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COMPUTING THE ELEMENTARY SYMMETRIC POLYNOMIALS OF THE MULTIPLIER SPECTRA OF THE MAPS $z^2 + c$

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Abstract. Let f be a complex quadratic rational map. The ith elementary symmetric polynomial of the formal n multiplier spectra of f is denoted $\sigma_i^{(n)}(f)$. The values of these polynomials are invariant under conjugation by the projective linear group $\mathrm{PGL}_2(\mathbb{C})$ and are interesting to the study of the moduli space of quadratic rational maps, $\mathcal{M}_2(\mathbb{C})$. For every $n \in \mathbb{Z}_{>0}$ and i in the appropriate range, $\sigma_i^{(n)}(f) \in \mathbb{Z}[\sigma_1, \sigma_2]$ where σ_1, σ_2 are $\sigma_1^{(1)}(f)$, $\sigma_2^{(1)}(f)$, respectively. Despite this, the $\sigma_i^{(n)}(f)$ are difficult to compute. By restricting our focus to the family of quadratic polynomials $z^2 + c$, computations become simpler. We determine an upper bound for the degrees of the $\sigma_i^{(n)}$ for the maps of the form $z^2 + c$ by arguing in terms of the growth rates of their periodic points and corresponding multipliers. We also include computations of the forms of the $\sigma_i^{(n)}$ for $n = 2, ..., 6$ for these maps.

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1 Introduction

We define a complex rational map to be a quotient of relatively prime complex polynomials. Let $\mathrm{Rat}_2(\mathbb{C})$ denote the space of complex quadratic rational maps. We define the projective linear group $\mathrm{PGL}_2(\mathbb{C})$ to be the group of Möbius transformations on $\mathbb{C}$, that is, the set of maps of the form $\frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ which forms a group under composition. Many dynamical properties of complex quadratic rational maps are preserved under conjugation by $\mathrm{PGL}_2(\mathbb{C})$. Because of this, it is interesting to study the quotient space $\mathcal{M}_2(\mathbb{C})$ of $\mathrm{Rat}_2(\mathbb{C})$, known as the moduli space of quadratic rational maps, that is formed from this conjugation action. Functions on $\mathrm{Rat}_2(\mathbb{C})$ that are $\mathrm{PGL}_2(\mathbb{C})$-invariant can be used to define functions on $\mathcal{M}_2(\mathbb{C})$ and provide a means through which we can describe the space. In this article, we investigate a particular subset of these functions and their restrictions to the set of maps of the form z^2+c in $\mathrm{Rat}_2(\mathbb{C})$. Before we define them and state our results, we briefly introduce needed concepts from dynamics. This preliminary material will be defined more formally in Sections 2 and 3.

Let f be a complex rational map. For $n \in \mathbb{Z}_{>0}$, we denote the nth iterate of f by f^n, defined $f^n = f \circ f^{n-1}$ where $f^0(z) = z$. We say $z_0 \in \mathbb{C}$ is a periodic point of period n of f if $f^n(z_0) = z_0$. The multiplier of such a point z_0 is defined to be $\lambda_{z_0}(f) = (f^n)'(z_0)$. A periodic point of period n of f is said to have formal period n if it is a root of the nth dynatomic polynomial of f, $\Phi_{f,n}$. The set of multipliers of periodic points of formal period n of f is denoted $\Lambda_n(f)$ and is called the formal n multiplier spectra of f. For each $n \in \mathbb{Z}_{>0}$, $\Lambda_n(f)$ is preserved under conjugation by elements of $\mathrm{PGL}_2(\mathbb{C})$. This allows for a more general study of conjugacy classes of rational maps instead of specific maps. Since the sets are preserved, the values of symmetric polynomials on these sets are also preserved by conjugation allowing us to assign a single, sometimes unique, complex number to each conjugacy class, an invariant.

We consider the case of quadratic rational maps. Let $f \in \mathrm{Rat}_2(\mathbb{C})$. The first two elementary symmetric polynomials of $\Lambda_1(f)$ are denoted σ_1, σ_2. These are functions on $\mathcal{M}_2(\mathbb{C})$ and it turns out that they induce an isomorphism between $\mathcal{M}_2(\mathbb{C})$ and affine 2-space $\mathbb{A}^2(\mathbb{C})$, each conjugacy class corresponding bijectively to a complex pair (σ_1, σ_2) [2, Lemma 3.1]. Our primary objects of study are generalizations of σ_1 and σ_2, namely the elementary symmetric polynomials of $\Lambda_n(f)$ for $n \in \mathbb{Z}_{>0}$. We denote the ith elementary symmetric polynomial of $\Lambda_n(f)$ as $\sigma_i^{(n)}(f)$. From Silverman [4, Corollary 5.2], we have that $\sigma_i^{(n)}(f) \in \mathbb{Z}[\sigma_1, \sigma_2]$.

For general quadratic rational f, finding the forms of the $\sigma_i^{(n)}(f)$ is difficult. Milnor [2] and Berker et al. [1] have computed the forms of $\sigma_i^{(n)}(f)$ for $n = 2, 3$. Computations are simpler for the family of quadratic polynomials of the form $z^2 + c$. In fact, for $\phi(z) = z^2 + c$, $\sigma_1 = 2$ and $\sigma_2 = 4c$, allowing us to write ϕ as $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$. Thus for $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$, $\sigma_i^{(n)}(\phi_{\sigma_2}) \in \mathbb{Z}[\sigma_2]$. The periodic points of formal period n of ϕ_{σ_2} depend continuously on σ_2 since they are roots of the nth dynatomic polynomial of ϕ_{σ_2}. For two continuous functions $f, g : \mathbb{C} \to \mathbb{C}$, we say $f = \Theta(g)$ if $\lim_{z\to\infty} \frac{f(z)}{g(z)}$ exists and is nonzero. By arguing in terms of growth rates we show the following:

Proposition 4.1. *Let $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$ and let $n \in \mathbb{Z}_{>0}$. Then for every P such that $\Phi_{\phi_{\sigma_2},n}(P) = 0$, we have $P = \Theta(\sigma_2^{\frac{1}{2}})$.*

Corollary 4.2. *Let $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$ and let $n \in \mathbb{Z}_{>0}$. Then for every P such that $\Phi_{\phi_{\sigma_2},n}(P) = 0$, we have $\lambda_P(\phi_{\sigma_2}) = \Theta(\sigma_2^{\frac{n}{2}})$.*

These growth rates along with the fact that the $\sigma_i^{(n)}$ are polynomials allow us to determine a bound on the degrees of the $\sigma_i^{(n)}$.

Theorem 4.3. *For the map $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$, and for every $n \in \mathbb{Z}_{>0}$ and every $i \in \{1, ..., |\Lambda_n(\phi_{\sigma_2})|\}$, we have $\deg(\sigma_i^{(n)}(\phi_{\sigma_2})) \leq \left\lfloor \frac{in}{2} \right\rfloor$ where $\lfloor \cdot \rfloor$ denotes the floor function.*

The argument we use to show this does not say anything about cancellation of terms in the elementary symmetric polynomials so we cannot conclude equality; however, we conjecture that the other direction holds.

Conjecture 4.4. For the conditions of Theorem 4.3, we have $\deg(\sigma_i^{(n)}(\phi_{\sigma_2})) = \left\lfloor \frac{in}{2} \right\rfloor$ for every $n \in \mathbb{Z}_{>0}$ and every $i \in \{1, ..., |\Lambda_n(\phi_{\sigma_2})|\}$.

The upper bound on the degrees of the $\sigma_i^{(n)}$ allows us to easily compute the general forms of $\sigma_i^{(n)}(\phi_{\sigma_2})$ with interpolation. We include our computations of $\sigma_i^{(n)}(\phi_{\sigma_2})$ for $n = 2, ..., 6$. We also include the same results using the multiplier spectra of ϕ_{σ_2} including the multipliers of all periodic points instead of the formal multiplier spectra. Computations were done using the Sage computer algebra system [6].

The remainder of this article is organized as follows. In Section 2, we introduce essential concepts and terminology from dynamics. In Section 3, we describe the conjugation action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathrm{Rat}_2(\mathbb{C})$ and then define the polynomials $\sigma_i^{(n)}$. We prove our main result in Section 4. In Sections 5 and 6, we provide an algorithm that can be used to compute the $\sigma_i^{(n)}$ and discuss its implementation using the Sage computer algebra system. We describe our computational results in Section 7 and list them in Appendix A. Finally, in Section 8 we briefly discuss a possible direction for future research.

2 Definitions from dynamics

We begin by first introducing needed concepts from dynamics.

Definition. A *discrete dynamical system* consists of a set S and a map f from that set to itself, that is, $f : S \to S$.

We consider systems of the form $f : \mathbb{C} \to \mathbb{C}$ for complex rational maps f which we define to be quotients relatively prime complex polynomials. We note that to be more precise, complex rational maps should be considered as self maps of the extended complex plane; however, since we eventually focus on a family of polynomial maps, we introduce needed dynamical concepts in terms of $\mathbb{C}$ instead. For such a rational map, we define its *degree* to be the maximum of the degrees of its numerator and denominator.

Example 2.1.

- $f(z) = \frac{z^2+2z+1}{z-1}$ is degree 2.
- $f(z) = \frac{z^2+z+2}{7z^5+2z^2+z+9}$ is degree 5.

For a nonnegative integer n, we define the *nth iterate* of a rational map f to be $f^n = f \circ f^{n-1}$. The 0th iterate is defined to be the identity map, that is, $f^0(z) = z$.

Example 2.2.

- For $f(z) = z^2$, $f^3(z) = ((z^2)^2)^2 = z^8$.
- For $f(z) = z^2 + 1, f^2(z) = f(f(z)) = (z^2+1)^2 + 1 = z^4 + 2z^2 + 2$.
- For $f(z) = \frac{2z-3}{z+1}, f^2(z) = f(f(z)) = \frac{2(\frac{2z-3}{z+1})-3}{\frac{2z-3}{z+1}+1} = \frac{z-9}{3z-2}$.

We say a point $z_0 \in \mathbb{C}$ is a *periodic point of period* n of f if it is fixed by the nth iterate of f, that is, if $f^n(z_0) = z_0$.

Example 2.3. Let $f(z) = z^2 - \frac{7}{4}$. Its periodic points of period 2 are the solutions to

$$f^2(z) - z = f(f(z)) - z = (z^2 - \frac{7}{4})^2 - z - \frac{7}{4} = z^4 - \frac{7}{2}z^2 - z + \frac{21}{16} = 0$$

which are $z = \frac{1}{2}, -\frac{3}{2}, \frac{1}{2} \pm \sqrt{2}$.

Since a periodic point of period n is a periodic point of period nm for every $m \in \mathbb{Z}_{>0}$ we can classify periodic points based on the smallest period for which they are periodic.

Definition. A point $z_0 \in \mathbb{C}$ is called a *periodic point of minimal period* n of a rational map f if $f^n(z_0) = z_0$ and $f^m(z_0) \neq z_0$ for every $m \in \{1, ..., n-1\}$.

We define the *forward orbit* of a point $z_0 \in \mathbb{C}$ by a rational map f to be the set of forward images of z_0 by f, $\{f^n(z_0) \mid n \in \mathbb{Z}_{\geq 0}\}$. The forward orbit of a periodic point is always finite. Also, if z_0 is a periodic point of period n of f, then its forward orbit by f will consist of other periodic points of period n of f.

Example 2.4. For the map $f(z) = z^2 - \frac{7}{4}$, the forward orbit of $\frac{1}{2}$ is $\{\frac{1}{2}, -\frac{3}{2}\}$. Both $\frac{1}{2}$ and $-\frac{3}{2}$ are periodic points of minimal period 2 of f. The points $\frac{1}{2} + \sqrt{2}$ and $\frac{1}{2} - \sqrt{2}$ are fixed points of f, that is, they are periodic points of period 1, and thus their orbits only contain one element.

We make an additional classification of periodic points for polynomial maps, the roots of the nth dynatomic polynomial. We define the *Möbius function* μ as:

Definition. Let $n \in \mathbb{Z}_{>0}$.

$$\mu(n) = \begin{cases} (-1)^r \text{ if } n \text{ is squarefree} \\ 0 \text{ if } n \text{ is not squarefree} \\ 1 \text{ if } n = 1 \end{cases}$$

where r is the number of distinct prime factors of n.

Example 2.5.

- $\mu(5) = -1$.
- $\mu(6) = 1$.
- $\mu(8) = 0$.

Definition. Let $f \in \mathbb{C}[z]$. The *nth dynatomic polynomial of f* [3] is defined to be

$$\Phi_{f,n}(z) = \prod_{k|n} (f^k(z) - z)^{\mu(\frac{n}{k})}.$$

Example 2.6. For $f(z) = z^2 + c$,

$$\Phi_{f,2} = \prod_{k|2} (f^k(z) - z)^{\mu(\frac{2}{k})} = \frac{f^2(z) - z}{f(z) - z} = z^2 + z + c + 1,$$

$$\begin{aligned}\Phi_{f,4} = \prod_{k|4} (f^k(z) - z)^{\mu(\frac{4}{k})} = \frac{f^4(z) - z}{f^2(z) - z} &= z^{12} + 6cz^{10} + z^9 + (15c^2 + 3c)z^8 + 4cz^7 \\ &+ (20c^3 + 12c^2 + 1)z^6 + (6c^2 + 2c)z^5 + (15c^4 + 18c^3 + 3c^2 + 4c)z^4 \\ &+ (4c^3 + 4c^2 + 1)z^3 + (6c^5 + 12c^4 + 6c^3 + 5c^2 + c)z^2 + (c^4 + 2c^3 + c^2 + 2c)z \\ &+ (c^6 + 3c^5 + 3c^4 + 3c^3 + 2c^2 + 1).\end{aligned}$$

There is a more general version of the dynatomic polynomial for rational maps [3], but in this article we only focus on a family of polynomial maps. For a polynomial map f, $\Phi_{f,n}$ is guaranteed to be a polynomial [3, Theorem 4.5], and all of the periodic points of minimal period n of f are among its roots. Additionally, its roots are always periodic points of period n of f. We can compute the degree of $\Phi_{f,n}$ as $\deg(\Phi_{f,n}) = \sum_{k|n}((\deg(f))^k \mu(\frac{n}{k}))$. It turns out that any simple root of $\Phi_{f,n}$ is a periodic point of minimal period n of f [5]. However, there are cases where there are higher multiplicity roots that are periodic points of period smaller than n.

Example 2.7. Consider $f(z) = z^2 - \frac{3}{4}$, $f(z) - z = (z + \frac{1}{2})(z - \frac{3}{2})$ and $f^2(z) - z = (z + \frac{1}{2})^3(z - \frac{3}{2})$ meaning that f has no periodic points of minimal period 2. As a result, the 2nd dynatomic polynomial of f has a higher multiplicity root, $\Phi_{f,2}(z) = \frac{f^2(z) - z}{f(z) - z} = (z + \frac{1}{2})^2$.

Definition. A point $z_0 \in \mathbb{C}$ is called a *periodic point of formal period* n of a polynomial map f if $\Phi_{f,n}(z_0) = 0$. For a general rational map, its periodic points of formal period n are the zeros of the corresponding generalized nth dynatomic polynomial.

We also define the multiplier of a periodic point:

Definition. Let $z_0 \in \mathbb{C}$ be a periodic point of period n of a rational map f. The *multiplier* of z_0 is defined to be $\lambda_{z_0}(f) = (f^n)'(z_0)$.

The multipliers of periodic points in the same orbit are the same. We can see this by applying the chain rule: $(f^n)'(z) = f'(f^{n-1}(z)) \cdots f'(z)$. If z is a periodic point of period n of f, then $(f^n)'(z)$ is the product of the first derivatives of f evaluated at each point in the forward orbit of z. Thus the value of $(f^n)'(z)$ is invariant with respect to the choice of point from the orbit.

Example 2.8.

- For $f(z) = z^2 - \frac{7}{4}$ we found $z_0 = \frac{1}{2} - \sqrt{2}$ to be a fixed point of f.

$$\lambda_{z_0}(f) = f'(z_0) = 2z_0 = 1 - 2\sqrt{2}.$$

- Additionally, we found $z_1 = \frac{1}{2}$ to be a periodic point of minimal period 2.

$$\lambda_{z_1}(f) = (f^2)'(z_1) = \frac{d}{dz}\Big|_{z_1}((z^2-\frac{7}{4})^2-\frac{7}{4}) = \frac{d}{dz}\Big|_{z_1}(z^4-\frac{7}{2}z^2+\frac{49}{16}) = (4z^3-7z)\Big|_{z_1} = \frac{1}{2}-\frac{7}{2} = -3.$$

- $z_2 = -\frac{3}{2}$ was also a periodic point of minimal period 2, and its multiplier is

$$\lambda_{z_2}(f) = (4z^3 - 7z)\Big|_{z_2} = -\frac{27}{2} + \frac{21}{2} = -3$$

which reflects the fact $\frac{1}{2}$ and $-\frac{3}{2}$ are in the same forward orbit.

3 The isomorphism and its invariants

We now introduce the projective linear group $\mathrm{PGL}_2(\mathbb{C})$ and describe how it acts on the space of rational quadratic maps $\mathrm{Rat}_2(\mathbb{C})$. This action induces an equivalence relation from which we construct the moduli space of quadratic rational maps $\mathcal{M}_2(\mathbb{C})$. We then define our objects of study, which are generalizations of the invariants of an isomorphism from this space.

We define a *Möbius transformation* on $\mathbb{C}$ to be a degree one rational map of the form $f(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. The set of Möbius transformations under composition forms the *projective linear group* $\mathrm{PGL}_2(\mathbb{C})$. We denote the space of complex quadratic rational maps as $\mathrm{Rat}_2(\mathbb{C})$. For a rational map f and a Möbius transformation g we define the *conjugate* of f by g to be $f^g = g^{-1} \circ f \circ g$.

Example 3.1. Let $f(z) = \frac{z^2+5}{z+1} \in \mathrm{Rat}_2(\mathbb{C})$, and let $g(z) = \frac{1}{z} \in \mathrm{PGL}_2(\mathbb{C})$, then

$$f^g(z) = \frac{(\frac{1}{z}) + 1}{(\frac{1}{z})^2 + 5} = \frac{z^2 + z}{5z^2 + 1}.$$

Möbius transformations are the automorphisms of the extended complex plane and are very useful in studying iteration of rational maps since conjugation with them commutes with iteration, $g^{-1} \circ f^n \circ g = (g^{-1} \circ f \circ g)^n$. We can use this conjugation action to partition $\mathrm{Rat}_2(\mathbb{C})$ into equivalence classes where two rational quadratic maps are in the same equivalence class if they are conjugates of each other by an element of $\mathrm{PGL}_2(\mathbb{C})$.

Definition. Let $f, h \in \mathrm{Rat}_2(\mathbb{C})$. We say f, h are *conjugate* if there exists a $g \in \mathrm{PGL}_2(\mathbb{C})$ such that $f = h^g$.

This is an equivalence relation and the resulting equivalence classes are called conjugacy classes. We denote the set of these conjugacy classes as $\mathcal{M}_2(\mathbb{C})$, called the *moduli space of rational quadratic maps* [2]. More precisely, $\mathcal{M}_2(\mathbb{C}) = \mathrm{Rat}_2(\mathbb{C})/\,\mathrm{PGL}_2(\mathbb{C})$, that is, $\mathrm{Rat}_2(\mathbb{C})$ modulo the action of conjugation by $\mathrm{PGL}_2(\mathbb{C})$.

For any $n \in \mathbb{Z}_{>0}$, and a rational quadratic map f, conjugation by $\mathrm{PGL}_2(\mathbb{C})$ preserves the set of multipliers of periodic points of formal period n of f. Before showing this, we introduce a convenient notation for the set of multipliers of periodic points of formal period n.

Definition. The *formal n multiplier spectra* $\Lambda_n(f)$ of a quadratic rational map f is the set of multipliers of the periodic points of formal period n of f included with appropriate multiplicity [3]. Since periodic points of the same forward orbit share the same multiplier, only one multiplier per orbit is included.

Example 3.2. Let $f(z) = z^2 - \frac{8}{9}$. The periodic points of formal period 2 of f are the roots of

$$\Phi_{f,2}(z) = z^2 + z + \frac{1}{9} = (z - (-\frac{1}{2} - \frac{\sqrt{5}}{6}))(z - (-\frac{1}{2} + \frac{\sqrt{5}}{6}))$$

and are thus $-\frac{1}{2} \pm \frac{\sqrt{5}}{6}$. These points are in the same forward orbit and so they share the same multiplier,

$$\lambda_{-\frac{1}{2}-\frac{\sqrt{5}}{6}}(f) = \lambda_{-\frac{1}{2}+\frac{\sqrt{5}}{6}}(f) = f^2(-\frac{1}{2} + \frac{\sqrt{5}}{6}) = \frac{11}{12} - \frac{17\sqrt{5}}{108}.$$

Since only one multiplier per orbit is included in the formal multiplier spectra, $\Lambda_2(f) = \{\frac{11}{12} - \frac{17\sqrt{5}}{108}\}$.

We first show how the set of fixed points of a rational map is acted upon by $\mathrm{PGL}_2(\mathbb{C})$ conjugation. This result is well known but we include it for completeness:

Lemma 3.1. *Let f be a rational map, and let $g \in \mathrm{PGL}_2(\mathbb{C})$. Then P is a fixed point of f if and only if $g^{-1}(P)$ is a fixed point of f^g.*

Proof. Let P be a fixed point of f. Then

$$f^g(g^{-1}(P)) = g^{-1}(f(g(g^{-1}(P)))) = g^{-1}(P).$$

Let $g^{-1}(P)$ be a fixed point of f^g. Then $f(P) = (g \circ f^g \circ g^{-1})(P) = P$. □

Since the periodic points of period n of a rational map f are the fixed points of f^n, the above result also holds for the set of periodic points of period n of f. Given any $g \in \mathrm{PGL}_2(\mathbb{C})$ and k dividing n we have $f^k(P) - P = 0$ if and only if $(f^g)^k(g^{-1}(P)) - g^{-1}(P) = 0$, and so it follows that P is a periodic point of formal period n of f if and only if $g^{-1}(P)$ is a periodic point of formal period n of f^g. We now want to show $\Lambda_n(f)$ is preserved under $\mathrm{PGL}_2(\mathbb{C})$ conjugation.

Proposition 3.2. *Let f be a rational map, let $g \in \mathrm{PGL}_2(\mathbb{C})$. Then $\Lambda_n(f) = \Lambda_n(f^g)$.*

Proof. Since we have P is a periodic point of formal period n of f if and only if $g^{-1}(P)$ is a periodic point of formal period n of f^g, we need only show $\lambda_P(f) = \lambda_{g^{-1}(P)}(f^g)$ for each periodic point P of formal period n of f. Here $\lambda_P(f) = (f^n)'(P)$.

Let $z \in \mathbb{C}$. For convenience let $h = f^n$. Then $h^g(z) = (g^{-1} \circ h \circ g)(z)$, and so $(h^g)'(z) = (g^{-1})'(h(g(z)))h'(g(z))g'(z)$. Let P be a periodic point of formal period n of f. Note that $\lambda_{g^{-1}(P)}(f^g) = (h^g)'(g^{-1}(P))$ since conjugation by g commutes with iteration. Thus we have:

$$\begin{aligned}\lambda_{g^{-1}(P)}(f^g) &= (h^g)'(g^{-1}(P)) = (g^{-1})'(h(g(g^{-1}(P))))h'(g(g^{-1}(P)))g'(g^{-1}(P)) \\ &= (g^{-1})'(P)h'(P)g'(g^{-1}(P)) = (g^{-1})'(P)g'(g^{-1}(P))h'(P) = h'(P) = \lambda_P(f)\end{aligned}$$

since $(g^{-1})'(g(z))g'(z) = (g^{-1} \circ g)'(z) = 1$ for any $z \in \mathbb{C}$. □

Though $\Lambda_n(f)$ is fixed by conjugation by $\mathrm{PGL}_2(\mathbb{C})$, it would be even nicer to have a single complex number corresponding to $\Lambda_n(f)$ that is preserved by conjugation rather than a set. This would allow us to identify each conjugacy class with a single complex value, an invariant.

Definition. Let $n \in \mathbb{Z}_{>0}$, $z_1, ..., z_n \in \mathbb{C}$. Consider the monic polynomial

$$(z - z_1) \cdots (z - z_n) = z^n - e_1 z^{n-1} + ... + (-1)^n e_n,$$

here e_i is the ith *elementary symmetric polynomial* of the n variables $z_1, ..., z_n$.

The values of elementary symmetric polynomials are invariant under permutation of the variables, meaning that the values of elementary symmetric polynomials evaluated over the elements of $\Lambda_n(f)$ will be invariant under conjugation by $\mathrm{PGL}_2(\mathbb{C})$.

A rational map of degree d has exactly $d+1$ fixed points [3]. Thus for a quadratic rational map we will have exactly 3 fixed points. We denote the multipliers of these fixed points as

$\lambda_1, \lambda_2, \lambda_3$ and let σ_1, σ_2 denote the first two elementary symmetric polynomials of $\lambda_1, \lambda_2, \lambda_3$. That is, $\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3$ and $\sigma_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$.

We also define *affine* 2*-space* over $\mathbb{C}$ to be $\mathbb{A}^2(\mathbb{C}) = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{C}\}$. We denote the conjugacy class of a rational quadratic map f in $\mathcal{M}_2(\mathbb{C})$ as $[f]$. From Milnor [2, Lemma 3.1], we have:

Theorem 3.3. *$\mathcal{M}_2(\mathbb{C}) \cong \mathbb{A}^2(\mathbb{C})$ with isomorphism $\pi : \mathcal{M}_2(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$ defined by $[f] \in \mathcal{M}_2(\mathbb{C}) \mapsto (\sigma_1, \sigma_2)$.*

We now define the objects of our study which are generalizations of σ_1, σ_2:

Definition. For a quadratic rational map f and for $n \in \mathbb{Z}_{>0}$ and $i \in \{1, ..., |\Lambda_n(f)|\}$, we define $\sigma_i^{(n)}(f)$ to be the ith elementary symmetric polynomial of the formal n multiplier spectra of f.

From Silverman [4, Corollary 5.2], we have a useful fact about these polynomials:

Theorem 3.4. *For $n \in \mathbb{Z}_{>0}$ and $i \in \{1, ..., |\Lambda_n(f)|\}$, $\sigma_i^{(n)}(f) \in \mathbb{Z}[\sigma_1, \sigma_2]$, that is, $\sigma_i^{(n)}(f)$ is a polynomial in σ_1, σ_2 with integer coefficients.*

From now on we focus our study on the family of quadratic polynomial maps of the form $z^2 + c$. The reason why we choose the family $z^2 + c$ instead of general quadratic polynomials is that any quadratic polynomial is $PGL_2(\mathbb{C})$ conjugate to $z^2 + c$ for some $c \in \mathbb{C}$ [3].

We first want to compute the set of coordinates (σ_1, σ_2) for maps of the form $\phi(z) = z^2 + c$. Notice that the point at infinity is always a fixed point of ϕ, but we need an alternate definition for its multiplier.

Definition. If the point at infinity is a fixed point of a rational map f, its multiplier is

$$\lambda_\infty(f) = \lim_{z \to 0} \frac{z^{-2} f'(z^{-1})}{f(z^{-1})^2}$$

which can be seen by conjugating f by $\frac{1}{z}$ [3].

Lemma 3.5. *For $\phi(z) = z^2 + c$, $\lambda_\infty(\phi) = 0$.*

Proof. $\lambda_\infty(\phi) = \lim_{z \to 0} \frac{z^{-2} 2z^{-1}}{((z^{-1})^2 + c)^2} = \lim_{z \to 0} \frac{z^{-2} 2z^{-1}}{((z^{-1})^2 + c)^2} \frac{z^4}{z^4} = \lim_{z \to 0} \frac{2z}{(1 + cz^2)^2} = \frac{0}{1} = 0$. □

Without loss of generality, let λ_3 correspond to the multiplier of the point at infinity. Then for ϕ, $\sigma_1 = \lambda_1 + \lambda_2$, $\sigma_2 = \lambda_1\lambda_2$, and $\sigma_3 = 0$. The two finite fixed points of ϕ can be found by solving $\phi(z) - z = z^2 - z + c = 0$. The roots are $z_1 = \frac{1+\sqrt{1-4c}}{2}$ and $z_2 = \frac{1-\sqrt{1-4c}}{2}$. So we have $\lambda_1 = \phi'(z_1) = 2z_1$, $\lambda_2 = \phi'(z_2) = 2z_2$, and thus $\sigma_1 = \lambda_1 + \lambda_2 = 2z_1 + 2z_2 = 2$, $\sigma_2 = \lambda_1\lambda_2 = 2z_1 2z_2 = 4c$. This means that the image of the family of maps of the form $z^2 + c$ by $\pi : \mathrm{Rat}_2(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$ that maps $f \in \mathrm{Rat}_2(\mathbb{C})$ to (σ_1, σ_2) is a line in $\mathbb{A}^2(\mathbb{C})$.

Since $\sigma_1 = 2$ for the family of maps $\phi(z) = z^2 + c$, we have that $\sigma_i^{(n)}(\phi) \in \mathbb{Z}[\sigma_2]$. This fact and the relation $\sigma_2 = 4c$ allows for easier computation of the $\sigma_i^{(n)}(\phi)$. We relabel $\phi(z) = z^2 + c$ as $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$ for convenience.

Note. The reason we define the multiplier spectra in terms of periodic points of formal period rather than those of minimal period is that we want the $\sigma_i^{(n)}$ to be polynomials. It turns out that if we define the multiplier spectra in terms of periodic points of minimal period we encounter discontinuities in the $\sigma_i^{(n)}$. To see this, consider the general map $\phi_c(z) = z^2 + c$. Its 2nd dynatomic polynomial is $\Phi_{\phi_c,2}(z) = z^2 + z + c + 1$ which has roots $z = -\frac{1}{2}\sqrt{-4c-3} - \frac{1}{2}, \frac{1}{2}\sqrt{-4c-3} - \frac{1}{2}$. These are distinct for $c \neq -\frac{3}{4}$ meaning that $\Phi_{\phi_c,2}$ has two simple roots, which are thus exactly the periodic points of minimal period 2. So then suppose we define the multiplier spectra in terms of periodic points of minimal period so that the $\sigma_i^{(n)}$ are elementary symmetric polynomials of the multipliers of the periodic points of minimal period. Then for $c \neq -\frac{3}{4}$, $\sigma_1^{(2)}(\phi_c) = (\phi_c^2)'(\frac{1}{2}\sqrt{-4c-3} - \frac{1}{2}) = 4c + 4$. We have $\lim_{c \to -\frac{3}{4}} \sigma_1^{(2)}(\phi_c) = 1$ and so if $\sigma_1^{(2)}(\phi_c)$ were continuous with respect to c we would expect $\sigma_1^{(2)}(\phi_{-\frac{3}{4}}) = 1$.

However, this is not the case. Recall that in Example 2.7, we saw that there can be deficiencies of periodic points of minimal period. In particular, $\phi_{-\frac{3}{4}}(z) = z^2 - \frac{3}{4}$ has no periodic points of minimal period 2 so $\sigma_1^{(2)}(\phi_{-\frac{3}{4}})$ would be an empty sum. Thus defining the $\sigma_i^{(n)}$ in terms of periodic points of minimal period introduces discontinuities, and in particular, the $\sigma_i^{(n)}$ cannot be polynomials in c so results such as those in Theorem 3.4 are not possible.

4 Results

By arguing in terms of the growth rates of periodic points with respect to σ_2 we can deduce an upper bound for the degrees of the $\sigma_i^{(n)}$ for $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$. We use asymptotic notation to denote growth rate relations:

Definition. Let $f, g : \mathbb{C} \to \mathbb{C}$ be continuous functions. We say $f = O(g)$ if $\lim_{z\to\infty} \frac{f(z)}{g(z)}$ exists. We say $f = \Theta(g)$ if $\lim_{z\to\infty} \frac{f(z)}{g(z)}$ exists and is nonzero. We say $f = o(g)$ if $\lim_{z\to\infty} \frac{f(z)}{g(z)} = 0$, and $f = \omega(g)$ if $g = o(f)$.

Proposition 4.1. *Let $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$ and let $n \in \mathbb{Z}_{>0}$. Then for every P such that $\Phi_{\phi_{\sigma_2},n}(P) = 0$, we have $P = \Theta(\sigma_2^{\frac{1}{2}})$.*

Proof. Let $n \in \mathbb{Z}_{>0}$ and $i \in \{1, ..., |\Lambda_n(\phi_{\sigma_2})|\}$ be given. The constant term of $\phi_{\sigma_2}^k$ has degree 2^{k-1} with respect to σ_2. Thus the constant term of the nth dynatomic polynomial has degree $\sum_{k|n} \mu(\frac{n}{k}) 2^{k-1} = \frac{1}{2}\sum_{k|n} \mu(\frac{n}{k}) 2^k = \frac{1}{2}\deg(\Phi_{\phi_{\sigma_2},n})$ with respect to σ_2. Let $N = \deg(\Phi_{\phi_{\sigma_2},n})$, then $\Phi_{\phi_{\sigma_2},n}$ has exactly N roots, and since it is a monic polynomial in z we may write $\Phi_{\phi_{\sigma_2},n}(z) = (z - P_1) \cdots (z - P_N)$ where $P_1, ..., P_N$ are the roots of $\Phi_{\phi_{\sigma_2},n}$. The product of the roots of a polynomial is its constant term, thus we have $P_1 \cdots P_N = \Theta(\sigma_2^{\frac{1}{2}N})$.

Let $P \in \{P_1, ..., P_N\}$, then P is a periodic point of formal period n so $\phi_{\sigma_2}^n(P) = P$, and also P is a continuous function of σ_2 since it is a root of $\phi_{\sigma_2}^n$. Suppose $P = \omega(\sigma_2^{\frac{1}{2}})$, then we have $P^2 = \omega(\sigma_2) \Rightarrow \phi_{\sigma_2}(P) = P^2 + \frac{\sigma_2}{4} = \Theta(P^2) = \omega(P)$. And so $\phi^n_{\sigma_2}(P) = \Theta(P^{2^n}) = \omega(P)$, but since $\phi^n_{\sigma_2}(P) = P$, this is a contradiction. Thus we must have $P = O(\sigma_2^{\frac{1}{2}})$.

We have from earlier that $P_1 \cdots P_N = \Theta(\sigma_2^{\frac{1}{2}N})$. Since each $P_i = O(\sigma_2^{\frac{1}{2}})$, we must have $P_i = \Theta(\sigma_2^{\frac{1}{2}})$ for every i since if for some i, $P_i = o(\sigma_2^{\frac{1}{2}})$, then $P_1 \cdots P_N = o(\sigma_2^{\frac{1}{2}N})$, a contradiction. $\square$

Corollary 4.2. *Let $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$ and let $n \in \mathbb{Z}_{>0}$. Then for every P such that $\Phi_{\phi_{\sigma_2},n}(P) = 0$, we have $\lambda_P(\phi_{\sigma_2}) = \Theta(\sigma_2^{\frac{n}{2}})$.*

Proof. We have $\lambda_P(\phi_{\sigma_2}) = (\phi^n_{\sigma_2})'(P) = \prod_{j=0}^{n-1} \phi'_{\sigma_2}(\phi^j_{\sigma_2}(P))$, the product of the first derivatives of ϕ_{σ_2} evaluated at each periodic point in the forward n-orbit of P, each of which is also a periodic point of formal period n of ϕ_{σ_2}. For $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$, $\phi'_{\sigma_2}(z) = 2z$, linear in z. Thus using the result of Proposition 4.1, $\lambda_P(\phi_{\sigma_2}) = \Theta(\sigma_2^{\frac{n}{2}})$. $\square$

Theorem 4.3. *For the map $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$, and for each $n \in \mathbb{Z}_{>0}$ and each $i \in \{1, ..., |\Lambda_n(\phi_{\sigma_2})|\}$, we have $\deg(\sigma_i^{(n)}(\phi_{\sigma_2})) \leq \lfloor \frac{in}{2} \rfloor$ where $\lfloor \cdot \rfloor$ denotes the floor function.*

Proof. $\sigma_i^{(n)}$ is the sum of products of length i of multipliers of periodic points of formal period n of ϕ_{σ_2}, and so $\sigma_i^{(n)} = O(\sigma_2^{\frac{in}{2}})$ after applying the result of Corollary 4.2. Since $\sigma_i^{(n)} \in \mathbb{Z}[\sigma_2]$, we have $\deg(\sigma_i^{(n)}) \leq \lfloor \frac{in}{2} \rfloor$. $\square$

The reason we cannot conclude equality from this argument is that it is possible for cancellation to occur in the elementary symmetric polynomials which can reduce growth rates. As far as we can compute, the other direction holds, however, we have not yet been able to formulate a lower bound for the possible amount of cancellation. We leave this as a conjecture:

Conjecture 4.4. For the conditions in Theorem 4.3, we have $\deg(\sigma_i^{(n)}(\phi_{\sigma_2})) = \lfloor \frac{in}{2} \rfloor$ for every $n \in \mathbb{Z}_{>0}$ and every $i \in \{1, ..., |\Lambda_n(\phi_{\sigma_2})|\}$.

Equality does indeed hold for the last elementary symmetric polynomial, $\sigma^{(n)}_{|\Lambda_n(\phi_{\sigma_2})|}$; since it is a product, no cancellation can occur.

5 Computing the $\sigma_i^{(n)}$

We can use the upper bound for the degrees of the $\sigma_i^{(n)}(\phi_{\sigma_2})$ to compute their general forms. To do this, we compute $\lfloor \frac{in}{2} \rfloor + 1$ distinct pairs $(\sigma_2, \sigma_i^{(n)}(\phi_{\sigma_2}))$ and then perform Lagrange interpolation on them. If the degree of $\sigma_i^{(n)}$ happens to be smaller than $\lfloor \frac{in}{2} \rfloor$, the terms that are too large will have zero coefficients.

Given a value of σ_2, we want to compute the corresponding value of $\sigma_i^{(n)}$. We can do this since given a value of σ_2, we know the map of the form $z^2 + c$ to which it corresponds, namely $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$. The task of computing the value of $\sigma_i^{(n)}$ at σ_2 becomes:

1. Compute the periodic points of formal period n of $\phi_{\sigma_2}(z) = z^2 + \frac{\sigma_2}{4}$, that is, the roots of $\Phi_{\phi_{\sigma_2},n}$.

2. Compute the multipliers of these periodic points.

3. Remove all duplicates in the collection of multipliers to emulate the formal n multiplier spectra of ϕ_{σ_2}.

4. If there aren't enough multipliers, choose a different value of σ_2.

5. Otherwise, compute $\sigma_i^{(n)}$ directly by evaluating the ith elementary symmetric polynomial of the collection of multipliers.

Although it is rare, there are cases where we have too few multipliers in our list after we remove duplicates. The first cause of this is that two distinct forward orbits of periodic points of formal period n may have the same multipliers. The other cause is when $\Phi_{\phi_{\sigma_2},n}$ has a higher multiplicity root. These are rare enough that we may simply skip the value of σ_2 corresponding to the deficit.

6 Sage computer algebra system

We used the Sage computer algebra system to compute the polynomials [6]. Sage has many tools for computing iterates of maps, multipliers, and dynatomic polynomials. Progress and goals for Sage development in the areas of arithmetic and complex dynamics is managed in the *Arithmetic and Complex Dynamics wiki* [7].

With any computer algebra system, computing the roots of polynomials of degree greater than four is problematic since most often these roots can only be approximated. Since our algorithm for computing the value of $\sigma_i^{(n)}$ for a given value of σ_2 requires the removal of duplicates in our list of multipliers, we need duplicate roots to be treated as equal by Sage. With only approximations, duplicate roots may be computed differently and thus will not pass equality tests.

A way around this is to treat the roots of a polynomial as intervals in the complex plane. To do this, only the general location within some error bound of the root needs to be known. With higher precision, the intervals are smaller. We can treat these intervals as points themselves and define equality of two points as the intersection of their respective intervals. With high enough precision duplicate roots will be computed accurately enough so that their respective intervals intersect, passing required equality tests.

7 The polynomials

When we defined the n multiplier spectra for the map $\phi(z) = z^2 + c$, it was possible to also define the n multiplier spectra to be the set of multipliers included with appropriate multiplicity for all of the periodic points of ϕ of period n instead of just those of the periodic points of formal period n. The periodic points of period n are the roots of $\phi^n(z) - z$ and the same upper bound holds for the degrees of the corresponding $\sigma_i^{(n)}$ so we can apply a similar algorithm to compute them. We include the results of our computations for both families of polynomials in Appendix A.

8 Future research

One goal for future research could be to compute more of the $\sigma_i^{(n)}$ for general quadratic rational maps by using an interpolation-based algorithm. This would be a more complex task as σ_1 is not constant for general maps. Unless upper bounds for the degrees of the $\sigma_i^{(n)}$ are found, interpolating the $\sigma_i^{(n)}$ would first require finding their degrees as multivariate polynomials in σ_1, σ_2. Additionally, with maps of the form $z^2 + c$ we had the relationship $\sigma_2 = 4c$ allowing us to easily determine the map corresponding to a given value of σ_2, $z^2 + \frac{\sigma_2}{4}$. The relationship for general quadratic rational maps is less straightforward, and is seen by conjugating the maps to their Milnor normal form [2].

References

[1] Selim Berker, Adam L. Epstein, and Kevin M. Pilgrim, *Remarks on the period three cycles of quadratic rational maps*. Nonlinearity. 2003.

[2] John Milnor, *Geometry and dynamics of quadratic rational maps*. Experiment. Math. 1993.

[3] Joseph Silverman, *The arithmetic of dynamical systems*. Graduate Texts in Mathematics, vol. 241, Springer, New York, 2007.

[4] Joseph Silverman, *The space of rational maps on* $\mathbb{P}^1$. Duke Math. J., 1998.

[5] Patrick Morton and Joseph Silverman, *Periodic points, multiplicities, and dynamical units*. J. Reine Angew. Math., 461:81122, 1995.

[6] Stein, William and Joyner, David, *SAGE: System for algebra and geometry experimentation*. Communications in Computer Algebra (SIGSAM Bulletin), 2005, http://www.sagemath.org

[7] *Arithmetic and Complex Dynamics wiki.* http://wiki.sagemath.org/dynamics/ArithmeticAndComplex

Appendix A

The following are the results of our computations for the $\sigma_i^{(n)}$ first using only the multipliers of the periodic points of a given formal period and then using the multipliers of all periodic points of a given period.

A.1 With only the multipliers of the periodic points of a given formal period

For the formal 2 multiplier spectra

$$\sigma_1^{(2)} = \sigma_2 + 4$$

For the formal 3 multiplier spectra

$$\begin{aligned}
\sigma_1^{(3)} &= 2\sigma_2 + 16\\
\sigma_2^{(3)} &= \sigma_2^3 + 8\sigma_2^2 + 16\sigma_2 + 64
\end{aligned}$$

For the formal 4 multiplier spectra

$$\begin{aligned}
\sigma_1^{(4)} &= -\sigma_2^2 + 48\\
\sigma_2^{(4)} &= -\sigma_2^4 - 4\sigma_2^3 + 16\sigma_2^2 + 768\\
\sigma_3^{(4)} &= \sigma_2^6 + 12\sigma_2^5 + 48\sigma_2^4 + 192\sigma_2^3 + 512\sigma_2^2 + 4096
\end{aligned}$$

For the formal 5 multiplier spectra

$$\begin{aligned}
\sigma_1^{(5)} &= -2\sigma_2^2 + 8\sigma_2 + 192\\
\sigma_2^{(5)} &= 3\sigma_2^5 + 12\sigma_2^4 - 96\sigma_2^3 - 128\sigma_2^2 + 1280\sigma_2 + 15360\\
\sigma_3^{(5)} &= -4\sigma_2^7 - 72\sigma_2^6 - 544\sigma_2^5 - 2688\sigma_2^4 - 6656\sigma_2^3 + 4096\sigma_2^2 + 81920\sigma_2 + 655360\\
\sigma_4^{(5)} &= 3\sigma_2^10 + 44\sigma_2^9 + 96\sigma_2^8 - 1280\sigma_2^7 - 10752\sigma_2^6 - 54272\sigma_2^5 - 151552\sigma_2^4 - 49152\sigma_2^3 + 524288\sigma_2^2\\
&\quad + 2621440\sigma_2 + 15728640\\
\sigma_5^{(5)} &= -2\sigma_2^{12} - 56\sigma_2^{11} - 640\sigma_2^{10} - 4224\sigma_2^9 - 20480\sigma_2^8 - 75776\sigma_2^7 - 172032\sigma_2^6 - 229376\sigma_2^5 + 131072\sigma_2^4\\
&\quad + 4718592\sigma_2^3 + 14680064\sigma_2^2 + 41943040\sigma_2 + 201326592\\
\sigma_6^{(5)} &= \sigma_2^{15} + 32\sigma_2^{14} + 448\sigma_2^{13} + 3840\sigma_2^{12} + 24064\sigma_2^{11} + 118784\sigma_2^{10}\\
&\quad + 466944\sigma_2^9 + 1540096\sigma_2^8 + 4521984\sigma_2^7 + 11534336\sigma_2^6\\
&\quad + 27262976\sigma_2^5 + 58720256\sigma_2^4 + 83886080\sigma_2^3 + 134217728\sigma_2^2\\
&\quad + 268435456\sigma_2 + 1073741824
\end{aligned}$$

For the formal 6 multiplier spectra

$$\sigma_1^{(6)} = \sigma_2^3 - 4\sigma_2^2 - 16\sigma_2 + 576$$

$$\sigma_2^{(6)} = -4\sigma_2^6 - 8\sigma_2^5 + 128\sigma_2^4 - 256\sigma_2^3 - 1536\sigma_2^2 - 8192\sigma_2 + 147456$$

$$\begin{aligned}\sigma_3^{(6)} = &-4\sigma_2^9 - 8\sigma_2^8 + 544\sigma_2^7 + 4032\sigma_2^6 + 13824\sigma_2^5 + 6144\sigma_2^4 - 172032\sigma_2^3 - 229376\sigma_2^2\\ &- 1835008\sigma_2 + 22020096\end{aligned}$$

$$\begin{aligned}\sigma_4^{(6)} = &\,6\sigma_2^{12} + 56\sigma_2^{11} - 176\sigma_2^{10} - 512\sigma_2^9 + 25856\sigma_2^8 + 196608\sigma_2^7 + 946176\sigma_2^6 + 2326528\sigma_2^5\\ &- 5439488\sigma_2^4 - 29360128\sigma_2^3 - 14680064\sigma_2^2 - 234881024\sigma_2 + 2113929216\end{aligned}$$

$$\begin{aligned}\sigma_5^{(6)} = &\,6\sigma_2^{15} + 80\sigma_2^{14} - 240\sigma_2^{13} - 7936\sigma_2^{12} - 48384\sigma_2^{11} - 190464\sigma_2^{10} - 200704\sigma_2^9 + 3473408\sigma_2^8\\ &+ 13631488\sigma_2^7 + 28573696\sigma_2^6 + 5242880\sigma_2^5 - 943718400\sigma_2^4 - 2348810240\sigma_2^3 - 18790481920\sigma_2\\ &+ 135291469824\end{aligned}$$

$$\begin{aligned}\sigma_6^{(6)} = &-4\sigma_2^{18} - 88\sigma_2^{17} - 352\sigma_2^{16} + 7168\sigma_2^{15} + 112896\sigma_2^{14} + 873472\sigma_2^{13} + 4739072\sigma_2^{12}\\ &+ 18726912\sigma_2^{11} + 50724864\sigma_2^{10} + 84410368\sigma_2^9 - 40894464\sigma_2^8 - 1291845632\sigma_2^7 - 5603590144\sigma_2^6\\ &- 16777216000\sigma_2^5 - 61740154880\sigma_2^4 - 90194313216\sigma_2^3 + 60129542144\sigma_2^2 - 962072674304\sigma_2\\ &+ 5772436045824\end{aligned}$$

$$\begin{aligned}\sigma_7^{(6)} = &-4\sigma_2^{21} - 120\sigma_2^{20} - 1280\sigma_2^{19} - 3648\sigma_2^{18} + 49920\sigma_2^{17}\\ &+ 751616\sigma_2^{16} + 5902336\sigma_2^{15} + 31801344\sigma_2^{14} + 123273216\sigma_2^{13}\\ &+ 352321536\sigma_2^{12} + 597688320\sigma_2^{11} - 528482304\sigma_2^{10} - 7834959872\sigma_2^9\\ &- 39795556352\sigma_2^8 - 151934468096\sigma_2^7 - 402653184000\sigma_2^6\\ &- 910533066752\sigma_2^5 - 1580547964928\sigma_2^4 - 962072674304\sigma_2^3\\ &+ 3848290697216\sigma_2^2 - 30786325577728\sigma_2 + 158329674399744\end{aligned}$$

$$\begin{aligned}\sigma_8^{(6)} = &\,\sigma_2^{24} + 40\sigma_2^{23} + 656\sigma_2^{22} + 5376\sigma_2^{21} + 15360\sigma_2^{20} - 158720\sigma_2^{19}\\ &- 2629632\sigma_2^{18} - 22069248\sigma_2^{17} - 133431296\sigma_2^{16} - 643563520\sigma_2^{15}\\ &- 2644508672\sigma_2^{14} - 9529458688\sigma_2^{13} - 30903631872\sigma_2^{12} - 92945776640\sigma_2^{11}\\ &- 247765925888\sigma_2^{10} - 575525617664\sigma_2^9 - 1245540515840\sigma_2^8 - 2267742732288\sigma_2^7\\ &- 4191888080896\sigma_2^6 - 7146825580544\sigma_2^5 + 1099511627776\sigma_2^4 + 35184372088832\sigma_2^3\\ &+ 105553116266496\sigma_2^2 - 562949953421312\sigma_2 + 2533274790395904\end{aligned}$$

$$\begin{aligned}\sigma_9^{(6)} = &\,\sigma_2^{27} + 52\sigma_2^{26} + 1248\sigma_2^{25} + 18752\sigma_2^{24} + 202752\sigma_2^{23}\\ &+ 1712128\sigma_2^{22} + 11845632\sigma_2^{21} + 69124096\sigma_2^{20} + 348192768\sigma_2^{19}\\ &+ 1544552448\sigma_2^{18} + 6126829568\sigma_2^{17} + 22053650432\sigma_2^{16}\\ &+ 72913780736\sigma_2^{15} + 222130339840\sigma_2^{14} + 625723047936\sigma_2^{13}\\ &+ 1637456281600\sigma_2^{12} + 3981434683392\sigma_2^{11} + 9208409882624\sigma_2^{10}\\ &+ 20478404067328\sigma_2^9 + 42606075576320\sigma_2^8 + 83562883710976\sigma_2^7\\ &+ 153931627888640\sigma_2^6 + 299067162755072\sigma_2^5 + 492581209243648\sigma_2^4\\ &+ 844424930131968\sigma_2^3 + 1125899906842624\sigma_2^2 - 4503599627370496\sigma_2 + 18014398509481984\end{aligned}$$

A.2 With the multipliers of all periodic points of a given period

For the 2 multiplier spectra

$$\begin{aligned}
\sigma_1^{(2)} &= -\sigma_2 + 8\\
\sigma_2^{(2)} &= -\sigma_2^2 - 4\sigma_2 + 16\\
\sigma_3^{(2)} &= \sigma_2^3 + 4\sigma_2^2
\end{aligned}$$

For the 3 multiplier spectra

$$\begin{aligned}
\sigma_1^{(3)} &= -4\sigma_2 + 24\\
\sigma_2^{(3)} &= 2\sigma_2^3 - 4\sigma_2^2 - 64\sigma_2 + 192\\
\sigma_3^{(3)} &= -4\sigma_2^4 - 24\sigma_2^3 - 32\sigma_2^2 - 256\sigma_2 + 512\\
\sigma_4^{(3)} &= \sigma_2^6 + 8\sigma_2^5 + 16\sigma_2^4 + 64\sigma_2^3
\end{aligned}$$

For the 4 multiplier spectra

$$\begin{aligned}
\sigma_1^{(4)} &= 2\sigma_2^2 - 8\sigma_2 + 80\\
\sigma_2^{(4)} &= -\sigma_2^4 + 4\sigma_2^3 + 48\sigma_2^2 - 512\sigma_2 + 2560\\
\sigma_3^{(4)} &= -4\sigma_2^6 + 16\sigma_2^5 + 336\sigma_2^4 + 64\sigma_2^3 - 768\sigma_2^2 - 12288\sigma_2 + 40960\\
\sigma_4^{(4)} &= -\sigma_2^8 + 8\sigma_2^7 + 240\sigma_2^6 + 1408\sigma_2^5 + 3584\sigma_2^4 - 1024\sigma_2^3\\
&\quad - 28672\sigma_2^2 - 131072\sigma_2 + 327680\\
\sigma_5^{(4)} &= 2\sigma_2^{10} + 24\sigma_2^9 + 32\sigma_2^8 - 448\sigma_2^7 - 2560\sigma_2^6 - 12288\sigma_2^5\\
&\quad - 28672\sigma_2^4 - 16384\sigma_2^3 - 196608\sigma_2^2 - 524288\sigma_2 + 1048576\\
\sigma_6^{(4)} &= \sigma_2^{12} + 20\sigma_2^{11} + 160\sigma_2^{10} + 768\sigma_2^9 + 2816\sigma_2^8 + 7168\sigma_2^7\\
&\quad + 12288\sigma_2^6 + 32768\sigma_2^5 + 65536\sigma_2^4
\end{aligned}$$

For the 5 multiplier spectra

$$\begin{aligned}
\sigma_1^{(5)} &= 8\sigma_2^2 - 32\sigma_2 + 224\\
\sigma_2^{(5)} &= 4\sigma_2^5 - 8\sigma_2^4 + 64\sigma_2^3 + 1408\sigma_2^2 - 6144\sigma_2 + 21504\\
\sigma_3^{(5)} &= 24\sigma_2^7 - 64\sigma_2^6 - 1696\sigma_2^5 + 256\sigma_2^4 + 8192\sigma_2^3 + 102400\sigma_2^2\\
&\quad - 491520\sigma_2 + 1146880\\
\sigma_4^{(5)} &= 6\sigma_2^{10} + 16\sigma_2^9 - 560\sigma_2^8 - 4096\sigma_2^7 - 16896\sigma_2^6 - 15360\sigma_2^5\\
&\quad + 69632\sigma_2^4 + 393216\sigma_2^3 + 3932160\sigma_2^2 - 20971520\sigma_2 + 36700160\\
\sigma_5^{(5)} &= 24\sigma_2^{12} + 192\sigma_2^{11} - 1888\sigma_2^{10} - 22144\sigma_2^9 - 80384\sigma_2^8 - 225280\sigma_2^7\\
&\quad + 221184\sigma_2^6 + 4259840\sigma_2^5 + 2490368\sigma_2^4 + 8388608\sigma_2^3 + 83886080\sigma_2^2 - 503316480\sigma_2\\
&\quad + 704643072\\
\sigma_6^{(5)} &= 4\sigma_2^{15} + 56\sigma_2^{14} + 64\sigma_2^{13} - 1664\sigma_2^{12} - 5120\sigma_2^{11} + 8192\sigma_2^{10}\\
&\quad + 241664\sigma_2^9 + 2146304\sigma_2^8 + 7208960\sigma_2^7 + 19136512\sigma_2^6 + 77594624\sigma_2^5\\
&\quad + 20971520\sigma_2^4 + 67108864\sigma_2^3 + 939524096\sigma_2^2 - 6442450944\sigma_2 + 7516192768\\
\sigma_7^{(5)} &= 8\sigma_2^{17} + 224\sigma_2^{16} + 2592\sigma_2^{15} + 17280\sigma_2^{14} + 80896\sigma_2^{13}\\
&\quad + 272384\sigma_2^{12} + 516096\sigma_2^{11} + 294912\sigma_2^{10} - 1310720\sigma_2^9 - 11534336\sigma_2^8\\
&\quad - 29360128\sigma_2^7 - 92274688\sigma_2^6 - 436207616\sigma_2^5 - 134217728\sigma_2^4 + 4294967296\sigma_2^2\\
&\quad - 34359738368\sigma_2 + 34359738368\\
\sigma_8^{(5)} &= \sigma_2^{20} + 32\sigma_2^{19} + 448\sigma_2^{18} + 3840\sigma_2^{17} + 24064\sigma_2^{16} + 118784\sigma_2^{15}\\
&\quad + 466944\sigma_2^{14} + 1540096\sigma_2^{13} + 4521984\sigma_2^{12} + 11534336\sigma_2^{11}\\
&\quad + 27262976\sigma_2^{10} + 58720256\sigma_2^9 + 83886080\sigma_2^8 + 134217728\sigma_2^7\\
&\quad + 268435456\sigma_2^6 + 1073741824\sigma_2^5
\end{aligned}$$

For the 6 multiplier spectra

$$\begin{aligned}
\sigma_1^{(6)} &= -2\sigma_2^3 + 32\sigma_2^2 - 32\sigma_2 + 832\\
\sigma_2^{(6)} &= -5\sigma_2^6 - 16\sigma_2^5 - 160\sigma_2^4 + 384\sigma_2^3\\
&\quad + 21248\sigma_2^2 - 24576\sigma_2 + 319488\\
\sigma_3^{(6)} &= 12\sigma_2^9 - 160\sigma_2^8 - 32\sigma_2^7 + 13888\sigma_2^6\\
&\quad - 11264\sigma_2^5 - 20480\sigma_2^4 + 679936\sigma_2^3 + 6307840\sigma_2^2 - 8650752\sigma_2 + 74973184\\
\sigma_4^{(6)} &= 9\sigma_2^{12} + 128\sigma_2^{11} - 640\sigma_2^{10} - 5184\sigma_2^9\\
&\quad + 149504\sigma_2^8 + 1377280\sigma_2^7 + 2768896\sigma_2^6 - 2162688\sigma_2^5 + 17629184\sigma_2^4\\
&\quad + 233308160\sigma_2^3 + 1095761920\sigma_2^2 - 1845493760\sigma_2 + 11995709440\\
\sigma_5^{(6)} &= -30\sigma_2^{15} + 192\sigma_2^{14} + 5056\sigma_2^{13} - 34432\sigma_2^{12}\\
&\quad - 543488\sigma_2^{11} - 645120\sigma_2^{10} + 15753216\sigma_2^9 + 119898112\sigma_2^8\\
&\quad + 271122432\sigma_2^7 - 282329088\sigma_2^6 + 60817408\sigma_2^5 + 7008681984\sigma_2^4\\
&\quad + 42278584320\sigma_2^3 + 121802588160\sigma_2^2 - 265751101440\sigma_2 + 1381905727488
\end{aligned}$$

$$\begin{aligned}
\sigma_6^{(6)} &= -5\sigma_2^{18} - 272\sigma_2^{17} - 992\sigma_2^{16} + 38720\sigma_2^{15} \\
&\quad + 217600\sigma_2^{14} - 1652736\sigma_2^{13} - 6823936\sigma_2^{12} + 117391360\sigma_2^{11} \\
&\quad + 1169489920\sigma_2^{10} + 5735710720\sigma_2^{9} + 15774777344\sigma_2^{8} - 5410652160\sigma_2^{7} \\
&\quad - 120930172928\sigma_2^{6} + 84288733184\sigma_2^{5} + 1230508130304\sigma_2^{4} \\
&\quad + 4806068404224\sigma_2^{3} + 8787503087616\sigma_2^{2} - 27212912787456\sigma_2 + 117922622078976 \\
\sigma_7^{(6)} &= 40\sigma_2^{21} + 192\sigma_2^{20} - 11072\sigma_2^{19} - 107392\sigma_2^{18} \\
&\quad + 381440\sigma_2^{17} + 10124288\sigma_2^{16} + 71208960\sigma_2^{15} + 364904448\sigma_2^{14} \\
&\quad + 1820721152\sigma_2^{13} + 10465574912\sigma_2^{12} + 57792266240\sigma_2^{11} \\
&\quad + 205466370048\sigma_2^{10} + 345711312896\sigma_2^{9} - 429496729600\sigma_2^{8} \\
&\quad - 5143760207872\sigma_2^{7} - 12235288084480\sigma_2^{6} + 14912126451712\sigma_2^{5} \\
&\quad + 128436702019584\sigma_2^{4} + 363663470886912\sigma_2^{3} + 380980779024384\sigma_2^{2} \\
&\quad - 2031897488130048\sigma_2 + 7547047813054464 \\
\sigma_8^{(6)} &= -5\sigma_2^{24} + 128\sigma_2^{23} + 5376\sigma_2^{22} + 40320\sigma_2^{21} \\
&\quad - 209152\sigma_2^{20} - 4520960\sigma_2^{19} - 24678400\sigma_2^{18} + 26427392\sigma_2^{17} \\
&\quad + 1790443520\sigma_2^{16} + 18180210688\sigma_2^{15} + 111976382464\sigma_2^{14} \\
&\quad + 500321746944\sigma_2^{13} + 1586352881664\sigma_2^{12} + 2577785683968\sigma_2^{11} \\
&\quad - 4343285678080\sigma_2^{10} - 50117973377024\sigma_2^{9} - 204930069561344\sigma_2^{8} \\
&\quad - 449768975237120\sigma_2^{7} - 377613524664320\sigma_2^{6} + 1400777813786624\sigma_2^{5} \\
&\quad + 8635564324552704\sigma_2^{4} + 18524571904770048\sigma_2^{3} + 5805421394657280\sigma_2^{2} \\
&\quad - 111464090777419776\sigma_2 + 362258295026614272 \\
\sigma_9^{(6)} &= -30\sigma_2^{27} - 608\sigma_2^{26} + 3680\sigma_2^{25} + 203328\sigma_2^{24} \\
&\quad + 2112000\sigma_2^{23} + 6812672\sigma_2^{22} - 34160640\sigma_2^{21} - 331415552\sigma_2^{20} \\
&\quad + 583467008\sigma_2^{19} + 27978366976\sigma_2^{18} + 260767219712\sigma_2^{17} \\
&\quad + 1489485430784\sigma_2^{16} + 5960340865024\sigma_2^{15} + 16322419228672\sigma_2^{14} \\
&\quad + 20760798167040\sigma_2^{13} - 70250632577024\sigma_2^{12} - 600427838046208\sigma_2^{11} \\
&\quad - 2396608931037184\sigma_2^{10} - 6581814042820608\sigma_2^{9} - 12725747579879424\sigma_2^{8} \\
&\quad - 9143538696585216\sigma_2^{7} + 19865976090656768\sigma_2^{6} + 80712949571780608\sigma_2^{5} \\
&\quad + 379569006094319616\sigma_2^{4} + 616430198996336640\sigma_2^{3} - 371546969258065920\sigma_2^{2} \\
&\quad - 4458563631096791040\sigma_2 + 12880294934279618560
\end{aligned}$$

$$\begin{aligned}
\sigma_{10}^{(6)} &= 9\sigma_2^{30} + 208\sigma_2^{29} - 1376\sigma_2^{28} - 104448\sigma_2^{27} \\
&\quad - 1676288\sigma_2^{26} - 13265920\sigma_2^{25} - 25321472\sigma_2^{24} + 768065536\sigma_2^{23} \\
&\quad + 12048793600\sigma_2^{22} + 107183341568\sigma_2^{21} + 695179673600\sigma_2^{20} \\
&\quad + 3505964187648\sigma_2^{19} + 13827596877824\sigma_2^{18} + 40978014535680\sigma_2^{17} \\
&\quad + 75617194213376\sigma_2^{16} - 37736656404480\sigma_2^{15} - 1024353995063296\sigma_2^{14} \\
&\quad - 5221941497561088\sigma_2^{13} - 18865901545717760\sigma_2^{12} - 54299656615624704\sigma_2^{11} \\
&\quad - 123143103287656448\sigma_2^{10} - 204025377650114560\sigma_2^{9} - 116495455986122752\sigma_2^{8} \\
&\quad + 626422560669564928\sigma_2^{7} + 2017612633061982208\sigma_2^{6} + 2913828958908710912\sigma_2^{5}
\end{aligned}$$

$$
\begin{aligned}
&+ 10565444725811183616\sigma_2^4 + 12069647001352929280\sigma_2^3 - 27742173704602255360\sigma_2^2 \\
&- 12682136550675316736 0\sigma_2 + 32973555031755823513 6
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{11}^{(6)} &= 12\sigma_2^{33} + 480\sigma_2^{32} + 5984\sigma_2^{31} - 21184\sigma_2^{30} \\
&- 1450496\sigma_2^{29} - 17976320\sigma_2^{28} - 92917760\sigma_2^{27} + 258981888\sigma_2^{26} \\
&+ 9218031616\sigma_2^{25} + 93886873600\sigma_2^{24} + 631098048512\sigma_2^{23} + 3144104804352\sigma_2^{22} \\
&+ 11778159280128\sigma_2^{21} + 30375820656640\sigma_2^{20} + 24305756798976\sigma_2^{19} \\
&- 271568634642432\sigma_2^{18} - 2056400276553728\sigma_2^{17} - 9574770592972800\sigma_2^{16} \\
&- 34930797219676160\sigma_2^{15} - 104753496435195904\sigma_2^{14} - 259983922434146304\sigma_2^{13} \\
&- 513977705520168960\sigma_2^{12} - 709967852194496512\sigma_2^{11} \\
&- 323625854473076736\sigma_2^{10} + 2253488663545511936\sigma_2^{9} \\
&+ 11961560610296037376\sigma_2^{8} + 35218149086037278720\sigma_2^{7} \\
&+ 62365847639826628608\sigma_2^{6} + 63122452377224871936\sigma_2^{5} \\
&+ 16832653967259965849 6\sigma_2^{4} + 89927877359334064128\sigma_2^{3} \\
&- 86238528544592153804 8\sigma_2^{2} - 24349702177296608133 12\sigma_2 + 57553841509973801041 92
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{12}^{(6)} &= -5\sigma_2^{36} - 256\sigma_2^{35} - 5248\sigma_2^{34} - 43328\sigma_2^{33} \\
&+ 289024\sigma_2^{32} + 13088768\sigma_2^{31} + 200683520\sigma_2^{30} + 2072690688\sigma_2^{29} \\
&+ 16492789760\sigma_2^{28} + 106902847488\sigma_2^{27} + 582085509120\sigma_2^{26} \\
&+ 2718030626816\sigma_2^{25} + 11060195098624\sigma_2^{24} + 39831593811968\sigma_2^{23} \\
&+ 129399848435712\sigma_2^{22} + 389398914924544\sigma_2^{21} + 1128287908659200\sigma_2^{20} \\
&+ 3315577313558528\sigma_2^{19} + 10355475388301312\sigma_2^{18} + 34717354524934144\sigma_2^{17} \\
&+ 120526265613549568\sigma_2^{16} + 410535651579002880\sigma_2^{15} + 1318006578447646720\sigma_2^{14} \\
&+ 3913628076184961024\sigma_2^{13} + 10707026639096643584\sigma_2^{12} + 27073389159937736704\sigma_2^{11} \\
&+ 63356639557848137728\sigma_2^{10} + 136278924724231208960\sigma_2^{9} + 265892521999954083840\sigma_2^{8} \\
&+ 462033292971193925632\sigma_2^{7} + 662929865148937011200\sigma_2^{6} + 719423018874672513024\sigma_2^{5} \\
&+ 112525138849628264857 6\sigma_2^{4} - 103301766812773489049 6\sigma_2^{3} - 138719515434295828152 32\sigma_2^{2} \\
&- 28334198897217871282176\sigma_2 + 61390764277305387778048
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{13}^{(6)} &= -2\sigma_2^{39} - 128\sigma_2^{38} - 3584\sigma_2^{37} - 55040\sigma_2^{36} \\
&- 414464\sigma_2^{35} + 1410048\sigma_2^{34} + 85966848\sigma_2^{33} + 1367965696\sigma_2^{32} \\
&+ 14761328640\sigma_2^{31} + 124932849664\sigma_2^{30} + 878374289408\sigma_2^{29} \\
&+ 5292678971392\sigma_2^{28} + 27886635450368\sigma_2^{27} + 130339573858304\sigma_2^{26} \\
&+ 546423993008128\sigma_2^{25} + 2073014283796480\sigma_2^{24} + 7164593860247552\sigma_2^{23} \\
&+ 22656450702606336\sigma_2^{22} + 65680289157873664\sigma_2^{21} + 174423500973408256\sigma_2^{20} \\
&+ 423288886949576704\sigma_2^{19} + 936172578400108544\sigma_2^{18} + 1880006553822560256\sigma_2^{17} \\
&+ 3402258412245876736\sigma_2^{16} + 5424304276191051776\sigma_2^{15} + 7043629817207455744\sigma_2^{14} \\
&+ 5332261958806667264\sigma_2^{13} - 6449154666394550272\sigma_2^{12} - 43018383640642977792\sigma_2^{11} \\
&- 147862182965828124672\sigma_2^{10} - 428886799713747075072\sigma_2^{9} - 986900807943461011456\sigma_2^{8} \\
&- 1715547198854988300288\sigma_2^{7} - 885443715538058477568\sigma_2^{6} + 2951479051793528258560\sigma_2^{5}
\end{aligned}
$$

$$
\begin{aligned}
&- 1180591620717411303424\sigma_2^4 - 18889465931478580854784\sigma_2^3 - 94447329657392904273920\sigma_2^2 \\
&- 151115727451828646838272\sigma_2 + 302231454903657293676544 \\
\sigma_{14}^{(6)} &= \sigma_2^{42} + 80\sigma_2^{41} + 3040\sigma_2^{40} + 73536\sigma_2^{39} \\
&+ 1281792\sigma_2^{38} + 17286144\sigma_2^{37} + 188989440\sigma_2^{36} + 1730805760\sigma_2^{35} \\
&+ 13596491776\sigma_2^{34} + 93260349440\sigma_2^{33} + 566373646336\sigma_2^{32} + 3080070365184\sigma_2^{31} \\
&+ 15142444072960\sigma_2^{30} + 67852832866304\sigma_2^{29} + 279114355310592\sigma_2^{28} \\
&+ 1060504734793728\sigma_2^{27} + 3741118378278912\sigma_2^{26} + 12304772065394688\sigma_2^{25} \\
&+ 37861270585606144\sigma_2^{24} + 109304924818374656\sigma_2^{23} + 296963797011136512\sigma_2^{22} \\
&+ 761763645955768320\sigma_2^{21} + 1851577581174784000\sigma_2^{20} + 4278982595955392512\sigma_2^{19} \\
&+ 9425752545109737472\sigma_2^{18} + 19824845559684923392\sigma_2^{17} + 39865863901483630592\sigma_2^{16} \\
&+ 76651265657845841920\sigma_2^{15} + 140224077997807763456\sigma_2^{14} + 241248824838982729728\sigma_2^{13} \\
&+ 386228704043293736960\sigma_2^{12} + 581072438321850875904\sigma_2^{11} + 903890459611768029184\sigma_2^{10} \\
&+ 1623313478486440542208\sigma_2^{9} + 2951479051793528258560\sigma_2^{8} + 4722366482869645213696\sigma_2^{7} \\
&+ 4722366482869645213696\sigma_2^{6}
\end{aligned}
$$