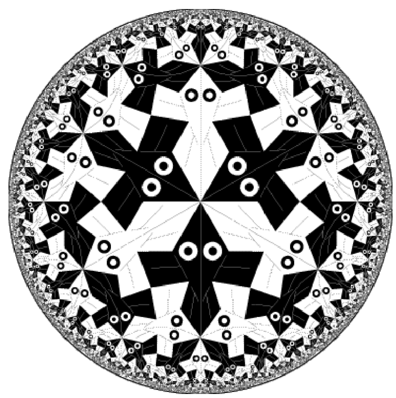

Figure 1: A picture by M.C. Escher [**E**]

This paper deals with tilings of hyperbolic surfaces rather than the hyperbolic plane. One way to construct a tiling of a plane is to begin with a triangle that has angle measures π/k, π/l and π/m. This triangle is called a *kaleidoscopic* triangle. Place one triangle on the plane. Next create a layer of triangles surrounding the first triangle by hyperbolically reflecting the triangle in its edges. Then create a third layer by reflecting in the edges of the second layer. Continuing in this fashion we obtain a tiling of the plane. A similar construction can be used to tile hyperbolic surfaces, but as you can imagine there are more constraints on the triangles and the surface for this process to successfully tile the surface.

Another reason that tilings are studied is that they offer a very rich blend of geometry and group theory. Once we have a kaleidoscopic tiling of a surface, we can introduce a group of elements that act on the tiles. Begin with a single tile and then reflect it over each of its three edges. Each reflection is a group element and we consider the group generated by these three reflections. We call this group of transformations the full tiling group. Just looking at those transformations that preserve the orientation of the original tile, we can form a subgroup of index 2 in the full tiling group. This subgroup is known as the orientation-preserving subgroup.

We are interested in a specific property of these tilings, called separability.

Let S be a surface and Θ a tiling of the surface by kaleidoscopic triangles. Define the set of fixed points by a reflection R in an edge e, as the mirror of R or F_R. That is $F_R = \{x \in S : Rx = x\}$. Then we say that a mirror, or a reflection, is separating if $S - F_R$ has more than one component. The easiest example of a separating mirror can be seen on the surface of the sphere. Place a kaleidoscopic triangle on the surface of the sphere, then the set of fixed points by the reflection in one of its edges is a great circle. The mirror is separating since the sphere minus the great circle has two components. This example shows us that every reflection of every tiling of the sphere by kaleidoscopic triangles is separating. This might lead one to believe that most tilings of surfaces have lots of separating reflections. Surprisingly, this is far from being correct!

So when is a reflection separating? We completely answer this case for an infinite family of tilings. We answer it when the orientation preserving subgroup of the tiling group is Abelian. We also give a conjecture and provide some evidence for a second infinite class of tilings, when the orientation preserving subgroup is simple.

2 Tilings

A tiling, Θ, of a surface, S, is a set of closed polygons such that interiors of the elements of Θ are pairwise disjoint, and S is the closure of union of all tiles in the tiling. In other words, a tiling is a collection of polygons that completely cover the surface without overlap. We consider tilings of hyperbolic surfaces by triangles. Additionally, we require that the tilings meet two conditions: the kaleidoscopic and geodesic conditions.

The kaleidoscopic condition requires that each triangle edge, e, be part of a closed curve, called a *geodesic*, on the surface such that there is a reflection in e, R_e, that fixes e and maps tiles to tiles.

The geodesic condition requires that for each edge, the set of fixed points of R_e is the union of edges in the tiling. In other words, each triangle edge is part of a long straight curve made up of edges.

The icosahedral tiling of the sphere (seen in Figure 2) meets both conditions. On the sphere a geodesic is a great circle. Here each edge of a triangle can be extended to a great circle and each of these great circles is a union of edges.

The tiling of the plane by hexagons meets the kaleidoscopic condition, but not the geodesic condition, as seen in Figure 3. Each edge can be extended to a geodesic (such as the dotted line in the figure), which is a reflection on the tiling that maps tiles to tiles. However, the geodesic is not a union of edges on the tiling.

These conditions force some very specific restrictions on the angle measures of the triangles that tile S. Let x and y be two edges of a tile meeting at the vertex V. The reflection in the edge y produces an adjacent triangle with edges $y' = y$ and x' meeting at V. Reflecting in x' produces a third triangle at vertex V. Since the triangles tile the surface we can repeat this process and obtain an integer number of triangles that completely surround the vertex. From the

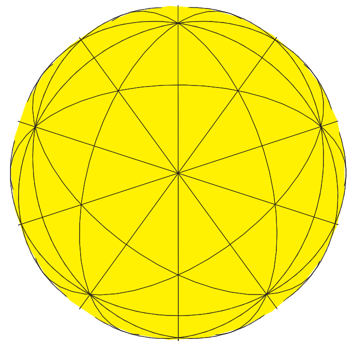

Figure 2: Icosahedral tiling of the sphere.

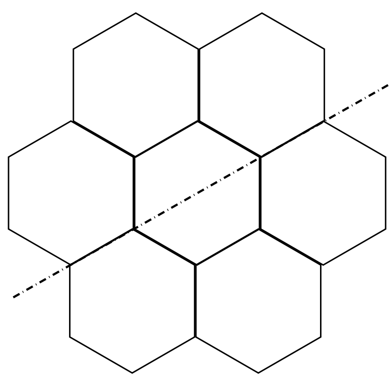

Figure 3: Hexagonal tiling of the plane.

geodesic condition there must be an even number of triangles surrounding V. So $2k$, $2l$, or $2m$ triangles meet at each vertex, for some integers k, l, and m. These angles are $\frac{\pi}{k}$, $\frac{\pi}{l}$, and $\frac{\pi}{m}$. As short hand, we refer to the angle measures of a triangle by the triple (k, l, m).

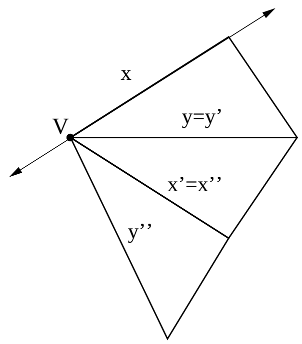

Figure 4: Triangles around a vertex

2.1 Tiling Groups

So far we have focused on the geometric aspect of tilings; however, tilings can also be described using group theory. Each edge of the tiling determines a reflection that maps S onto itself. We use the reflections in the edges of a tile to construct a group of symmetries G^*. Select a tile Δ_0, as shown in Figure 5, and call this tile the *master tile*. Let p, q, and r denote the three sides of the triangle and also the corresponding reflections in the sides of Δ_0. The images of Δ_0 in the sides p, q and r, $p\Delta_0, q\Delta_0$, and $r\Delta_0$, have been drawn in the figure. The product $a = pq$ is a counter clockwise rotation through $\frac{2\pi}{k}$ radians. To obtain $a\Delta_0$, the master tile is first reflected over edge q or e_q. The resulting tile, $q\Delta_0$ is then reflected over the geodesic containing e_p of the master tile. This gives us $a\Delta_0$. Similarly $b = qr$ and $c = rp$ are counterclockwise rotations through $\frac{2\pi}{l}$ radians and $\frac{2\pi}{m}$ radians respectively.

From these geometric observations and the fact that reflections have order 2, we have:

$$a^k = b^l = c^m = 1$$

and

$$abc = pqqrrp = 1.$$

The group $G^* = \langle p, q, r \rangle$ is called the *tiling group* of S.

The elements in G^* are transformations which map the master tile to other tiles. The kaleidoscopic and geodesic conditions ensure that G^* acts simply transitively on the tiling Θ [Br1]. That is, for any triangle in the tiling, there is exactly one transformation in G^* which will map the master tile onto it and

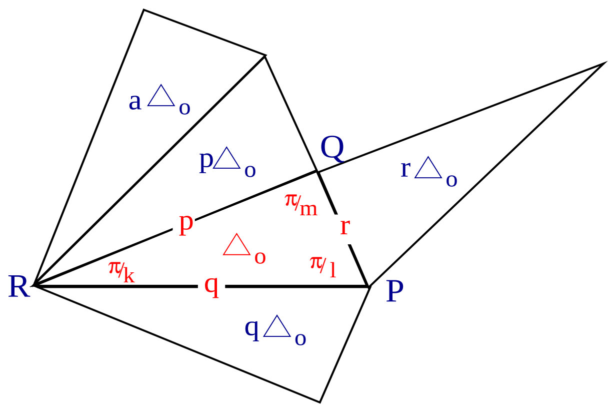

Figure 5: Triangles produced by group elements

every element in G^* takes the master tile to exactly one tile in Θ. This one-to-one correspondence allows us to study the tiling by studying the tiling group!

The orientation of a triangle can be determined by ordering the edges of the triangle p, q, and then r. Following the edges around in this order will result in either a clockwise or a counterclockwise movement.

When a triangle Δ is acted upon by a single reflection g, the orientation of the resulting triangle $g\Delta$ is reversed. For instance, if Δ is oriented counterclockwise, then $g\Delta$ is oriented clockwise. When $g\Delta$ is reflected once again (either by g or by a different reflection), the resulting triangle has the same orientation as the original. Thus any transformation which is a sequence of an even number of reflections preserves the orientation of the master tile.

We can define a subgroup G of index 2 in G^* which consists of all the orientation-preserving transformations. This group $G = \langle a, b, c \rangle = \langle a, b \rangle$ is referred to as the *conformal tiling group* or the *orientation preserving tiling group*.

Let's define an automorphism $\theta : G \to G$ such that $\theta(g) = qgq^{-1} = qgq$. Since q has order 2, $q = q^{-1}$. In other words, θ is conjugation by the reflection in q. Note that

$$\theta(a) = qaq = qpqq = qp = a^{-1}$$

and similarly,

$$\theta(b) = qbq = qqrq = rq = b^{-1}.$$

In fact, a surface S has a tiling represented by an orientation-preserving group G if and only if there exists an automorphism θ of order 2 such that $\theta(a) = a^{-1}$ and $\theta(b) = b^{-1}$.

2.2 Separability

We are interested in a particularly fascinating property of tilings called *separability* . Define the set of points fixed by a reflection R, $F_R = \{x \in S|\, Rx = x\}$, to be the *mirror* of R. A reflection is said to be *separating* or *splitting* if $S \backslash F_R$ has two components. For example the icosahedral tiling of the sphere in Figure 1 is separating along any mirror. In fact, every kaleidoscopic tiling of triangles on the sphere separates at every mirror since no matter the tiling, every mirror is a great circle on the sphere. This leads one to believe that it is common for a mirror to be separating. However, on higher genus surfaces, separating reflections become much harder to find.

The question we are interested in answering is: Given a tiling Θ of a surface S, when does the tiling have a separating reflection? We completely answer this question when the conformal tiling group is Abelian. Additionally, we give a conjecture in the case when G is simple.

To answer this question we will need to understand what the mirrors of a tiling look like.

2.3 Ovals

For spherical surfaces a mirror is always a great circle but on higher genus surfaces, things are more complicated. Many mirrors have multiple components. In all cases, a mirror is a disjoint set of one or more closed curves. We call these curves *ovals*. An oval is sequence of edges which meet at angles of π radians to form a circle embedded in the surface.

To determine whether a tiling separates, it is very useful to know exactly what the ovals of a mirror look like. An oval can be described by its *pattern*. The pattern is a sequence of edge-types which repeats itself. Possible edge types are p^+, q^+, r^+, p^-, q^-, and r^-, where $^+$ and $^-$ refer to the orientation of the edges. The pattern of an oval may depend on which direction we traverse the oval.

Two ovals are equivalent if they have the same pattern. Every oval is equivalent to one of $\mathcal{O}_p$, $\mathcal{O}_q$, and $\mathcal{O}_r$: the ovals containing e_p, e_q, and e_r of the master tile. Since all tiles are equivalent, the same set of ovals will be obtained no matter which tile is chosen as the master tile. Thus, all ovals containing an edge of type p are equivalent to $\mathcal{O}_p$. For two ovals $\mathcal{O}_p$ and $\mathcal{O}_q$ to be equivalent means that a group element g can be found such that $\mathcal{O}_p = g\mathcal{O}_q$.

For a given tiling, there are at most three different types of ovals. The three types of ovals are not necessarily distinct, however. Two ovals $\mathcal{O}_p$ and $\mathcal{O}_q$ are considered equivalent if there exists a group element g such that $\mathcal{O}_p = g\mathcal{O}_q$. If $\mathcal{O}_p$ contains $q-$type edges, then $\mathcal{O}_p$ is equivalent to $\mathcal{O}_q$. This is true since any $q-$type edge is contained in an oval equivalent to the oval containing e_q of the master tile. For example, it is possible that there will be only one oval type for a given tiling. This occurs when the oval containing e_p also contains q- and r-type edges so $\mathcal{O}_p$ is equivalent to $\mathcal{O}_q$ and $\mathcal{O}_r$.

The pattern of the edges in the oval can be determined by looking at the *parity* of the vertices. *Parity* refers to whether k, l, and m are even or odd. Consider a vertex on a geodesic. When an even number of triangles meet at that vertex on one side of the geodesic, the two edges that lie along the geodesic will be of the same type, only with opposite orientation. When an odd number of triangles meet at a vertex on one side of the geodesic, the two edges that lie along the geodesic will be of different types.

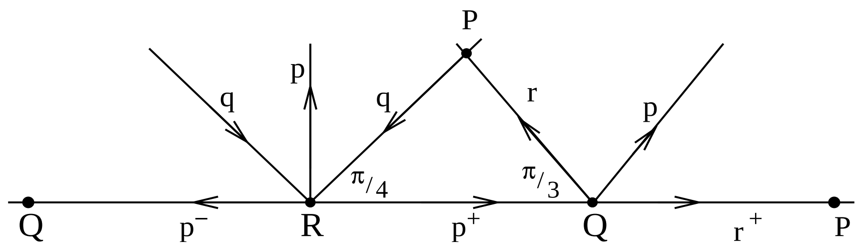

Figure 6: Parity and the oval pattern

Suppose we are trying to determine what $\mathcal{O}_p$ looks like. Obviously p^+ (p^-) is in the pattern. When the vertex R is even (for instance $k = 4$), the next edge in the pattern is p^- (p^+): the p-edge with the opposite orientation. When the vertex Q is odd (for instance $m = 3$), the previous edge in the pattern is r^+ (r^-).

The oval pattern can be determined by tracing along the edges of the master tile. The pattern "bounces" off the vertex P when k is even and "passes through" P when k is odd. The same holds true for Q and R.

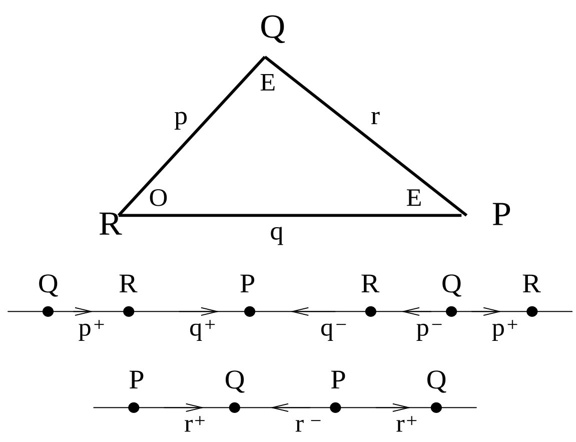

Figure 7: Oval patterns for an OEE triangle.

We consider the example when the parity is OEE. To determine the oval patterns, lets begins at vertex Q and begin travelling counterclockwise along edge p towards vertex R. The first edge in the pattern of $\mathcal{O}_p$ is p^+. Since vertex R is odd, we "pass through it" and begin heading along edge q. We are still moving counterclockwise, so q^+ is the next edge in the pattern. When we get to the even vertex P, we "bounce", and begin moving back along edge q in the opposite direction so we can add q^- to the pattern. We pass through vertex R again and continue moving clockwise, adding p^- to the pattern. At vertex Q, we bounce, and the pattern begins to repeat itself. Thus, we have determined that $\mathcal{O}_p$ is equivalent to $\mathcal{O}_q$.

Let's now look at $\mathcal{O}_r$. Beginning at vertex P, we move counterclockwise along edge r towards vertex Q. Since Q is even, we bounce and begin moving back along edge r in the opposite direction. At vertex P, we again bounce and the pattern begins to repeat itself.

The following table shows the edge patterns of the ovals for all possible parities.

Table 1. Edge patterns of ovals

parity	$\mathcal{O}_p$	$\mathcal{O}_q$	$\mathcal{O}_r$
OOO	$p^+q^+r^+, r^-q^-p^-$	$p^+q^+r^+, r^-q^-p^-$	$p^+q^+r^+, r^-q^-p^-$
EOO	$q^+r^+p^+p^-r^-q^-$	$q^+r^+p^+p^-r^-q^-$	$q^+r^+p^+p^-r^-q^-$
OEO	$r^+p^+q^+q^-p^-r^-$	$r^+p^+q^+q^-p^-r^-$	$r^+p^+q^+q^-p^-r^-$
OOE	$p^+q^+r^+r^-q^-p^-$	$p^+q^+r^+r^-q^-p^-$	$p^+q^+r^+r^-q^-p^-$
OEE	$p^+q^+q^-p^-$	$p^+q^+q^-p^-$	r^+r^-
EOE	p^+p^-	$q^+r^+r^-q^-$	$q^+r^+r^-q^-$
EEO	$r^+p^+p^-r^-$	q^+q^-	$r^+p^+p^-r^-$
EEE	p^+p^-	q^+q^-	r^+r^-

3 Criterion for a Separating Mirror

Mirrors are key in determining whether or not a reflection is separating. If the mirror is a *boundary*, then the reflection is separating. A *boundary* is a union of edges that encloses a region of the surface. In a separating reflection, the mirror is actually a boundary for two disjoint regions, where each is exactly half the surface and is a reflection of the other. Conversely, if a reflection is separating, then the mirror is a boundary. The example of the icosahedral tiling of the sphere illustrates is idea very nicely.

We understand what mirrors and boundaries look like geometrically. In order to take full advantage of the power of the above theorem, however, we need to translate this understanding into group theory.

In this section we develop a method that allows us to first represent regions, mirrors, and boundaries of surfaces in terms of the group G. We will then use this machinery in the following plan of attack:

1. Determine what the general form of a boundary is.
2. Determine what a particular mirror looks like.
3. Check whether the mirror fits the form of a boundary.

3.1 Group Algebra

The tool that allows us to represent parts of the surface and their boundaries in terms of the group, is the group algebra of G over R. Let $\mathcal{A} = \mathcal{R}[G]$, the $\mathcal{R}$-group algebra of G. $\mathcal{R}$ is an arbitrary ring, and G is the orientation-preserving tiling group. Recall that a group algebra is both a ring and a vector space, and that the elements of G serve as the basis for the vector space. Thus, the elements of $\mathcal{A}$ are linear combinations of orientation-preserving transformations. This allows us to define the following $\mathcal{A}$-modules:

$$
\begin{aligned}
C_2 =& \mathcal{A}\Delta_0 \oplus \mathcal{A}q\Delta_0 \\
C_1 =& \mathcal{A}e_p \oplus \mathcal{A}e_q \oplus \mathcal{A}e_r
\end{aligned}
$$

Recall that a G-module or V is an $\mathcal{R}$-module for which left multiplication by the group on V has been defined so that the module become an $\mathcal{A}$-module: $\sum_{g\in G} a_g gv = \sum_{g\in G} a_g(gv)$

The module C_2 consists of all linear combinations of oriented tiles. Any tile on the surface can be represented by an element of G acting on either the master tile or on a reflection of the master tile. Since we are developing the method to check the $q-$ type mirror, we will use the reflection over q as the most intuitive choice. However, it is important to note that the reflection may be arbitrarily chosen, and p or r would work just as well.

Any tile with counterclockwise orientation can be represented by $g\Delta_0$ for some $g \in G$, and any tile with clockwise orientation can be represented by $gq\Delta_0$. By taking a linear combination of tiles, it is possible to represent any surface or region of a surface as $\zeta\Delta_0 + \eta q\Delta_0 \in C_2$, where $\zeta, \eta \in \mathcal{A}$.

The module C_1 consists of all linear combinations of oriented edges. Any edge on the surface can be represented by an element of G acting on an edge of the master tile.

Now that we can represent any region of a surface or sequence of edges algebraically, we are ready to begin describing boundaries.

3.2 Boundary Requirements

Suppose we have a region $R = \zeta\Delta_0 + \eta q\Delta_0$, in a surface S and we want to determine the boundary of R. Since R can be represented by a linear combination of the master tile Δ_0 and its reflection $q\Delta_0$, we really only need to know the boundaries of these two tiles.

The boundary of the master tile is

$$\partial(\Delta_0) = e_p + e_q + e_r$$

and the boundary of it's reflection is

$$\partial(q\Delta_0) = a^{-1}e_p + e_q + be_r.$$

Recall that a^{-1} is the rotation clockwise through $\frac{2\pi}{k}$ radians, and b is the rotation counterclockwise through $\frac{2\pi}{l}$ radians.

Since the boundary operation ∂ is a G-module homomorphism, we have

$$\begin{aligned}\partial(R) =&\partial(\zeta\Delta_0 + \eta q\Delta_0)\\ =&\zeta\partial(\Delta_0) + \eta\partial(q\Delta_0)\\ =&\zeta(e_p + e_q + e_r) + \eta(a^{-1}e_p + e_q + be_r)\\ =&(\zeta + \eta a^{-1})e_p + (\zeta + \eta)e_q + (\zeta + \eta b)e_r\end{aligned}$$

This is the form that all boundaries take. However, we also know that any boundary is a sequence of edges and can be represented as an element of C_1. Combining these facts, we obtain the following theorem.

Theorem 1 *A linear combination of edges, $c = \alpha e_p + \beta e_q + \gamma e_r \in C_1$, is a boundary for a part of a surface if and only if there exist some $\zeta, \eta \in \mathcal{A}$ such that*

$$\eta(1 - a^{-1}) = \beta - \alpha \tag{1}$$

$$\eta(1 - b) = \beta - \gamma \tag{2}$$

Proof. We have shown that c is a boundary if and only if there exists ζ and η such that

$$\alpha = \zeta + \eta a^{-1} \tag{3}$$

$$\beta = \zeta + \eta \tag{4}$$

$$\gamma = \zeta + \eta b \tag{5}$$

Subtracting (1) from (2) and (3) from (2) we obtain the two equations in the theorem. Then we can choose $\zeta = \beta - \eta$ and this finishes the proof. ■

3.3 Mirrors

We are now ready to determine what the mirror of a reflection looks like. Recall that the mirror is the set of those points which remain fixed by the reflection.

A mirror is made up of one or more ovals. These ovals may be of the same type (for instance, two $\mathcal{O}_p-$type ovals might form a mirror) or of different types (a mirror might consist of an $\mathcal{O}_q-$type oval and an $\mathcal{O}_r-$type oval, where $\mathcal{O}_q$ is not equivalent to $\mathcal{O}_r$).

However, all the edge-types in a particular mirror and exactly those edges are conjugate to one another. Two edges p and q are conjugate to one another if an element g can be found such that $q = gpg^{-1}$ or $p = g^{-1}qg$. Multiplying each side of the equation by q we get $qp = qg^{-1}qg$. Thus, p is conjugate to q if and only if:

$$a^{-1} = \theta(g^{-1})g,$$

for some $g \in G$ Similarly, q is conjugate to r if and only if

$$b = \theta(h^{-1})h,$$

for some $h \in G$ and p is conjugate to r if and only if

$$c^{-1} = \theta(j^{-1})j$$

for some $j \in G$.

As we saw in Table 1 edges meeting at an odd vertex are always conjugate to one another. For example, consider the case where vertex R is odd, so $k = 2\lambda+1$ for some integer λ. Now, p is conjugate to q if and only if we can find some g such that $q = gpg^{-1}$, or equivalently, $1 = gpg^{-1}q$. Let $g = a^{\lambda}$. We get that

$$\begin{aligned} gpg^{-1}q =& a^{\lambda}pa^{-\lambda}q \\ =& (pq)^{\lambda}p(qp)^{\lambda}q = (pq)^{2\lambda+1} \\ =& a^{2\lambda+1} = 1. \end{aligned}$$

All the edges in an oval are conjugate to one another, but not all conjugate edges are in the same oval. Two edges that are conjugate to one another, but not in the same oval, will still be in the same mirror. This is when a mirror has ovals of more than one type.

Let $\mathcal{M}_q$ be the part of the mirror containing edges of type-q. In other words, $\mathcal{M}_q$ is the union of all $\mathcal{O}_q-$type ovals in a particular mirror.

Table 2 shows what the mirror containing the q-edge of the master tile looks like. The mirror definitely contains $\mathcal{M}_q$, but it may also contains ovals of another type. For instance, an $\mathcal{O}_p-$type oval may be in the mirror, where $\mathcal{O}_q$ is not equivalent to $\mathcal{O}_p$. In this case, the mirror is said to be *spanned* by $\varepsilon_g g\mathcal{M}_p + \mathcal{M}_q$. In our notation, ε indicates the orientation of the oval and is always ±1. The group elements g and h are those from above.

Table 2. Mirrors containing e_q

Mirror	Cases
$\mathcal{M}_q$	all cases, except as in the lines below and in *
$\varepsilon_g g\mathcal{M}_p + \mathcal{M}_q$	EOE, EEO, EEE and $a^{-1} = \theta(g^{-1})g$ **
$\mathcal{M}_q + \varepsilon_h h\mathcal{M}_r$	OEE, EEE and $b = \theta(h^{-1})h$ ***
$\varepsilon_g g\mathcal{M}_p + \mathcal{M}_q + \varepsilon_h h\mathcal{M}_r$	EEE and $a^{-1} = \theta(g^{-1})g$ and $b = \theta(h^{-1})h$

* but q not conjugate to p or r unless forced by parity considerations
** in case EEE, q is not conjugate to r
*** in case EEE, q is not conjugate to p

This table gives us a broad understanding of what an entire mirror looks like, and what oval types are contained in it. For a finer understanding of what each oval-type looks like, we must study the centralizer of q.

Let r_e be the reflection over edge e. It is known that $r_{ge_q} = gr_{e_q}g^{-1}$. Also, if two reflections fix the same edge, they are the same reflection. Using these two facts, it can be proven that all the q-type edges in $\mathcal{M}_q$ are in one-to-one correspondence with the centralizer H_q of q in G. Recall that the centralizer $H_q = \{g \in G |\, qg = gq\}$. For each q-type edge in the mirror, there is exactly one group element in G that transforms the q-edge in the master tile to this q-type edge. Additionally, this group element commutes with q.

We can define a sign function $sgn_q : H_q \longrightarrow \{1, -1\}$ such that $sgn_q(h) = 1$ if $h \in H_q$ restricted to the mirror is orientation preserving and $sgn_q(h) = -1$ if h restricted to the mirror is not orientation preserving.

This allows us to define a signed sum of the centralizer as

$$\mathcal{H}_q = \sum_{h \in H_q} sgn_q(h)h.$$

We use this sum to represent the part of the mirror containing q-type edges as:

$$\mathcal{M}_q = \mathcal{H}_q(xe_p + e_q + ye_r)$$

where the x and y are elements of the group algebra $\mathcal{A}$. First note that $\mathcal{H}_q e_q$ represents all the q-type edges in $\mathcal{M}_q$. The coefficients x and y can be determined by how e_p and e_r are mapped into $\mathcal{O}_q$. If there are no p-type edges in $\mathcal{O}_q$, then $x = 0$, and if there are no r-type edges in $\mathcal{O}_q$, then $y = 0$.

Similarly,

$$\begin{aligned} \mathcal{M}_p =& \mathcal{H}_p(e_p + xe_q + ye_r) \\ \mathcal{M}_r =& \mathcal{H}_r(xe_p + ye_q + e_r) \end{aligned}$$

We write $k = 2\lambda$ or $2\lambda+1$, $l = 2\mu$ or $2\mu+1$ and $m = 2\nu$ or $2\nu+1$ depending on whether the k, l, and m are even or odd. For short hand we record the parities as a triple, like OOE. The values for x and y are found in Table 3.

Table 3.a The elements $xe_p + e_q + ye_r$

parity	$xe_p + e_q + ye_r$
OOO	$a^\lambda e_p + e_q + b^{\mu+1}e_r$
EOO	$b^{\mu+1}c^{\nu+1}e_p + e_q + b^{\mu+1}e_r$
OEO	$a^\lambda e_p + e_q + a^\lambda c^\nu e_r$
OOE	$a^\lambda e_p + e_q + b^{\mu+1}e_r$
OEE	$a^\lambda e_p + e_q$
EOE	$e_q + b^{\mu+1}e_r$
EEO	e_q
EEE	e_q

Table 3.b The elements $e_p + xe_q + ye_r$ and $xe_p + ye_q + e_r$

parity	$e_p + xe_q + ye_r$	$xe_p + ye_q + e_r$
OOO	$e_p + a^{\lambda+1}e_q + c^{\nu}e_r$	$c^{\nu+1}e_p + b^{\mu}e_q + e_r$
EOO	$e_p + c^{\nu}b^{\mu}e_q + c^{\nu}e_r$	$c^{\nu+1}e_p + b^{\mu}e_q + e_r$
OEO	$e_p + a^{\lambda+1}e_q + c^{\nu}e_r$	$c^{\nu+1}e_p + c^{\nu+1}a^{\lambda+1}e_q + e_r$
OOE	$e_p + a^{\lambda+1}e_q + a^{\lambda+1}b^{\mu+1}e_r$	$b^{\mu}a^{\lambda}e_p + b^{\mu}e_q + e_r$
OEE	$e_p + a^{\lambda+1}e_q$	e_r
EOE	e_p	$b^{\mu}e_q + e_r$
EEO	$e_p + c^{\nu}e_r$	$c^{\nu+1}e_p + e_r$
EEE	e_p	e_r

This gives us a much more detailed view of what the individual components of the mirror look like.

Now that we know the form that a boundary takes and how to algebraically calculate a particular mirror, we just need to check whether that mirror can be represented as a boundary. If so, the tiling separates. Otherwise, the reflection is not separating.

4 Abelian Conformal Tiling Groups

We have developed a criterion which allows us to check whether a mirror is separating or not. However, completing some of the calculations along the way could be difficult. For instance, determining when p and q are conjugate or deciding whether an η exists to solve the equations in the previous section might be troublesome.

Assuming that G is abelian, however, simplifies many of the calculations. This allows us to completely solve the case in which G is abelian.

The first condition that becomes easier is checking whether or not two sides are conjugate. Recall that to check if p is conjugate to q we need to find a $g \in G$ such that $a = \theta(g^{-1})g$. But when G is abelian we have the following Proposition.

Proposition 2 *An automorphism satisfies $\theta(g) = g^{-1}$ for all $g \in G$ if and only if G is abelian.*

Proof. The map $i : g \to g^{-1}$ is an automorphism if and only if G is abelian. The automorphism $\theta^{-1} \circ i$ fixes a and b, and hence it fixes $G = \langle a, b \rangle$. ∎

So for an abelian group, p is conjugate to q if and only if $a = g^2$ for some $g \in G$.

The second simplification is that the elements in the

$$Cent_G(q) = H_q = \{g \in G | gq = qg\}$$

are exactly those elements in G of order 2. Why is this true? We are looking for all g such that $g = qgq^{-1} = \theta(g)$. However, we have just shown that $\theta(g) = g^{-1}$. Therefore, the elements in the centralizer H_q are the $g \in G$ such that $\theta(g) = g = g^{-1}$, which are the elements of order 2. Similarly, H_q and H_r are also those elements in G of order 2, and $H_p = H_q = H_r$. As a result, $\mathcal{H}_p = \mathcal{H}_q = \mathcal{H}_r$.

The final major simplification comes from the following theorem:

Theorem 3 *Let $G = \langle x, y \rangle$ be a finite abelian group and suppose that x and y have order s and t respectively. Let $\mathcal{R}$ be an arbitrary ring and $\mathcal{A} = \mathcal{R}[G]$ be its group algebra. There exists $\alpha, \beta, \eta \in \mathcal{A}$ that satisfy the equations*

$$\eta(1-x) = \alpha \tag{6}$$

$$\eta(1-y) = \beta. \tag{7}$$

if and only if α, β, and η satisfy

$$\alpha\left(1 + x + x^2 + \cdots + x^{s-1}\right) = 0 \tag{8}$$

$$\beta\left(1 + y + y^2 + \cdots + y^{t-1}\right) = 0 \tag{9}$$

and

$$\alpha(1-y) = \beta(1-x). \tag{10}$$

Proof. We know that $(1-w)(1+w+w^2+\cdots+w^{u-1}) = 1-w^u$ for any $w \in G$ and integer u.

$$\alpha(1+x+x^2+\cdots+x^{s-1}) = \eta(1-x)(1+x+x^2+\cdots+x^{s-1}) \tag{11}$$

$$= \eta(1-x^s) \tag{12}$$

$$= \eta(1-1) \tag{13}$$

$$= 0 \tag{14}$$

Similarly, $\beta(1+y+y^2+\cdots+y^{t-1}) = 0$.

Now, consider that when G is abelian

$$\alpha(1-y) = \eta(1-x)(1-y) \tag{15}$$

$$= \eta(1-y)(1-x) \tag{16}$$

$$= \beta(1-x) \tag{17}$$

The proof of the reverse direction is found in **[Br3]**. ■

This is an important theorem because instead of having to determine whether or not there exists an η that satisfies equations (1) and (2), we can just check whether two known quantities are equal! This is a key simplification.

We are now ready to crack the case where the orientation preserving tiling group G is abelian. There are only two possible forms that an abelian G can take, namely $\mathbb{Z}_n$ or $\mathbb{Z}_n \times \mathbb{Z}_{dn}$ for some $d, n \in \mathbb{Z}$. Because $G = \langle a, b \rangle$, it must generated by at most two elements. Hence, it can be the product of at most two cyclic groups. So G is either cyclic or it is the direct product of two cyclic groups. We can use the fundamental theorem of finitely generated abelian groups to rearrange the ordering of the component groups to obtain $\mathbb{Z}_n$ and $\mathbb{Z}_{dn}$. So in solving the Abelian case there are two main cases: $G = \mathbb{Z}_n$ and $G = \mathbb{Z}_n \times \mathbb{Z}_{dn}$.

4.1 Cyclic Conformal Tiling Groups

We now present necessary and sufficient conditions for a tiling to have a separating mirror when G is abelian. The case when G is cyclic was proved by Baeth, Deblois, and Powell [**DBP**]. We restate their theorem and provide a shorter and perhaps more elegant proof using the methods developed in this paper.

Note that we may assume that $k \le l \le m$, since if they are not in this order we can relabel the sides to put them in this order.

Theorem 4 *Let $G = \mathbb{Z}_n$, $n \ge 4$, with a (k,l,m)- generating triple $[a,b,c]$, such that $k \le l \le m$ and G is the orientation preserving tiling group for some surface. Then the tiling splits at a mirror if and only if*

$$b = c$$
$$k = \frac{n}{2}.$$

Furthermore, the tiling splits at all p-type and q-type mirrors, but at no r-type mirror.

It is worth noting that this implies n must be even.

Proof. We consider the 8 parities, OOO, OOE, etc... and in each case prove that a mirror is separating if and only if the above conditions are met. In each case let $G = \langle b, c\rangle = \mathbb{Z}_n = \langle x\rangle$, $(a,b,c) = \left(x^{-(s+t)}, x^t, x^s\right)$. Then $(k,l,m) = \left(\frac{n}{(n,t+s)}, \frac{n}{(n,t)}, \frac{n}{(n,s)}\right)$. Also since b and c generate G, $(s,t,n) = 1$. Throughout the proof we let $n = 2^\sigma e$, where e is an odd integer.

Case OOO If n is even then s, t, and $s+t$ must be even, but this is impossible. Thus n is odd. So the only element that when squared is the identity is the identity itself. Hence, $\mathcal{H}_q = 1$. Now $\mathcal{O}_q = \mathcal{O}_p = \mathcal{O}_r$, so we need only to check if $\mathcal{M}_q = \partial(\zeta\Delta_0 + \eta q\Delta_0)$. So we need to check

$$(1 - a^{-1})(1 - b^{\nu+1}) = (1 - b)(1 - a^{\lambda}).$$

Which is easily seen to be invalid.

Cases OOE, OEO, and EOO In each case n must be even and exactly one of s, t, and $s+t$ is not divisible by 2^σ. But this cannot happen.

Case OEE n must be even, $2^\sigma|(t+s)$, and 2^σ does not divide t, s. Since $(s,t,n) = 1$, it must be that both s and t are odd. Thus there does not exist $g \in G$, such that $a^{-1} = g^2$ or $b = g^2$. So we have no conjugacies besides the one induced by the odd vertex. According to Table 1, we must check two mirrors the one containing the p and q-type edges and the one containing the r-type edges. As a result we must check the two equations

$$(1 - a^{-1})\mathcal{H}_r = (1 - b)\mathcal{H}_r \tag{18}$$
$$(1 - a^{-1})\mathcal{H}_q = (1 - b)(1 - a^{\lambda})\mathcal{H}_q. \tag{19}$$

If there is a solution to equation (18) then for some h and h' in H_r, $a^{-1}h = bh'$. Since $h^2 = h'^2 = 1$, we have $a^{-2} = b^2$. So $c^2 = b^{-2}a^{-2} = 1$. Therefore $k \leq l \leq m = 2$ and we have that $(k, l, m) = (2, 2, 2)$ which is a tiling of the sphere, which we are not interested in. So the r-type mirror is never separating.

Expanding (19) we obtain

$$-a^{-1}\mathcal{H}_q = -b\mathcal{H}_q - a^{\lambda}\mathcal{H}_q + ba^{\lambda}\mathcal{H}_q.$$

Now for some $h, h' \in H_q$ we have $a^{-1}h = bh'$ or $a^{\lambda}h'$ or $ba^{\lambda}h'$. If $a^{-1}h = bh'$, then $a^{-2} = b^2$ and we have the (2,2,2) tiling of the sphere, as before. If $a^{-1}h = a^{\lambda}h'$, then $a^{2\lambda+2} = a = 1$, clearly a contradiction. Finally if $a^{-1}h = ba^{\lambda}h'$, then $b^2 = a^{-(2\lambda+2)} = a^{-1}$. So $c = a^{-1}b^{-1} = b$. Hence $G = \langle c \rangle$. So $(k, l, m) = (\frac{n}{2}, n, n)$. These are the conditions that we require the mirror to have in order to split Therefore, the p- and q- type mirrors never splits except under the specified conditions.

EOE Again n must be even. Also, $2^{\sigma}|t$. As in the previous case, s and $s+t$ must both be odd. So again there are no conjugacies, except for the conjugacies from the odd vertex. So we need to determine whether or not η and ζ exist to solve either of the following equations:

$$\partial(\zeta\Delta_0 + \eta q\Delta_0) = \mathcal{M}_q = \mathcal{H}_q(e_q + b^{\mu+1})e_r \tag{20}$$

$$\partial(\zeta\Delta_0 + \eta q\Delta_0) = \mathcal{M}_p = \mathcal{H}_p e_p, \tag{21}$$

since q and p are in the same oval. If equation (?) has a solution, then we have

$$(1 - a^{-1})\mathcal{H}_q(1 - b^{\mu+1}) = (1 - b)\mathcal{H}_q.$$

Then for some h and h' we must have one of $bh' = a^{-1}h, b^{\mu+1}h'$, or $a^{-1}b^{\mu+1}h'$. If $bh' = a^{-1}h$, then $b^2 = a^{-2}$ and we have the (2,2,2) tiling of the sphere as before. If $bh' = b^{\mu+1}h'$, then we have $b = 1$ which can not happen. Finally, if $bh = a^{-1}b^{\mu+1}h'$, then we have $a^{-2} = b^{-2\mu} = b$. So $c = a^{-1}b^{-1} = a$. But then $k = m$, so $k = l = m$, however this cannot be since $b = a^{-2}$.

Now if the p-type mirror is separating we have,

$$(1 - a^{-1})\mathcal{H}_p = (1 - b)\mathcal{H}_p.$$

But then $a^{-1}h = bh'$ for some h and h'. So $a^{-2} = b^2$ and we have the (2,2,2) tiling of the sphere, as before. In each case we have a problem and there are no separating mirrors.

EEO Again n must be even. As in the previous two cases, we have no conjugacies except for the one induced by the odd vertex. We must check for ζ and η that satisfy either of

$$\partial(\zeta\Delta_0 + \eta q\Delta_0) = \mathcal{M}_q = \mathcal{H}_q e_q \tag{22}$$

or

$$\partial(\zeta\Delta_0 + \eta q\Delta_0) = \mathcal{M}_p = \mathcal{H}_r(e_p + c^\nu e_r). \tag{23}$$

From Table 1, the p-type and $r-$ type ovals are the same. There do not exist solutions to (22), since

$$(1 - a^{-1})\mathcal{H}_q \neq (1 - b)\mathcal{H}_q.$$

If there were then this equation says we must have the (2,2,2) tiling of the sphere. If equation (22) has a solution then we have

$$(1 - a^{-1})\mathcal{H}_p(1 - c^\nu) = (1 - b)\mathcal{H}_q.$$

This equation gives us similar contradictions to the ones we saw in the EOE case.

Again, we see that there is never a separating mirror.

EEE This case has three subcases. We have no conjugacies induced by the parities. However we must have either 1) p conjugate to q, 2) p conjugate to r, or 3) q conjugate to r.

1) Suppose p is conjugate to q then we have a g such that $a^{-1} = g^2$. We need to check the q mirror and the r mirror. We have $\mathcal{M}_q = \mathcal{H}_q e_q + g\mathcal{H}_q e_p$ and $\mathcal{M}_r = \mathcal{H}_r e_r$.

To check the q mirror we must see if

$$(1 + a^{-1})\mathcal{H}_q = (1 + b)(1 + g)\mathcal{H}_q.$$

So we must have $a^{-1}h = bh', gh'$, or gbh' for some $h, h' \in H_q$. If $a^{-1}h = bh'$, then $a^{-2} = b^2$ and we have the (2,2,2) tiling of the sphere. If $a^{-1}h = gh'$, then we have $a^{-2} = g^2 = a$. So $a^3 = 1$, which can not happen. If $a^{-1}h = gbh'$, then $a^{-2} = g^2b^2 = a^{-1}b^2$. Thus, $a^{-1} = b^2$. So $b = c$ and $(k, l, m) = (n/2, n, n)$, and we have all the conditions necessary for the mirror to separate.

Similar to the q mirror in the previous case the r mirror does not separate.

2) Suppose q is conjugate to r then we have a g such that $b = g^2$. We need to check the q mirror and the p mirror. We have $\mathcal{M}_q = \mathcal{H}_q e_q + g\mathcal{H}_q e_r$ and $\mathcal{M}_p = \mathcal{H}_p e_p$.

In this case we obtain the same contradictions as in the **1)**, however in the case that works we find that $(k, l, m) = (n, n/2, n)$. So we just have a permutation of the previous case.

3) As in **2)** we have a permutation of the sides and the (k, l, m) triple does not meet the condition $k \leq l \leq m$.

This completes the proof of the cyclic theorem. ■

4.2 Abelian Non-cyclic Conformal Tiling Groups

The sufficient condition of the following theorem was proved by Deblois, Baeth and Powell [**DBP**].

Theorem 5 *Let $G = \mathbb{Z}_n \times \mathbb{Z}_{dn}$ be the conformal tiling group with a (k,l,m)-generating triple $[a,b,c]$, such that $k \le l \le m$ and G is the conformal tiling group for a surface. Then the tiling splits at a mirror if and only if*

$$k = n = 2$$
$$l = m = dn$$

with $d > 1$. Furthermore, the tiling splits at the r-type but not at the p-type or q-type mirrors.

Proof. Throughout the proof we let $(a,b,c) = (x^\alpha y^\beta, x^\gamma y^\delta, x^{-(\alpha+\gamma)}y^{-(\beta+\delta)})$, $n = 2^\sigma e$, and $d = 2^\tau f$, for f and e odd integers. Then we have $(n,\alpha,\gamma) = (dn,\beta,\delta) = 1$,

$$k = \mathrm{lcm}\left[\frac{n}{(n,\alpha)}, \frac{dn}{(dn,\beta)}\right],$$
$$l = \mathrm{lcm}\left[\frac{n}{(n,\gamma)}, \frac{dn}{(dn,\delta)}\right],$$

and

$$m = \mathrm{lcm}\left[\frac{n}{(n,\alpha+\gamma)}, \frac{dn}{(dn,\beta+\delta)}\right].$$

We proceed in this proof as we did in the Cyclic theorem, by treating each parity as its own case.

EEE There are no conjugacies from odd vertices. But there may be conjugacies. As a result we have four subcases **1)** no conjugacies, **2)** p is conjugate to q, **3)** p is conjugate to r, and **4)** q is conjugate to r. We will see that the only case that gives us a separating mirror is **2)** and, furthermore, the conditions in the theorem are satisfied.

1) Let $g = x^s y^t \in G$ such that $g^2 = a^{-1}$. We have two mirrors to check $\mathcal{M}_q + \epsilon_g g\mathcal{M}_q = \mathcal{H}_q(e_q + ge_p)$ and $\mathcal{M}_r = \mathcal{H}_r e_r$. If the p and q type mirror is separating then we have

$$(1 - a^{-1})\mathcal{H}_q = (1-b)(1-g)\mathcal{H}_q.$$

So we have for some $h, h' \in H_q$, $a^{-1}h = bh', gh'$, or bgh'. If $a^{-1}h = bh'$, then $a^{-2} = b^2$ and we have the (2,2,2) tiling of the sphere. If $a^{-1}h = gh'$, then $a^{-2} = g^2 = a^{-1}$ and we have $a = 1$, which can not happen. If $a^{-1}h = gbh'$, then we have $a^{-2} = b^2a^{-1}$. So $b^2 = a^{-1}$. Thus we get $c = b$. Which forced G to be cyclic. Therefore the q type and p type mirror is never separating.

Next we check the r type mirror. To show that the r-type mirror separates and when it does we have the required conditions, we will need to know $\partial r\Delta_0$.

As in the boundary section we find that $\partial r\Delta_0 = ce_p + b^{-1}e_q + e_r$. If the r type mirror is separating then we know that

$$\begin{aligned}\mathcal{M}_r = \mathcal{H}_r e_r =& \partial(\zeta\Delta_0 + \eta r\Delta_0)\\ =& (\zeta + \eta c)e_p + (\zeta + \eta b^{-1})e_q + (\zeta + \eta)e_r.\end{aligned}$$

Matching up coefficients and subtracting as before we have

$$\eta(1 - c) = \eta(1 - b^{-1}) = \mathcal{H}_q.$$

Using Theorem 3, we see that the mirror is separating if and only if

$$(1 - c)\mathcal{H}_q = (1 - b^{-1})\mathcal{H}_q.$$

Thus for some $h, h' \in H_q$, $ch = b^{-1}h'$. Hence $c^2 = b^{-2}$, which implies $a^2 = 1$. Furthermore since the l and m are both even, $l = m$. Additionally, since $G = \langle a, b\rangle = \mathbb{Z}_n \times \mathbb{Z}_{dn}$ and the maximum size group that a and b can generate has size $2dn$, we have $dn^2 \leq 2dn$. So $n \leq 2$, thus $n = 2$. It suffices to prove that $m = dn$. The order of the group generated by a and b is at most $4d$, which occurs when b has order $2d$ and a is not generated by b, but the order of G is $4d$. Hence $l = m = 2d$, as desired.

2) and 3) Each of these cases is a relabelling of **2)**, but since we require $k \leq l \leq m$, we see that no mirror is separating in these cases.

4) Assume there are no conjugacies, then we must check each mirror, $\mathcal{M}_q = \mathcal{H}_q e_q, \mathcal{M}_p = \mathcal{H}_p e_p$, and $\mathcal{M}_r = \mathcal{H}_r e_r$. As in **1)** we see that the mirror is separating if and only if we have the conditions from the Theorem. Additionally, with the q and p type mirrors we get contradictions because we find that $l(m) = 2$ and $k = m(k = l)$, which gives us the (2,2,2) tiling of the sphere, which we are not interested in.

Therefore, the p and q type mirrors are never separating and the r type edge is separating when we have $n = k = 2$ and $dn = l = m$.

OOO From table 1. we know that $\mathcal{M}_q = \mathcal{H}_q(a^\lambda e_p + e_q + b^{\mu+1})e_r$. So we must check whether or not

$$(1 - a^{-1})(1 - b^{\mu+1})\mathcal{H}_q = (1 - b)(1 - a^\lambda)\mathcal{H}_q.$$

If it does then we have one of $a^{-1}h = b^{\mu+1}h', a^{-1}b^{\mu+1}h', bh', a^\lambda h'$, or $ba^\lambda h'$ for some $h, h' \in H_q$. If $a^{-1} = b^{\mu+1}h'$, then $a^{-2} = b^{2\mu+2} = b$. Thus $c = b^{-1}a^{-1} = a$. So the group is cyclic, since $G = \langle a, c\rangle = \langle a\rangle$. If $a^{-1} = a^{-1}b^{\mu+1}h'$, then $b^{2\mu+2} = b = 1$, which cannot happen. If $a^{-1} = bh'$, then $a^{-2} = b^2$. So we have the (2,2,2) tiling of the sphere as before. If $a^{-1} = a^\lambda h'$, then $a^{-2} = a^\lambda = a^{-1}$. Thus $a = 1$, which is not possible. If $a^{-1} = ba^\lambda h'$, then $a^{-2} = b^2a^{2\lambda} = b^2a^{-1}$. So we have $a^{-1} = b^2$. Hence $b = c$ and we have a cyclic group. Therefore, the mirror is never separating.

EOO, OEO, and OOE As in the cyclic case we find that these parities are not possible in a abelian group.

OEE Again n must be even. Furthermore, $2^{\sigma}|\alpha$, $2^{\sigma+\tau}|\beta$, and 2^{σ} does not divide γ or $(\gamma+\alpha)$, and $2^(\sigma+\tau)$ does not divide δ or $(\delta+\beta)$. There are two oval types one containing the r edges and then the other contains both the p and q type edges. Additionally there are no conjugacies besides the one from the odd vertex. Suppose, for a contradiction, that q is conjugate to r, then there exists $g = x^s y^t$, such that $b = x^{\gamma} y^{\delta} = x^{2s} y^{2t}$. But then γ and δ are even. Thus α and γ which is not possible. Similarly, p and r are never conjugate.

Therefore we need to only check the two mirrors $\mathcal{M}_q = \mathcal{H}_q(a^{\lambda}e_p + e_q)$ and $\mathcal{M}_r = \mathcal{H}_q e_r$. If $\mathcal{M}_q$ is separating then $(1-a^{-1})\mathcal{H}_q = (1-b)\mathcal{H}_q(1-a^{\lambda})$. So for some $h, h' \in H_q$ we have $a^{-1}h = bh', a^{\lambda}h'$ or $ba^{\lambda}h'$. If $a^{-1}h = bh'$, then we have the (2,2,2) tiling of the sphere as before. If $a^{-1}h = a^{\lambda}h'$, then $a^{2\lambda+2} = a = 1$, which is not possible. If $a^{-1}h = ba^{\lambda}h'$, then $a^{-1} = b^2$ and we have $b = c$. Thus the group is cyclic, since $G = \langle b, c\rangle = \langle c\rangle$, but this contradicts the structure of G. Therefore the p and q type mirror is not separating.

Next we must check the r type mirror. Proceeding as in **EEE** we see that $k = 2$, but this contradicts the odd parity of k. So the r type mirror is not separating.

EOE and EEO Each of these cases is a permutation of the edges in the **OEE** case. Therefore, no mirror is separating in either of these cases. ■

We have completely solved the case in which G is abelian.

5 Further Work

In this section we present two different paths that could be explored to attack this problem further.

5.1 Systems of Equations in the Group Algebra

No matter what the structure of the group G we are always able to get to equations (1) and (2). However, the major difficulty and determining if a mirror is splitting or not is in determining when an η exists that satisfies these equations. When we assume that G is abelian we obtained Theorem 3. This theorem allowed us to determine whether or not an η existed by simply checking whether or not two things that we can write out are equal.

An area of future work would be to develop methods to determine whether or not η existed without assuming that G is abelian. We offer the following theorem as a starting point for further exploration in this direction.

Theorem 6 [Br2] *Let α_1, α_2 and β_1, β_2 be elements of $\mathcal{A} = \mathcal{R}[G]$. Then there exists $\eta \in \mathcal{A}$ satisfying*

$$\eta\alpha_1 = \beta_1$$
$$\eta\alpha_2 = \beta_2$$

if and only if for every pair $(\gamma_1, \gamma_2) \in \mathcal{A}^2$ *if*

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 = 0,$$

then

$$\beta_1\gamma_1 + \beta_2\gamma_2 = 0.$$

Proof. We prove the sufficient condition of the theorem in this paper, the necessary condition is much more difficult and is proved in [**Br2**]. Suppose $\eta\alpha_1 = \beta_1$, $\eta\alpha_2 = \beta_2$ and that for some (γ_1, γ_2) we have $\alpha_1\gamma_1 + \alpha_2\gamma_2 = 0$. Then

$$\begin{aligned}\beta_1\gamma_1 + \beta_2\gamma_2 =& \eta\alpha_1\gamma_1 + \eta\alpha_2\gamma_2\\ =& \eta(\alpha_1\gamma_1 + \alpha_2\gamma_2)\\ =& \eta 0 = 0.\end{aligned}$$

■

We now describe one way this theorem could be useful. Suppose we have a mirror $\alpha e_p + \beta e_q + \gamma e_r$ and to determine whether it is separating or not we must solve

$$\begin{aligned}\eta(1 - a^{-1}) =& \beta - \alpha\\ \eta(1 - b) =& \beta - \gamma.\end{aligned}$$

Then if we can find $\delta_1, \delta_2 \in \mathcal{A}$, such that $(1 - a^{-1})\delta_1 + (1 - b)\delta_2 = 0$ and $(\beta - \alpha)\delta_1 + (\beta - \gamma)\delta_2 \neq 0$, then we know the mirror is not separating. Since we suspect that most mirrors are not separating it is reasonable to try to find δ_i that will give these results.

The following example demonstrates how this theorem in conjunction with Theorem 1 can be used to prove that a mirror is not separating.

Example Let $G = \mathbb{Z}_n = \langle x \rangle$, with n even. Also let $(a, b, c) = (x^{-2}, x, x)$. We know from Theorem 4, that the r-type mirror does not separate. But we will use Theorem 6 to prove it as well. Note that $\mathcal{M}_r = \mathcal{H}_r e_r$. So we need to determine if there exists η such that

$$\eta(1 - a^{-1}) = 0 \tag{24}$$

and

$$\eta(1 - b) = -\mathcal{H}_r. \tag{25}$$

Now $(1-a^{-1})(1+x^2+x^4+\cdots+x^{n-2})+(1-b)(1+x+\cdots+x^{n-1}) = 0+0 = 0$, but $\mathcal{H}_r(1+x+\cdots+x^{n-1}) \neq 0$, since $\mathcal{H}_r = 1$ or $1 \pm x^{n/2}$. Therefore by Theorem 6 we know there does not exist an η that satisfies (24) and (25). Hence the r-type mirror is not separating.

5.2 Fixed Points as a Way to Test for Separability

In this section we introduce a totally different way to attack the problem of determining separating reflections.

Suppose that a has even order, say $k = 2k'$. Then $a^{k'}$ is a rotation through π radians. Then we can count the number of fixed points by $a^{k'}$ in two different ways. In one way we assume that the reflection in the edge p is separating, therefore if the two numbers are not the same then we know that the p type edge is not separating.

It turns out that calculating all of the fixed points by $a^{k'}$ is not difficult. First note that if a point is fixed it must be a vertex of a triangle. Let x_0 be a point on the surface and $H = \{g \in G : gx_0 = x_0\}$. Then we can define a map, α from $S \to G/H$, which is both one to one and onto. Then for any element g we can calculate the number of fixed R-vertices by looking at the permutation representation of G/H. We want to know when $dR = gdR$ for $d \in G$, but through the use of α we can look at when $dH = gdH$. This makes the calculation of the fixed points much easier. Through these calculations we find the total number of fixed points by $a^{k'}$.

On the other hand if we assume that the mirror is separating we can do a different calculation to get the number of fixed points. Say that the mirror separates the surface into two pieces S^+ and S^-. Then $a^{k'}$ maps S^+ to S^-, so all of the fixed points must be on the mirror. Furthermore, each oval in the mirror can have either 0 or 2 fixed points. Therefore, we can obtain an upper bound on the number of fixed points if a mirror is separating, namely twice the number of ovals. So it remains to find the number of ovals in a mirror. But again there are methods to easily compute this.

Methods similar to the one described here have been used to study the case when $G = PSL(2, q)$, where q is a prime, and $(k, l, m) = (2, 3, 7)$. In this case all of the edges are conjugate and it was proved that the tiling has no separating reflections [**PSL**].

Notice that each $PSl(2, q)$ is a simple group. We explored all simple groups of order less than or equal to 6000 and found no separating reflections. This evidence leads us to the following conjecture.

Conjecture 7 *Let G be the conformal tiling group for some surface S with tiling Θ. If G is simple then the tiling has no separating mirrors.*

We believe that the methods described above can be used to prove that when G is simple and there is a vertex of even order, then the tiling has no separating reflections.

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