Rose-Hulman Undergraduate Mathematics Journal

Volume 16
Issue 2

Article 13

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Jason Hempstead
University of Washington

Rose-Hulman Undergraduate Mathematics Journal

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Volume 16, No. 2, Fall 2015

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Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
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[a]University of Washington

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Apollonian Circle Packings and the Riemann Hypothesis

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Abstract. In this paper, we describe how one can state the Riemann hypothesis in terms of a geometric problem about Apollonian circle packings. We use, as a black box, results of Zagier, and describe numerical experiments which were used in a recent paper by Athreya, Cobeli, and Zaharescu.

Acknowledgements: I would like to thank Professor Jayadev Athreya for his help with the non-Euclidean geometry used and for all his help in writing this paper, Dr. Joseph Vandehey for his help in parsing through the literature involved, Danni Sun for her help in constructing sphere packings and working with Eisenstein integers, and Kaiyue Hou for her help in writing some of the code. Additionally, I would like to thank Somesh Kumar and Brainwave Magazine (`www.bwmag.in`) for use of their illustration of an Apollonian circle packing.

1 Introduction

An *Apollonian circle packing* is constructed as follows: draw any three mutually tangent circles. According to Apollonius's theorem, there are two circles tangent to all three: one outside the initial three and one inside them. Construct a set consisting of the three initial circles and the two tangent to all of them. Take any three circles from that set and add the mutually tangent circle not already present to the set. As the iteration continues, this becomes an Apollonian circle packing. An example of a packing is given in Figure 1.

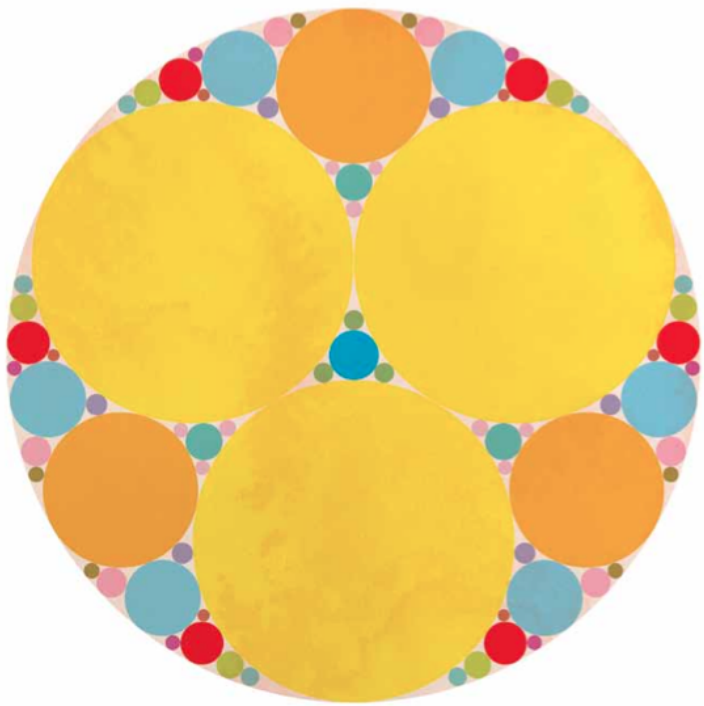

Figure 1: An Apollonian circle packing. Picture created by Somesh Kumar for the December 2011 issue of Brainwave magazine.

We now discuss the radial density problem in Apollonian circle and sphere packings, as studied by Athreya, Cobeli, and Zaharescu [ACZ15]. In an Apollonian circle packing, choose one circle C_0 with the radius r. Draw a concentric circle C_ϵ with a slightly larger radius $r+\epsilon$, $\epsilon > 0$. Then, C_ϵ intersects the circles tangent to C_0. The radial density problem examines the proportion μ_ϵ of the circumference of C_ϵ that is contained within those circles that are tangent to C_0 as $\epsilon \to 0$. The main result of Athreya, Cobeli, and Zaharescu's work is

$$\lim_{\epsilon\to 0} \mu_\epsilon = \frac{3}{\pi}.$$

We describe Athreya, Cobeli, and Zaharescu's proof strategy, using as a black box results of Zagier [Za81], on equidistribution on the modular surface. The importance of this work

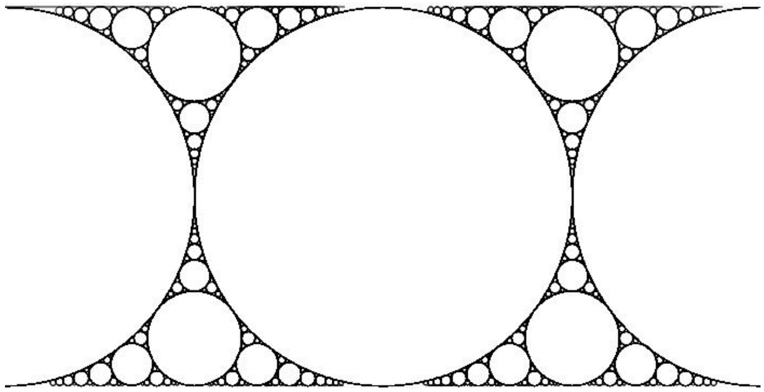

Figure 2: A circle packing between the x-axis and the line $y = 1$.

becomes clear when analyzing the rate at which the radial density approaches its eventual limit. Zagier's theorem relates the rate of approach to the Riemann hypothesis [Za81]. The numerical work described in this paper will therefore have direct applications to the validity of the Riemann hypothesis.

We will transform the radial density problem into the complex plane using Möbius transformations. We view our circles as subsets of $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$, where $\mathbb{C}$ is the complex plane. Each circle can be uniquely described by its radius and center point. Circles that include ∞ are taken to be straight lines. By working in $\hat{\mathbb{C}}$, Möbius transformations can be used to manipulate the packing. A Möbius transformation $f(z)$ is defined by

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. A Möbius transformation preserves the points of tangency and maps circles to circles (or lines, equivalent to circles through ∞). Because of this, any general packing can be transformed to be constructed between the lines $y = 0$ and $y = 1$ as in Figure 2. While Möbius transformations do alter the lengths of arcs (and even their ratios), this affect will disappear in the limit that we will discuss. For details, see Athreya, Cobeli, and Zaharescu [ACZ15].

In Section 2, we will introduce Zagier's Theorem and its connection the the Riemann hypothesis. Afterward in Section 3, we use a Möbius transformation on a circle packing so that we can apply Zagier's Theorem to solve the radial density problem. In Section 5 we show numerical evidence related to the radial density problem and analyze the rate of convergence, noting a connection to the Riemann hypothesis. Still in Section 5, we will move up one dimension and analyze the radial density problem in sphere packings.

2 Equidistribution Theorems

In this section we introduce Zagier's Theorem to calculate the radial density in Apollonian circle packings and to relate the radial density problem to the Riemann hypothesis.

2.1 Zagier's Theorem

Theorem 2.1. *Let $f(z)$ be a continuous function in $\mathbb{C}$. Assume that there exist $t_1, t_2 \in \mathbb{R}$ such that*

$$f(z) = \begin{cases} 0 & Im(z) > t_1 \\ g(z) & t_2 \leq Im(z) \leq t_1 \\ 0 & Im(z) < t_2 \end{cases}$$

where $g(z)$ is some continuous function in $\mathbb{C}$. Also assume that $f(\frac{az+b}{cz+d}) = f(z)$ for all $a, b, c, d \in \mathbb{Z}$ with the property that $ad - bc = 1$. Let Ω be the region

$$\Omega = \{z \in \mathbb{C} : |Re(z)| < \frac{1}{2}, |z| > 1\}.$$

Then:

$$\lim_{\epsilon \to 0} \int_0^1 f(x + i\epsilon)dx = \frac{3}{\pi} \int_\Omega \frac{f(z)}{Im(z)^2} dz.$$

2.2 Connection to the Riemann Hypothesis

Furthermore, Zagier describes a connection of the *rate* of convergence in the above result to the Riemann hypothesis.

Theorem 2.2. *The Riemann hypothesis is equivalent to the statement: For all $\delta > 0$, we have*

$$\left| \int_0^1 f(z + i\epsilon)dz - \frac{3}{\pi} \int_\Omega \frac{f(z)}{Im(z)^2} dz \right| \leq O\left(\epsilon^{\frac{3}{4} - \delta}\right).$$

3 Applying Zagier's Theorem to the Farey-Ford Packing

We will now transform our circle packing to the complex plane and use the results of Zagier's Theorem to solve the radial density problem.

3.1 The Farey-Ford Packing

The packing shown in Figure 2 is a transformation of the one in Figure 1. One of the large yellow circles is mapped to the x-axis while the outermost circle is mapped to the line $y = 1$. Returning to the radial density problem, remove all circles from the packing in Figure 2 that are not tangent to the x-axis. What remains is known as the Farey-Ford packing.

The radial density problem in the context of the Farey-Ford packing is studied by taking a horizontal line at $y = \epsilon > 0$, which is analogous to increasing the radius of the inner circle by ϵ. Color the portion of that line that lies within circles red. This is shown in Figure 3. The radial density problem then asks what proportion of the line at $y = \epsilon$ is red [ACZ15].

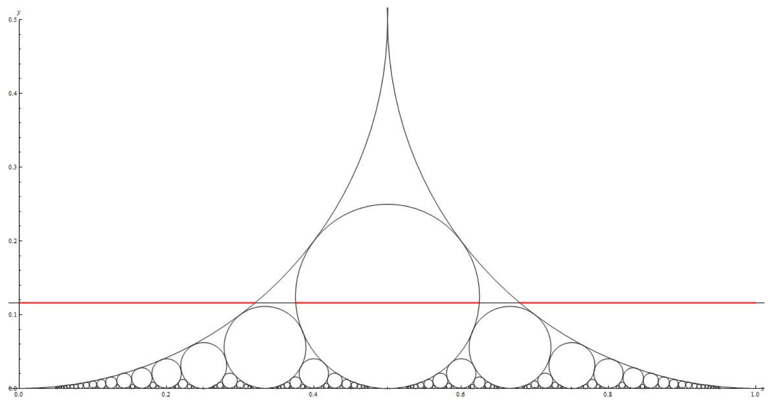

Figure 3: The Farey-Ford Packing between $x = 0$ and $x = 1$. The line at $y = \epsilon$ is shown. Red indicates intersection with a circle tangent to the x-axis.

3.2 Using Zagier's Theorem

We describe how to use Zagier's theorem to compute radial density in the Farey-Ford packing. For convenience, we calculate the proportion of the line *outside* the circles. Define the function

$$Q(x,y) = \begin{cases} 0 & y > 1 \\ 0 & \text{Inside Circles} \\ 1 & x < 1 \text{ and Outside Circles} \\ 0 & y < 0 \end{cases}$$

If the horizontal line is at $y = \epsilon$, the proportion of the line not contained within circles is $\int_0^1 Q(x,\epsilon)dx$. This function meets the conditions of Zagier's Theorem, so this proportion converge to $\frac{3}{\pi}\int_\Omega \frac{Q(x,y)}{y^2}dxdy$ where Ω is the same region defined above as $\epsilon \to 0$. To compute this integral, consider the hyperbolic area of the region. The region Ω is a hyperbolic triangle with vertices at ∞, $\frac{\sqrt{3}}{2}i + \frac{1}{2}$, and $\frac{\sqrt{3}}{2}i - \frac{1}{2}$ with respective angles 0, $\frac{\pi}{3}$, and $\frac{\pi}{3}$. The area of Ω is therefore

$$A_\Omega = \pi - (0 + \frac{\pi}{3} + \frac{\pi}{3}) = \frac{\pi}{3}$$

by the Gauss-Bonnet Formula. Using the definition of $Q(x,y)$,

$$\begin{aligned} \int_\Omega \frac{Q(x,y)}{y^2} &= A_\Omega - \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_1^\infty \frac{1}{y^2} dydx \\ \int_\Omega \frac{Q(x,y)}{y^2} &= \frac{\pi}{3} - 1. \end{aligned}$$

By Zagier's Theorem,

$$\int_0^1 Q(x,\epsilon)dx = \frac{3}{\pi}\int_\Omega \frac{Q(x,y)}{y^2} = \frac{3}{\pi}(\frac{\pi}{3} - 1) = 1 - \frac{3}{\pi}.$$

So, the proportion of the line *not* in the circles is $1 - \frac{3}{\pi}$ meaning that the proportion *in* the circles is given by $1 - (1 - \frac{3}{\pi}) = \frac{3}{\pi}$. This says that the proportion of the line that is red in Figure 3 is $\frac{3}{\pi}$ as the height of the horizontal line drops to 0.

4 Numerical Experiments

To test the limit established earlier in this paper and by Athreya, Cobeli, and Zaharescu [ACZ15] and the rate of convergence, numerical calculations of the radial density problem are presented.

4.1 Circle Packings

The Farey-Ford packing has a well-defined production algorithm. Circles can be drawn with center $(\frac{p}{q}, \frac{1}{2q^2})$ and radius $\frac{1}{2q^2}$ where $p, q \in \mathbb{N}$, $p < q$, and $\gcd(q, p) = 1$.

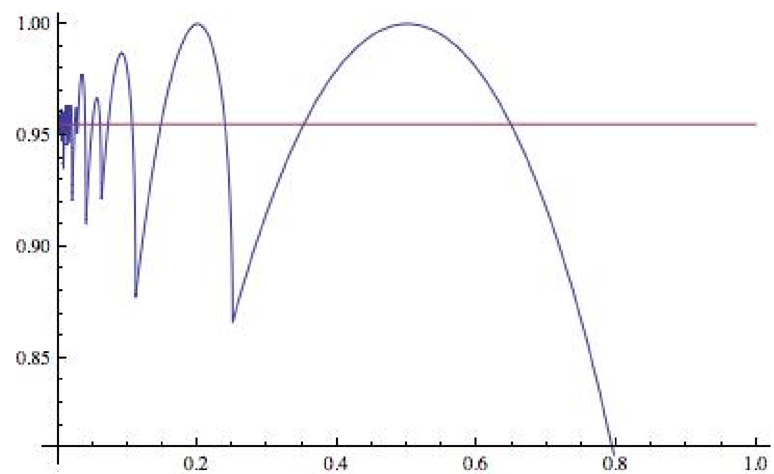

Figure 4: Plot of the total intersection length for $h \in [0, 1]$. The intersection length with circles is shown for a given value of h. Along with the plot of the function $f(h)$ is shown (in red) the expected asymptote at $h = 0$ of $3/\pi$.

4.1.1 Main Term

Having this formula to produce the circles makes calculating the proportion of the line which intersects the circles very easy. Since the Farey-Ford packing is periodic, it is only necessary to look at the included line segments where $x \in [0, 1]$. Let $f(h)$ be the total length of these line segments at height h, where $f_r(h)$ represents the length of the line through a circle of radius r. Then,

$$f(h) = \sum_{\frac{1}{q^2} \geq h} \sum_{0 \leq p < q} f_{\frac{1}{2q^2}}(h) = \sum_{1 \leq q \leq h^{-\frac{1}{2}}} \left(2\sqrt{h(\frac{1}{q^2} - h)} \cdot \varphi(q) \right)$$

where $\varphi(n)$ is Euler's totient function.

4.1.2 Error Term

The difference between the expected value of $\frac{3}{\pi}$ and the function $f(h)$ was taken for $h \to 0$. Zagier's Theorem states that if the Riemann hypothesis is true, this error function should be bounded by a function of the form $h^{\frac{3}{4}}$. To see the properties near $h = 0$ more clearly, the height is parametrized by $\frac{1}{x}$ is used. Fix $\delta > 0$.

$$\begin{aligned}
\left|f(h) - \lim_{h\to 0} f(h)\right| &\leq h^{\frac{3}{4}-\delta} \\
\left|f\left(\frac{1}{x}\right) - \lim_{h\to 0} f(h)\right| &\leq \left(\frac{1}{x}\right)^{\frac{3}{4}-\delta} \\
\log\left|f\left(\frac{1}{x}\right) - \lim_{h\to 0} f(h)\right| &\leq \log\left(\frac{1}{x}\right)^{\frac{3}{4}-\delta} \\
\log\left|f\left(\frac{1}{x}\right) - \lim_{h\to 0} f(h)\right| &\leq \left(\frac{3}{4}-\delta\right)\log\left(\frac{1}{x}\right) \\
\frac{\log\left|f\left(\frac{1}{x}\right) - \lim_{h\to 0} f(h)\right|}{\log\left(\frac{1}{x}\right)} &\geq \frac{3}{4}-\delta
\end{aligned}$$

The last line is possible because $\frac{1}{x} < 1$, so $\log\frac{1}{x} < 0$. Figure 5 shows the function $\frac{\log\left|f\left(\frac{1}{x}\right)-\frac{3}{\pi}\right|}{\log\frac{1}{x}}$ along with the expected lower bound of $\frac{3}{4}$ (in red). The apparent strength of the bound provides evidence that the Riemann hypothesis is true.

In fact, recently Kontorovich [Ko14] pointed out that convergence at rate $h^{3/4-\delta}$ in the radial density problem is sufficient for the Riemann hypothesis. Thus, the above plot is excellent numerical evidence for the Riemann hypothesis.

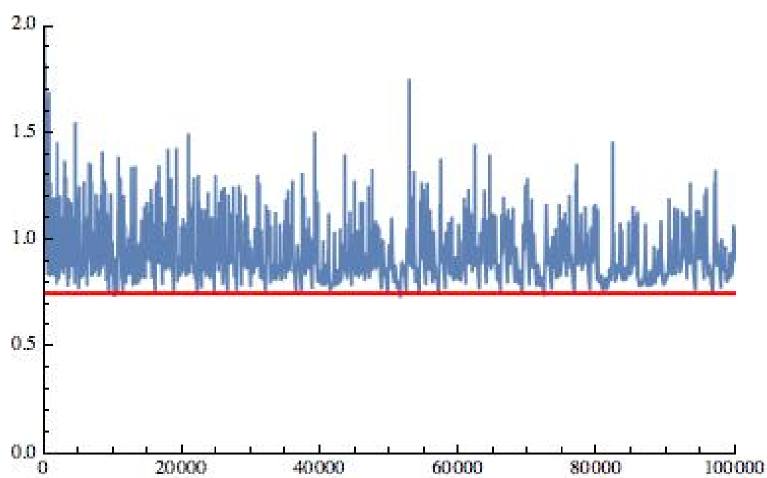

Figure 5: Plot of the natural logarithm of the error over the natural logarithm of the inverse of the height. The expected lower bound of $\frac{3}{4}$ is shown in red.

4.2 Sphere Packings

The idea of circle packings is generalized to a higher dimension to explore the problem further. We can extend the radial density problem to 3-dimensions. This time, start with four mutually tangent spheres. There are exactly two spheres tangent to all four. Draw those two spheres. Choose any four spheres from the set and draw the tangent sphere that is missing. Iterating this process produces a sphere packing. An example can be seen in Figure 6.

Similar to the 2-dimensional case, consider increasing the radius of an inner sphere by a small amount $\epsilon > 0$. What proportion of this slightly larger sphere intersects the spheres tangent to the original sphere [ACZ15]?

A generalized Möbius transformation can be used to transform the packing to map the spheres between the planes $z = 0$ and $z = 1$. The problem now becomes what proportion of the plane at height $\epsilon > 0$ intersects spheres.

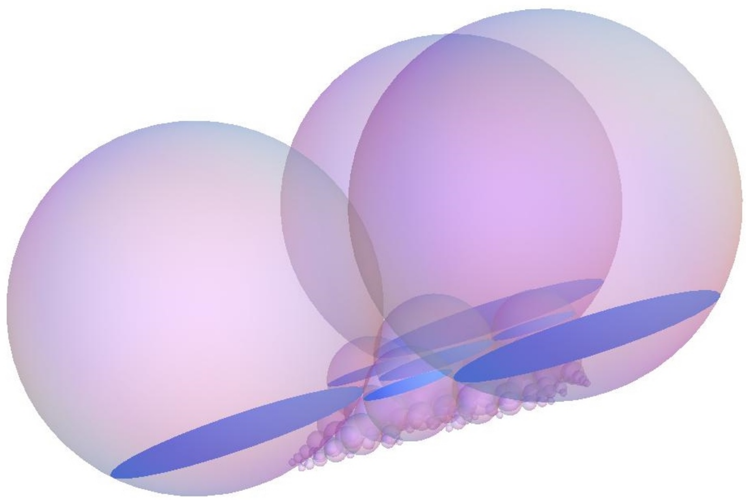

Figure 6: Example sphere packing. An example showing the spheres tangent to the plane $z = 0$. A plane at $z = h$ is shown intersecting the spheres.

Analogous to the Farey-Ford packing in two-dimensions, an algorithm was developed to produce all the spheres tangent to the plane $z = 0$. Since the packing will once again be periodic across the plane, only spheres with centers in the triangle with vertices at $0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i$ or on the perimeter were generated.

4.2.1 Eisenstein Integers

Eisenstein integers are a set of numbers of the form $a + b\omega$ where $a, b \in \mathbb{Z}$ and $\omega = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$.

The complex plane is used for the xy-plane. The centers of the spheres take the form

$\left(\frac{a+b\omega}{q}, \frac{1}{2\left|\frac{q}{\gcd(a+b\omega,q)}\right|^2}\right)$ where $a, b, q \in \mathbb{N}$, $b < a \leq q$, and gcd denotes the greatest common divisor in the Eisenstein integers [Ko13]. The radii of the spheres are the same as the z-components of the centers, since they are all tangent to the plane $z = 0$.

4.2.2 Main Term

The area of the plane $z = h$ within spheres in the triangle described above was then calculated. This total area intersected was then divided by the area of the triangle to find the proportion. Due to computational constraints, a minimum radius was required of each sphere, so the height of the plane could be taken only to a finite height above the $z = 0$ plane. If we denote the proportion of the plane contained within spheres as $f(h)$, a similar equidistribution result [ACZ15, Theorem 1.3] tells us that

$$\lim_{h \to 0} f(h) \approx 0.853.$$

Figure 7 shows the calculation of $f(h)$ and the expected limit as the height of the plane drops to 0.

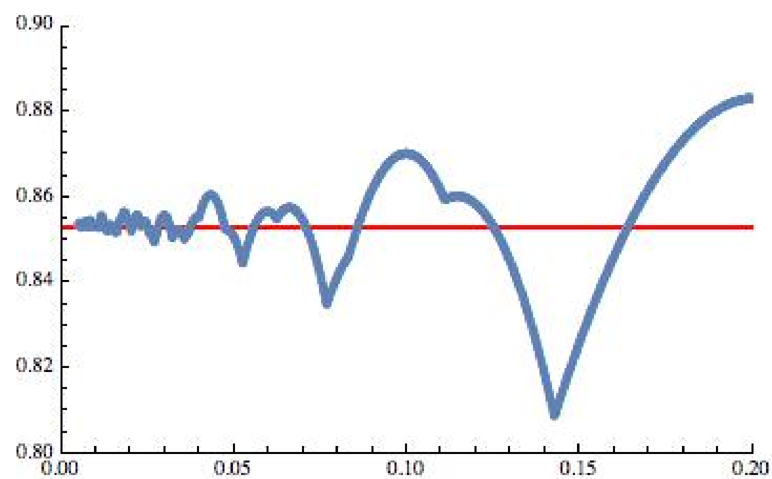

Figure 7: Plot of the proportion of the plane at $z = h$ included within spheres. Shown (in red) is the expected limit at $h = 0$ of 0.853.

4.2.3 Error Term

The difference between the area function and the limit at $h = 0$ is again interesting. As in the two-dimensional case, $\frac{\log\left|f\left(\frac{1}{x}\right) - 0.853\right|}{\log \frac{1}{x}}$, is plotted in Figure 8. The plot appears to have a soft lower bound at 1.5.

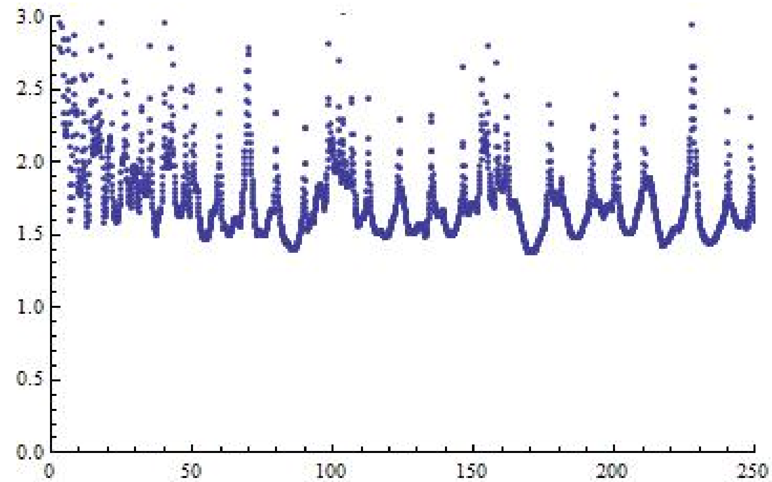

Figure 8: Plot of the natural logarithm of the error over the natural logarithm of the height, parametrizing the height as $h = 1/x$.

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