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Determining the Length of a One-Dimensional Bar

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July 8, 2004

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Abstract

In this paper we examine the inverse problem of determining the length of a one-dimensional bar from thermal measurements (temperature and heat flux) at one end of the bar (the “accessible” end); the other inaccessible end of the bar is assumed to be moving. We develop two different approaches to estimating the length of the bar, and show how one approach can also be adapted to find unknown boundary conditions at the inaccessible end of the bar.

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1 Introduction

In this paper we will study the problem of determining the time-dependent length of a one-dimensional bar from thermal measurements at one end of the bar. The interest for this problem originates from monitoring blast furnaces. We can regard the blast furnace as a large container filled with molten metal. Due to various chemical or mechanical reactions the thickness of the walls of the blast furnace can change with time. For example, corrosive effects can cause the walls to become thinner, and similarly, deposits on the walls can result in increasing thickness. Since the extreme thinning of the walls can be hazardous, it is important to know the thickness of the walls at all times. However, for practical and economical reasons, we cannot directly measure the thickness of the walls, and therefore an approximation of the thickness is desired. We may attempt estimation of the wall thickness or inner wall profile using measurements of the outer wall temperature and heat flux, and knowledge of the inner wall temperature (determined by the temperature of the molten interior, assumed known). This results in an inverse problem for the heat equation.

Such problems have been considered using traditional optimization-based fit-to-data approaches (see [4]), or by converting the problem to a version of the so-called “sideways heat equation” (see [2]). The situation is not entirely unlike coefficient recovery problems for parabolic equations, e.g., see [1] and the references therein.

In this paper we consider a simplified version of the general problem, in which the domain of interest is a one-dimensional bar. We also show how one approach can be used to determine the governing boundary conditions at some inaccessible portion of the domain.

2 Forward Problem

For the one-dimensional version of the problem, we use x to indicate the position along the bar, with $x = 0$ the accessible left end of the bar and $x = L(t)$ the inaccessible right end of the bar. Physically, $x = 0$ is the outer portion of the furnace wall and $x = L(t)$ the inner portion, in contact with the molten interior of the furnace. We use t to denote time, and $u(t, x)$ to represent the temperature of the bar at time t and position x . After appropriate rescaling we assume that $u(t, x)$ satisfies the usual heat equation

$$u_t - u_{xx} = 0, \quad 0 < x < L(t), \quad t > 0 \tag{1}$$

$$-u_x(t, 0) = g(t), \tag{2}$$

$$u(t, L(t)) = A(t), \tag{3}$$

$$u(0, x) = 0. \tag{4}$$

Here $g(t)$ is the outward heat flux at $x = 0$, assumed known (perhaps even capable of being controlled to some extent). We use $A(t)$ for the temperature at the right end of the

bar and assume that $A(t)$ is known (and typically we assumed this is constant). Note we are using zero initial conditions, although this is not essential.

We also assume that $u(t, 0)$ can be measured, and we use $a(t) = u(t, 0)$ to denote the temperature “data” at the left end.

It should be noted that for any given continuous functions $g(t)$ and $A(t)$, the equations (1)-(4) possess a unique solution $u(t, x)$, say in the space of continuous functions.

3 Method 1: A Special Identity and Test Functions

In the following subsections we illustrate how a certain identity, together with carefully chosen “test functions”, can be used to extract information about the inaccessible end of the bar from Cauchy data at the accessible end of the bar. We begin with an illustration of how one can recover the length of the bar, under the assumption that the length is constant. We then illustrate how one can recover boundary condition at the inaccessible end, using the same identity. Finally, we illustrate how one can approximately determine the changing length of the bar, at least providing certain assumptions are met.

3.1 The Identity

In this section we illustrate our first method for recovering the function $L(t)$, as well as recovering unknown boundary conditions at the right end of the bar. For recovery of $L(t)$ we assume that $L(t)$ varies by a “small” amount around some fixed length L_0 .

In fact, for the moment let us assume that L is constant. If $\psi(t, x)$ satisfies the backward heat equation $\psi_t + \psi_{xx} = 0$ for $x \in (0, L)$ and $t > 0$ then

$$\begin{aligned} - \int_0^T (\psi(t, 0)g(t) + \psi_x(t, 0)a(t)) dt + \int_0^L \psi(T, x)u(T, x) dx \\ = \int_0^T (\psi(t, L)u_x(t, L) - \psi_x(t, L)u(t, L)) dt \end{aligned} \tag{5}$$

as shown in the Appendix. Recall that $a(t) = u(t, 0)$ and $g(t) = -u_x(t, 0)$.

This identity will be used to help solve our inverse problem. With suitably chosen test functions for ψ , it allows us to extract information about $u(t, L)$ and $u_x(t, L)$ from knowledge of $u(t, 0)$, $u_x(t, 0)$, $u(T, x)$.

3.2 Illustration 1: Recovering Constant L

As a simple illustration of how equation (5) can be used to recover information about the other end of the bar, we consider the special case in which L is an unknown *constant*. In this section we will illustrate how one can determine L from knowledge of $a(t)$ and $g_0(t)$ by using suitably chosen test functions in equation (5).

First note, however, that we typically don't have data for $u(T, x)$, $0 < x < L$, so the last integral on the left in equation (5) presents a problem. But for simplicity we will make the assumption that $A(t) \equiv 0$ at the right end of the bar—note that by rescaling this is entirely equivalent to $A(t)$ equal to any constant at the right end. In this case it's reasonable to approximate $u(T, x) \approx 0$, at least for T large, and assuming that g decays to zero (a more general approximation is possible, however, especially if g does not decay). In this case the last integral on the left in equation (5) vanishes, and the entire left side of equation (5) may be considered known.

We now define special test functions which allow us to extract information about L from left side bar data. Specifically, suppose some function $\psi(t, x)$ satisfies the backwards heat equation $\psi_t(t, x) + \psi_{xx}(t, x) = 0$ and also that $\psi(t, L) = 0$ (we treat L here as some unknown constant). One can easily check that the functions

$$\begin{aligned} \phi_\alpha(t, x) = & \frac{1}{2\sqrt{2\alpha}} \left(-e^{-\sqrt{\frac{\alpha}{2}}(x-L)} \cos(\alpha t + \sqrt{\frac{\alpha}{2}}(x-L)) + e^{\sqrt{\frac{\alpha}{2}}(x-L)} \cos(\alpha t - \sqrt{\frac{\alpha}{2}}(x-L)) \right. \\ & \left. + e^{-\sqrt{\frac{\alpha}{2}}(x-L)} \sin(\alpha t + \sqrt{\frac{\alpha}{2}}(x-L)) - e^{\sqrt{\frac{\alpha}{2}}(x-L)} \sin(\alpha t - \sqrt{\frac{\alpha}{2}}(x-L)) \right) \end{aligned} \quad (6)$$

satisfy these conditions for any real α , and so are legitimate choices for ψ in equation (5). Note also that $\frac{\partial \phi_\alpha}{\partial x}(t, L) = \cos(\alpha t)$.

Note that we cannot take $\alpha = 0$ in equation (6), but one can easily check that as α approaches zero the function $\phi_\alpha(t, x)$ approaches $\phi_0(t, x) = x - L$, which itself satisfies the backward heat equation with $\phi_0(t, L) = 0$.

To approximate the length of the one-dimensional bar (or the thickness of the blast furnace wall) we will use the identity (5) and our special class of test functions (6). We also make use of the assumption that L is a constant, that $A(t) = 0$, and use the approximation that $u(T, x) \approx 0$ for reasonably large T . With this we find that, with $\psi = \phi_\alpha$ in equation (5), we obtain

$$\int_0^T (\phi_\alpha(t, 0)g(t) + \frac{\partial \phi_\alpha}{\partial x} \phi_\alpha(t)) dt = 0. \quad (7)$$

Note that the left side of above is computable from the measured boundary data $a(t)$ and $g(t)$.

Therefore, for the correct value of L , the left side of the above equation should be equal to zero. To estimate L we choose some $T > 0$ and some α , compute the integral on the left in equation (7) using the trapezoidal rule for multiple values of L , until we get a result that is approximately zero (actually, we use an intelligent root-finding scheme based on bisection). This value of L will be our approximation of the actual length.

Here is an example of the procedure. In this case the true value for the length of the bar is 1.32. We use input flux $g(t) = e^{-t}$ on the left, $u(t, L) = 0$ on the right, zero initial data; under these conditions the solution u does in fact decay to zero in time, over the entire bar. We use equation (7) with $\alpha = 0.1$, with the trapezoidal rule to evaluate the integral. We vary the value of L in the definition (6) from $L = 0$ to $L = 2$ and graph the

right side of (7).

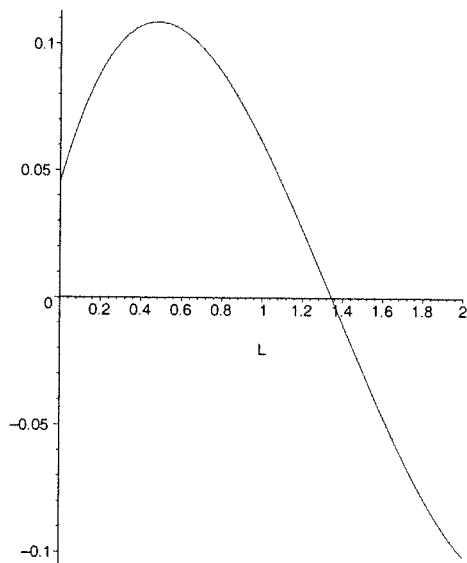


Figure 1: Approximation of L .

The graph crosses the axis at very nearly the exact value, $L = 1.32$.

We have not yet determined under what conditions such an approach is guaranteed to uniquely determine L .

3.3 Illustration 2: Finding Unknown Boundary Conditions at $x = L$

In this section we illustrate how the identity (5) can be used, with suitably chosen test functions, to recover an unknown boundary condition at the right end of the bar.

In what follows we assume that L is a known constant, that $u(t, x)$ satisfies the heat equation for $0 < x < L$, $t > 0$, with measured boundary data $u(t, 0) = a(t)$, $-u_x(t, 0) = g(t)$ at the left end of the bar an unknown boundary condition of the form

$$u_x(t, L) = K(t)(A(t) - u(t, L)) \quad (8)$$

so the heat flux at the right end of the bar satisfies a variation of Newton's law of cooling. Here $A(t)$ is some possibly unknown time-dependent temperature of the molten metal and $K(t)$ is some possibly unknown constant of proportionality that relates to how fast heat travels through the material of the wall. From these conditions, our goal is to determine the values of both $K(t)$ and/or $A(t)$ over some time range.

There are many variations of the problem which one can consider— $K(t)$ unknown and $A(t)$ known, or $K(t)$ known and $A(t)$ unknown, etc.—but a similar approach works for

each. As an illustration we'll consider the case in which K and A are unknown *constants*, both to be estimated from measurements at the left end of the bar.

It can easily be shown that if K is not constant and $u(t, L)$ is not constant in time then both K and A can be found and are unique, for if we have values for $u(t, L)$ and $u_x(t, L)$ at two different times t_1 and t_2 , then we can solve for K and A independently using the following equations

$$K = \frac{u_x(t_2, L) - u_x(t_1, L)}{u(t_1, L) - u(t_2, L)} \quad (9)$$

$$A = \frac{u_x(t_2, L)u(t_1, L) - u_x(t_1, L)u(t_2, L)}{u_x(t_2, L) - u_x(t_1, L)} \quad (10)$$

Assuming that K is not zero (and noise-free data) we see that the numerator on the right in equation (9) is not zero, and hence so is the denominator on the right in equation (10), so A is recovered.

We will assume there are zero initial conditions and that we have measurements of $u(t, 0) = a(t)$ and $-u_x(t, 0) = g(t)$ for a given time interval $t \in (0, T)$. We will first use the identity (5) with the ψ as our special $\phi_\alpha(t, x)$ test functions defined by equation (6), to solve for $u(t, L)$ for $0 < t < T$. In fact, with $\alpha = \alpha_j = \frac{\pi j}{T}$, $j = 0, 1, 2, \dots$ and using the facts that using $\phi_\alpha(t, L) = 0$ and $\frac{\partial \phi_\alpha}{\partial x}(t, L) = \cos(\alpha t)$ we have

$$-\int_0^T (\phi_{\alpha_j}(t, 0)g(t) + \frac{\partial \phi_{\alpha_j}}{\partial x}(t, 0)a(t)) dt = \int_0^T \cos(\frac{j\pi}{T}t)u(t, L) dt \quad (11)$$

where we have again assumed that the integral for $x \in (0, L)$ is negligible.

Let a_j denote the left side of equation (11), which is entirely computable from measured boundary data. Now recall that the Fourier series of a function $f(t)$ is defined as

$$f(t) = \sum_{j=0}^{\infty} a_j \cos(\frac{j\pi t}{T})$$

where

$$a_j = \frac{2}{T} \int_0^T \cos(\frac{j\pi t}{T})f(t)dt, \quad j = 1, 2, 3, \dots$$

$$a_0 = \frac{1}{T} \int_0^T f(t)dt$$

Therefore, the right side of (11) times $\frac{2}{T}$ for $j \geq 1$, or $\frac{1}{T}$ for $j = 0$ gives minus the Fourier coefficients of $u(t, L)$. We can thus reconstruct $u(t, L)$ for $0 < t < T$.

Similarly, we can find the Fourier coefficients of $u_x(t, L)$, by using a slightly different class of test functions. Specifically, define

$$\psi_\alpha(t, x) = \frac{1}{2}e^{-\sqrt{\frac{\alpha}{2}}(x-L)} \cos(\alpha t + \sqrt{\frac{\alpha}{2}}(x-L)) + \frac{1}{2}e^{-\sqrt{\frac{\alpha}{2}}(L-x)} \cos(\alpha t + \sqrt{\frac{\alpha}{2}}(L-x)) \quad (12)$$

It's easy to check that the ψ_α satisfy the backwards heat equation and also that $\frac{\partial \psi_\alpha}{\partial x}(t, L) = 0$. We also have that $\psi_\alpha(t, L) = \cos(\alpha t)$ for any real constant α . Using these in identity (5) with $\alpha = \alpha_j = \frac{j\pi}{T}$, $j = 0, 1, 2, \dots$ produces

$$\int_0^T (\psi_{\alpha_j}(t, 0)g(t) + \frac{\partial \psi_{\alpha_j}}{\partial x}(t, 0)a(t)) dt = \int_0^T \cos(\frac{j\pi}{T}t)u_x(t, L) dt \quad (13)$$

Having recovered $u_x(t, L)$ for $0 < t < T$, we need only choose two times t_1 and t_2 where $u(t_1, L) \neq u(t_2, L)$ and use equations (10) and (9) to estimate A and K . It seems likely that one would want to choose times for which the difference $u(t_1, L) - u(t_2, L)$ is as large in magnitude as possible, to stabilize the procedure.

Below is an example, in which the bar has length 1. The input flux is $g(t) = e^{-t}$, zero initial condition, and at the right the boundary condition is $u_x(t, 1) = K(A - u(t, 1))$ with $K = 0.54$ and $A = 0$. With 15 Fourier modes for $0 < t < 15$, the reconstructed temperature and flux are shown in Figure 3.3. Using these reconstructed functions with $t_1 = 1$ and $t_2 = 6$ in equations (10) and (9) produces estimates $k = 0.544$ and $A = 0.003$, quite good.

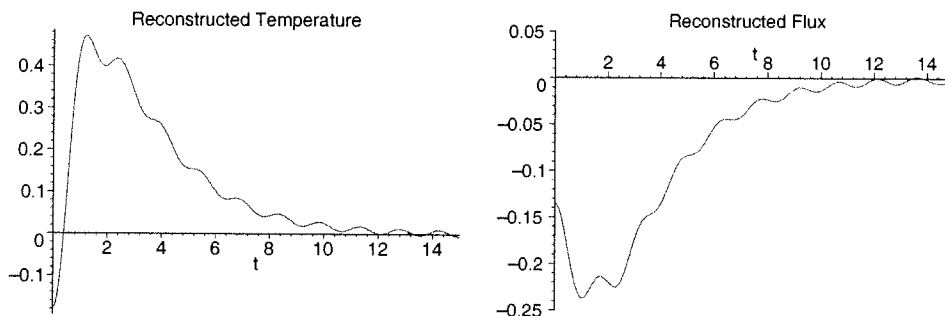


Figure 3: Reconstruction of temperature and flux.

It's also quite easy to see that if A and/or K are time-dependent, but one know either one, then the other can be recovered using this procedure, for $t \in (0, T)$.

3.4 Illustration 3: Recovering $L(t)$

We now consider the full problem of recovering a time-dependent length $L(t)$ on some interval $0 < t < T$. The particular approach we use makes the assumptions that $L(t)$ varies by only a small amount about a known constant value, and that the rate of variation is small. The central idea is the use of a slight variation of identity (5), again with appropriate test functions.

The fact that $u(t, x)$ is defined by a PDE (1) on a variable domain presents some difficulty, so we'll make a change of variable to fix the spatial domain. Define a function

$v(t, y) = u(t, x)$ with $y = \frac{x}{L(t)}$ (so $v(t, y) = u(t, L(t)y)$). It's simple to check that $v(t, y)$ satisfies

$$v_t(t, y) - \frac{yL'(t)}{L(t)}v_y(t, y) - \frac{1}{L^2(t)}v_{yy}(t, y) = 0, \quad 0 < y < 1, \quad t > 0 \quad (14)$$

$$-v_y(t, 0) = L(t)g(t), \quad (15)$$

$$v(t, 1) = A(t), \quad (16)$$

$$v(0, y) = 0 \quad (17)$$

with $v(t, 1) = a(t)$. Note that $v(t, y)$ lives on the fixed domain $0 < y < 1$, but $L(t)$ now appears in the PDE itself.

We now perform a linearization of the problem with respect to $L(t)$, by supposing that $L(t) = L_0 + q(t)$, where L_0 is some known initial constant and $q(t)$ is some function which is small, say as measured in supremum norm. For simplicity we'll consider the case $L_0 = 1$. Now let $v(t, y) = V(t, y) + r(t, y)$, where $V(t, y)$ satisfies the heat equation and is the solution to (14)-(17) when $q(t) = 0$ (so V is the solution to the heat equation on the constant length bar). The function $r(t, y)$ is the perturbation caused by $q(t)$. Since we are assuming $q(t)$ is small, we expect that $r(t, y)$ is also small in magnitude.

Now go back to the equations that $v(t, y)$ satisfies. With $A(t) = 0$ and zero initial conditions we get that $V(t, y)$ satisfies $-V_y(t, 0) = g(t)$ and $V(t, 1) = 0$ as boundary conditions. By substituting $v(t, y) = V(t, y) + r(t, y)$ into equations (14)-(17) we obtain

$$\begin{aligned} V_t(t, y) + r_t(t, y) &- \frac{yq'(t)}{1+q(t)}(V_y(t, y) + r_y(t, y)) \\ &- \frac{1}{(1+q(t))^2}(V_{yy}(t, y) + r_{yy}(t, y)) = 0, \end{aligned} \quad (18)$$

$$-(V_y(t, 0) + r_y(t, 0)) = (1+q(t))g(t), \quad (19)$$

$$V(t, 1) + r(t, 1) = 0, \quad (20)$$

$$V(0, y) + r(0, y) = 0 \quad (21)$$

and also that $V(t, 0) + r(t, 0) = a(t)$.

Using the fact that $V(t, y)$ satisfies equations (18)-(21) with $q(t) \equiv 0$, we can simplify the above equations to obtain the boundary conditions of $r(t, y)$. Equations (19)-(21) reduce to

$$-r_y(t, 0) = q(t)g_0(t) \quad (22)$$

$$r(t, 1) = 0 \quad (23)$$

$$r(0, y) = 0 \quad (24)$$

Note also that

$$r(t, 0) = a(t) - V(t, 0). \quad (25)$$

Now look at equation (18)—it's clear that the relation between r and q is nonlinear, so we will linearized this relationship. Specifically, notice that the term $\frac{1}{1+q(t)}$ can be

expressed as the Taylor series $1 - q(t) + q^2(t) - q^3(t) + \dots$, and since $q(t)$ is assumed to be small we can drop off the terms $q^2(t)$ and higher, leaving us with

$$\frac{1}{1 + q(t)} \approx 1 - q(t)$$

Similarly, we can reduce the term $\frac{1}{(1+q(t))^2}$ in (18) to obtain

$$\frac{1}{(1 + q(t))^2} \approx 1 - 2q(t).$$

Putting this back into equation (18) gives us

$$V_t(t, y) + r_t(t, y) - yq'(t)(1 - q(t))(V_y(t, y) + r_y(t, y)) - (1 - 2q(t))(V_{yy}(t, y) + r_{yy}(t, y)) = 0$$

Now we linearize by assuming that $q(t)$, and hence $r(t, y)$, are small. Moreover, we also suppose that $q'(t)$ is small (the bar does not change length rapidly), so we can drop out any second order and higher terms to get

$$V_t(t, y) + r_t(t, y) - yq'(t)V_y(t, y) - V_{yy}(t, y) - r_{yy}(t, y) + 2q(t)V_{yy}(t, y) = 0$$

Using the fact that $V(t, y)$ satisfies the heat equation ($V_t(t, y) - V_{yy}(t, y) = 0$) and rearranging the above equation we obtain

$$r_t(t, y) - r_{yy}(t, y) = yq'(t)V_y(t, y) - 2q(t)V_{yy}(t, y) \quad (26)$$

Here $V(t, y)$, $g(t)$, and $a(t)$ are all known and our goal is to find $q(t)$. Equation (26), together with boundary/initial conditions (22)-(24) and “measured” data in equation (25) constitute the linearized version of the problem.

Now let ψ be a function which satisfies the backward heat equation for $0 < x < 1$ and $t > 0$. In the Appendix we prove

$$\begin{aligned} - \int_0^T (\psi(t, 0)q(t)g(t) + \psi_y(t, 0)(a(t) - V(t, 0))) dt = \\ - \int_0^L \psi(T, y)r(T, y)dy + \int_0^T \psi(t, L)r_y(t, L) dt \\ + \int_0^T \int_0^L \psi(t, y)(yq'(t)V_y(t, y) - 2q(t)V_{yy}(t, y)) dy dt \quad (27) \end{aligned}$$

which is merely a slight variation of equation (5). As before, we will assume the term $-\int_0^L \psi(T, y)r(T, y)dy$ decays to zero, which is reasonable if either $q(t)$ or the input flux $g(t)$ decays. Also we will use the fact that we will choose our special $\psi(t, y)$ function such that $\psi(t, 1) = 0$. This leaves us with

$$\begin{aligned} - \int_0^T (\psi(t, 0)q(t)g(t) + \psi_y(t, 0)(a(t) - V(t, 0))) dt \\ = \int_0^T \int_0^L \psi(t, y)(yq'(t)V_y(t, y) - 2q(t)V_{yy}(t, y)) dy dt \end{aligned}$$

Now we invoke the assumption that $L(t)$ is “slowly” varying, so that $q'(t)$ is small, and we can drop the term $\int_0^T \int_0^L \phi(t, y) y q'(t) V_y(t, y) dy dt$ from the above equation to get

$$\int_0^T (\psi(t, 0)q(t)g(t) + \psi_y(t, 0)(a(t) - V(t, 0)))dt = 2 \int_0^T \int_0^L \psi(t, y)q(t)V_{yy}(t, y) dy dt$$

Also notice that since $V(t, y)$ satisfies the heat equation ($V_t - V_{yy} = 0$), then $V_{yy} = V_t$.

A little rearrangement of the above equation yields

$$\int_0^T q(t) \left(\psi(t, 0)g(t) - 2 \int_0^L \psi(t, y)V_t(t, y)dy \right) dt = - \int_0^T \psi_y(t, 0)(a(t) - V(t, 0)) dt. \quad (28)$$

Notice that we can rewrite (28) in the form

$$\int_0^T M(t)q(t)dt = \int_0^T D(t) dt \quad (29)$$

where

$$M(t) = \psi(t, 0)g(t) - 2 \int_0^L \psi(t, y)V_t(t, y) dy$$

and

$$D(t) = -\psi_y(t, 0)(a(t) - V(t, 0)).$$

Differentiating both sides of equation (29) with respect to T shows that $M(t)q(t) = D(t)$. Therefore, we can approximate

$$q(t) = \frac{D(t)}{M(t)} \quad (30)$$

Hence, we now have a way to find $q(t)$. Everything in (30) is known and can be computed for any chosen test function which satisfies the backward heat equation with $\psi(t, 1) = 0$.

In the reconstruction below in Figure 2 we use the test functions defined by equation (6). The actual length of the bar is give by $L(t) = 1 + 0.2 \sin(\frac{2}{3}t)$ (so $q(t) = 0.2 \sin(\frac{2}{3}t)$), which we attempt to recover from the data $g(t)$ and $a(t)$ for $0 < t < 10$ using measurements of $a(t)$ and $g(t)$ at time $t_i = i/10$, $0 \leq i \leq 100$. The boundary condition we used is $g(t) = e^{-t}$ as the input flux on the left end of the bar, and $u(t, 1) = 0$ on the right (so the solution does indeed decay to zero as we approximated in our derivation). For a small α in $\phi_\alpha(t, x)$, we obtain a relatively good approximation of $q(t)$. In this case we used $\alpha = 0.1$. Therefore our time dependent length of the bar is equivalent to $1 + q(t)$ for any given time in the interval 0 to T.

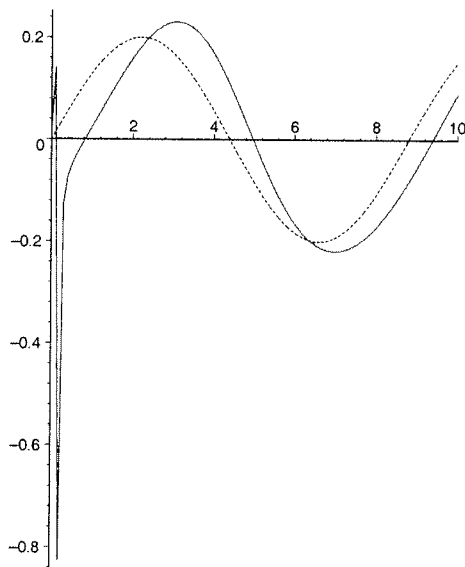


Figure 2: Approximation of $q(t)$.

In the above figure, the dashed curve is the actual $q(t)$, while the solid line represents our approximation of the actual $q(t)$.

4 The Integral Equation Approach for Estimating $L(t)$

4.1 Representation of Solution via Neumann Kernel

The problem (1)-(4) has a unique solution that can be found using different approaches. One method for solving the forward problem involves representing the solution via the Neumann Kernel for this PDE and boundary condition. We will extensively use the

fundamental solution to the heat equation (the Green's Function)

$$G(t, x) = \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}}$$

and the Neumann kernel for the problem (1)-(4)

$$N(t, x) = G(t, x) + \sum_{k=1}^{\infty} (-1)^k (G(t, x + 2Lk) + G(t, x - 2Lk)).$$

One can check that at $t = 0$, $x = 0$, $N(t, x)$ has the same singularity as $G(t, x)$, and also that $N_x(t, 0) = 0$ for all $t > 0$, and $N(0, x) = N(t, L) = 0$ for all $t > 0$. As a consequence, we can write

$$\frac{1}{2}a(t) = \int_0^t N(t-s, 0)g(s)ds - \int_0^t N_x(t-s, L)A(s)ds.$$

See [3], chapter 7, section 1(c) for the details.

For simplicity in what follows we'll consider $A(t) = 0$ and define

$$N_0(t) = N(t, 0) = \frac{1}{\sqrt{\pi t}} \left(\frac{1}{2} + \sum_{k=1}^{\infty} (-1)^k e^{-\frac{L^2 k^2}{t}} \right)$$

to get a simplified identity

$$\frac{1}{2}a(t) = \int_0^t N_0(t-s)g(s)ds \tag{31}$$

Equation (31) will form the basis of our analysis.

4.2 Change of time variables

In section 3.4 we performed a change of variables $y = x/L(t)$. To further simplify things, we will perform a change of time variables as well. Suppose $\tau = \phi(t)$ for some function ϕ . We will also define $w(\tau, y) = w(\phi(t), y) = v(t, y)$, where $v(t, y)$ satisfies equation (14), so that

$$v_t = \phi'(t)w_\tau, \quad v_y = w_y, \quad v_{yy} = w_{yy}.$$

From equation (14), we find that w satisfies

$$\phi'(t)w_\tau - \frac{yL'(t)}{L(t)}w_y - \frac{1}{L^2}w_{yy} = 0, \quad 0 < y < 1, \quad \tau > 0.$$

Now we choose $\phi(t)$ such that $\phi'(t) = \frac{1}{L^2(t)}$ and $\phi(0) = 0$, i.e., let

$$\phi(t) = \int_0^t \frac{1}{L^2(s)} ds \tag{32}$$

Note that ϕ is strictly increasing and, therefore invertible. We can rewrite the above PDE substituting for $\phi(t)$ and then multiply both sides by L^2

$$w_\tau - yL'(t)L(t)w_y - w_{yy} = 0, \quad 0 < y < 1, \quad \tau > 0.$$

If we let $M(\tau) = L'(t)L(t) = L'(\phi^{-1}(\tau))L(\phi^{-1}(\tau))$, then we have the following PDE

$$w_\tau - yM(\tau)w_y - w_{yy} = 0, \quad 0 < y < 1, \quad \tau > 0 \quad (33)$$

with boundary conditions

$$\begin{aligned} -w_y(\tau, 0) &= L(\phi^{-1}(\tau))g(\phi^{-1}(\tau)) \\ w(\tau, 1) &= A(\phi^{-1}(\tau)), \end{aligned}$$

measured data $w(\tau, 0) = a(\phi^{-1}(\tau))$, and initial condition $w(0, y) = 0$. For simplicity, let's assume that L changes relatively slowly, that is, $L' \approx 0$. Then $M(\tau) \approx 0$, the second term in equation (33) can be dropped, and w now satisfies the usual heat equation. If again we assume that $A(\phi^{-1}(\tau))$ is identically zero, and that $L(0) = 1$ (our initial length of the bar), we can use the integral equation formula to obtain

$$\frac{1}{2}a(\phi^{-1}(\tau)) = \int_0^\tau N_0(\tau - s)L(\phi^{-1}(s))g(\phi^{-1}(s))ds \quad (34)$$

We can simplify equation (34) a bit. Let $r = \phi^{-1}(s)$, so that $s = \phi(r)$, and $ds = \phi'(r)$. Also note $\tau = \phi(t)$, and note that $L(r) = \frac{1}{\phi'(r)}$. Then equation (34) becomes

$$\frac{1}{2}a(t) = \int_0^t N_0(\phi(t) - \phi(r))\sqrt{\phi'(r)}g(r)dr \quad (35)$$

In the above integral equation, we know the values for $a(t)$ and $g(t)$ on some interval $0 < t < T$ (we can measure the discrete temperature data on the left end of the bar and we control the heat input). Our goal is to find $\phi(t)$, from which we can recover $L(t) = 1/\sqrt{\phi'(t)}$ from equation (32).

4.3 Discretization and Recovery of $L(t)$

Since we cannot solve equation (35) for $\phi(t)$ explicitly, we shall use numerical approximation. Let $t_i = ih$ and $\phi_i = \phi(t_i)$ for some small stepsize h . Let $t = Mh$, also note that $\phi(0) = 0$, $\phi' > 0$, $\phi'(0) = 1/L^2(0)$. Now, using the trapezoidal rule and a centered difference for ϕ' , we get the following discretized version of equation (35):

$$\begin{aligned} \frac{1}{2}a(t_M) &= \frac{h}{2} \frac{N_0(\phi_M)}{L(0)} g(0) + \sum_{i=1}^{M-2} N_0(\phi_M - \phi_i) g(t_i) \sqrt{h(\phi_{i+1} - \phi_{i-1})/2} \\ &+ \frac{1}{2} N_0(\phi_M - \phi_{M-1}) g(t_{M-1}) \sqrt{h(\phi_M - \phi_{M-2})/2} \\ &+ g(t_M) \sqrt{\frac{h}{\pi}} \end{aligned} \quad (36)$$

The last term in the above equation needs special explanation. Since we can't evaluate $N_0(0)$, the trapezoidal rule works only from $r = 0$ to $r = t - h$, thus on the last subinterval $t - h < r < t$ we have to do something different to approximate the integral. For small enough h we can get good results by just using a linear approximation $\phi(r) = D(r - (M - 1)h) + \phi_{M-1}$ with $D = \frac{\phi_M - \phi_{M-1}}{h} \approx \phi'(t)$. Also, approximate $N_0(t) \approx \frac{1}{2\sqrt{\pi t}}$ since for small t we have that $\sum_{k=1}^{\infty} (-1)^k e^{-\frac{L^2 k^2}{t}} \approx 0$. Therefore, the last term turns out to be

$$\int_{t-h}^t N_0(\phi(t) - \phi(r)) \sqrt{\phi'(r)} g(r) dr$$

$$\approx \int_{t-h}^t \frac{1}{2\sqrt{\pi(\phi_M - \phi_{M-1} - D(r - (M - 1)h))}} \sqrt{D} g(t_M) dr = \sqrt{\frac{h}{\pi}} g(t_M)$$

where $\phi_M \approx \phi(t)$.

Now we know that $\phi_0 = 0$ and we let $\phi_1 = h$ (this is a reasonable extrapolation since the $\phi'(0) = 1/L^2(0) = 1$). To find ϕ_2 , consider equation (36) in the case $M = 2$. The resulting equation implicitly determines the value of ϕ_2 (in terms of ϕ_0 and ϕ_1). We can find ϕ_2 using a standard bisection method. Having found ϕ_2 we then consider equation (36) for $M = 3$, which implicitly determines ϕ_3 in terms of the previous found quantities ϕ_0, ϕ_1, ϕ_2 , and of course the data. We can clearly continue this procedure, using root bracketing and bisection methods, to approximate all values ϕ_i for as long as we have data $a(t)$ and $g(t)$. Once we know these values, we can use the definition of $\phi(t)$ to recover $L(t)$.

Below are examples in which the bar length is given by $L(t) = 1 + 0.2 \sin(t/1.3)$, as in Figure 2, with input flux $g(t) = e^{-t}$ and right hand side boundary condition $A(t) = 0$.

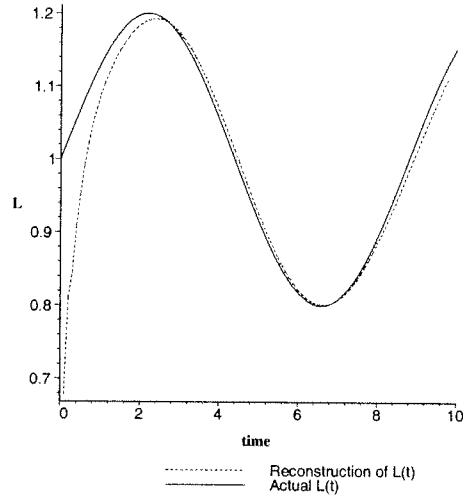


Figure 4: Reconstruction of $L(t)$ (assumed $L' \approx 0$).

Figure 5 shows the result of a reconstruction in which the condition that L' is small is violated. Here we have $L(t) = 0.7 \sin(t/2) + e^{t/10}$, flux $g(t) = e^{-t}$, with $A(t) = 0$. Despite the fairly rapidly changing value of L , the method still works well.

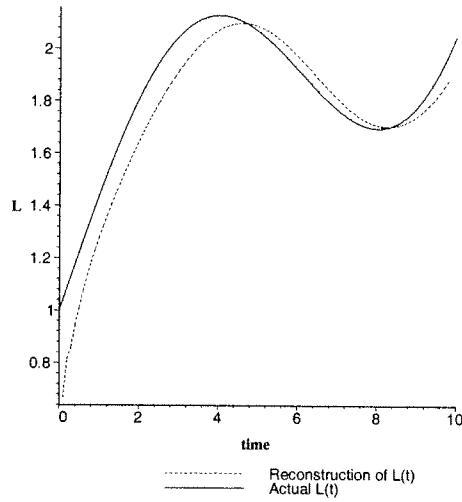


Figure 5: Reconstruction of $L(t)$ (assumption $L' \approx 0$ violated).

5 Conclusion

The first reconstruction method, though making more assumptions and giving slightly less accurate reconstructions, involves much less computation. The second method suffers less when the assumption of small (and slow) changes in the length is violated, but at the cost of more intensive numerical computations.

There is still plenty of future work to be done involving both methods for finding the length of the one-dimensional bar. For example, we have not done a thorough analysis of the error introduced by linearizing and assuming that $L(t)$ changes slowly in the first reconstruction method. Figure 2 also illustrates a curious phenomena which we do not fully understand—the reconstruction “lags” the true behavior of the bar; this holds true for both methods, though it is less pronounced for the second method. We also have not established solvability of the integral equation (35), nor stability properties for the solution.

6 Appendix: Two Identities

Let $u(t, x)$ denote a solution to the heat equation $u_t - u_{xx} = 0$ on a fixed domain $0 < x < L$ (so L is constant), with zero initial condition $u(0, x) = 0$. Let $\psi(t, x)$ be suitably smooth function for $0 < x < L$, $0 < t < T$ for some $T > 0$. Then we obviously have

$$\int_0^T \int_0^L \psi(t, x)(u_t(t, x) - u_{xx}(t, x)) dx dt = 0 \quad (37)$$

We can split this integral into two pieces to get,

$$\int_0^L \int_0^T \psi(t, x)u_t(t, x) dt dx - \int_0^T \int_0^L \psi(t, x)u_{xx}(t, x) dx dt = 0 \quad (38)$$

Integrate the first integral on the left above by parts in t and use $u(0, x) = 0$ to find,

$$\int_0^L \int_0^T \psi(t, x)u_t(t, x) dt dx = \int_0^L \psi(T, x)u(T, x) dx - \int_0^L \int_0^T \psi_t(t, x)u(t, x) dt dx \quad (39)$$

Now consider the second integral on the left in (38) and integrate it by parts in x :

$$\begin{aligned} \int_0^L \psi(t, x)u_{xx}(t, x) dx dt &= \int_0^T (\psi(t, L)u_x(t, L) - \psi(t, 0)u_x(t, 0)) dt \\ &\quad - \int_0^T \int_0^L \psi_x(t, x)u_x(t, x) dx dt \end{aligned}$$

Integrating by parts in x again on the second integral above yields

$$\int_0^L \psi(t, x)u_{xx}(t, x) dx dt = \int_0^T (\psi(t, L)u_x(t, L) - \psi(t, 0)u_x(t, 0)) dt$$

$$\begin{aligned}
& - \int_0^T (\psi_x(t, L)u(t, L) - \psi_x(t, 0)u(t, 0)) dt \\
& + \int_0^T \int_0^L \psi_{xx}(t, x)u(t, x) dx dt \tag{40}
\end{aligned}$$

Using the right sides of equations (39) and (40) to replace the integrals in (38) and rearranging, we find

$$\begin{aligned}
\int_0^L \psi(T, x)u(T, x) dx & - \int_0^L \int_0^T u(t, x)(\psi_t(t, x) + \psi_{xx}(t, x)) dt dx \\
& - \int_0^T (\psi(t, L)u_x(t, L) - \psi(t, 0)u_x(t, 0)) dt \\
& + \int_0^T (\psi_x(t, L)u(t, L) - \psi_x(t, 0)u(t, 0)) dt = 0
\end{aligned}$$

Now let us assume that $\psi(t, x)$ satisfies $\psi_t + \psi_{xx} = 0$, the so-called backwards heat equation, so that we obtain

$$\begin{aligned}
\int_0^T (\psi(t, 0)u_x(t, 0) - \psi_x(t, 0)u(t, 0)) dt + \int_0^L \psi(T, x)u(T, x) dx \\
= \int_0^T (\psi(t, L)u_x(t, L) - \psi_x(t, L)u(t, L)) dt
\end{aligned}$$

which is exactly equation (5).

We now show the derivation of identity (27). First we will start with the equation (26), multiply both sides by our special function $\psi(t, y)$ and integrate to get

$$\int_0^T \int_0^L \psi(t, y)(r_t(t, y) - r_{yy}(t, y)) dy dt = \int_0^T \int_0^L \psi(t, y)(yq'(t)V_y(t, y) - 2q(t)V_{yy}(t, y)) dy dt$$

The only difference between the above equation and (37) is that the right side does not equal zero. Therefore, doing the same steps as before, we get

$$\begin{aligned}
\int_0^T (\psi(t, 0)r_y(t, 0) - \psi_y(t, 0)r(t, 0)) dt & = - \int_0^L \phi(T, y)r(T, y) dy \\
& + \int_0^T (\psi(t, L)r_y(t, L) - \psi_y(t, L)r(t, L)) dt \\
& + \int_0^T \int_0^L \psi(t, y)(yq'(t)V_y(t, y) - 2q(t)V_{yy}(t, y)) dy dt
\end{aligned}$$

We can simplify this by using the boundary conditions of $r(t, y)$ to obtain

$$- \int_0^T (\psi(t, 0)q(t)g(t) + \psi_y(t, 0)(a(t) - V(t, 0)) dt =$$

$$\begin{aligned}
& - \int_0^L \psi(T, y)r(T, y)dy + \int_0^T \psi(t, L)r_y(t, L) dt \\
& + \int_0^T \int_0^L \psi(t, y)(yq'(t)V_y(t, y) - 2q(t)V_{yy}(t, y)) dy dt
\end{aligned}$$

which is exactly (27).

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